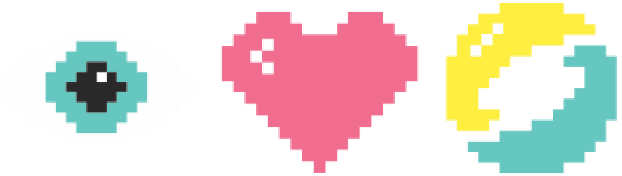
30 JULY–3 AUGUST Los Angeles

SIGGRAPH2017

AT THE ♥ of COMPUTER GRAPHICS & INTERACTIVE TECHNIQUES



κ-Curves: Interpolation at Local Maximum Curvature

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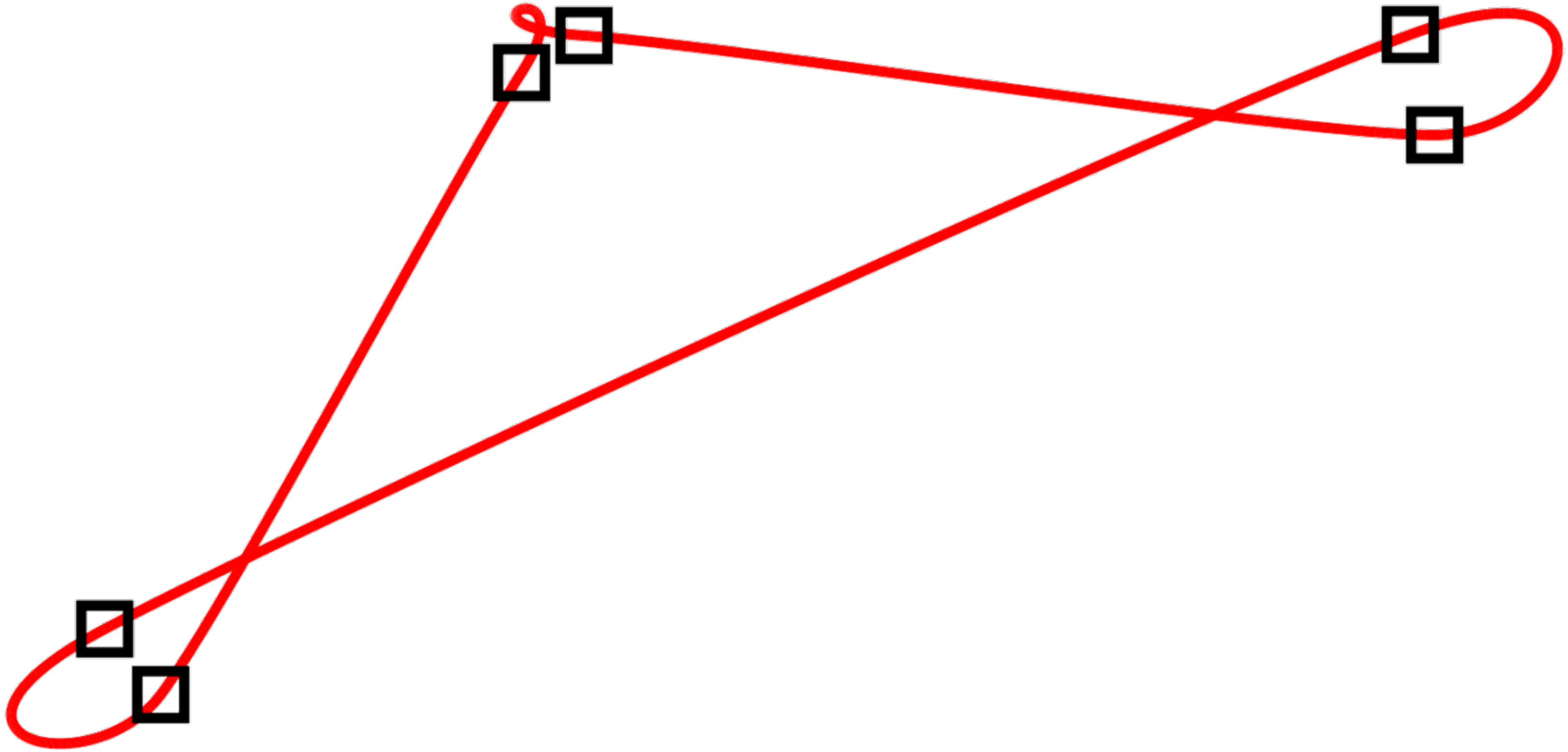
[1]Texas A&M University, [2]Adobe Research, [3]Adobe

Problems with Interpolation

Catmull-Rom Spline



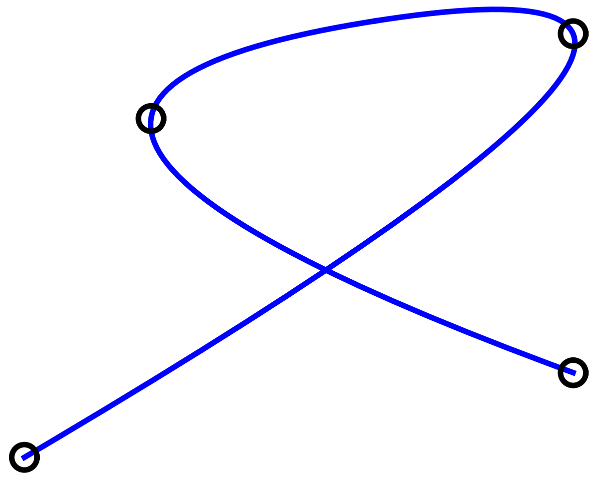
“… human visual system is sensitive to minima and maxima of curvature …”

“… all curvature extrema should coincide with the given control points …”

[Levien 2009]

Desired Properties

1. Interpolate all control points
2. Max curvatures (salient points) only occur at control points
   ---- No cusps except at control points
3. Curvature continuous

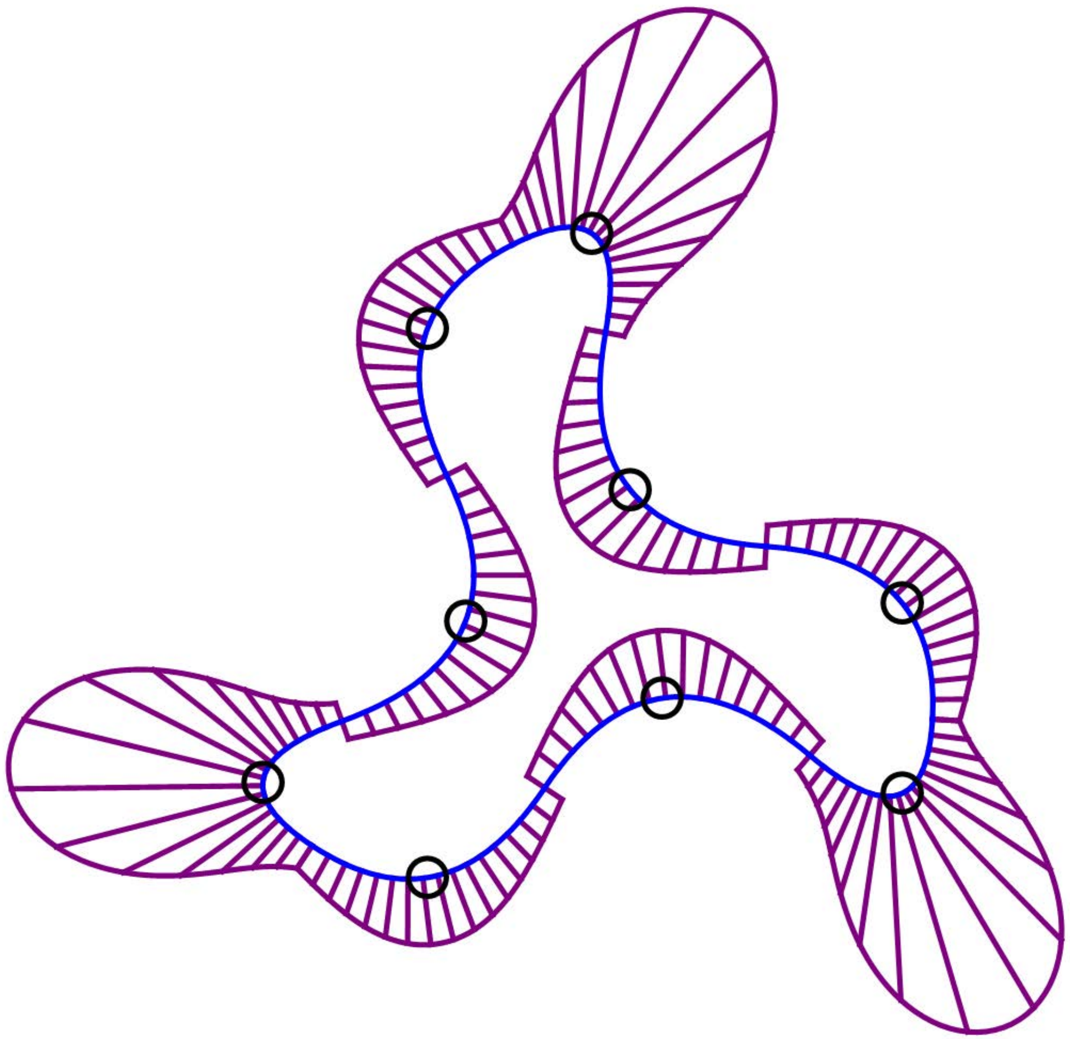


Clothoid

Linear curvature

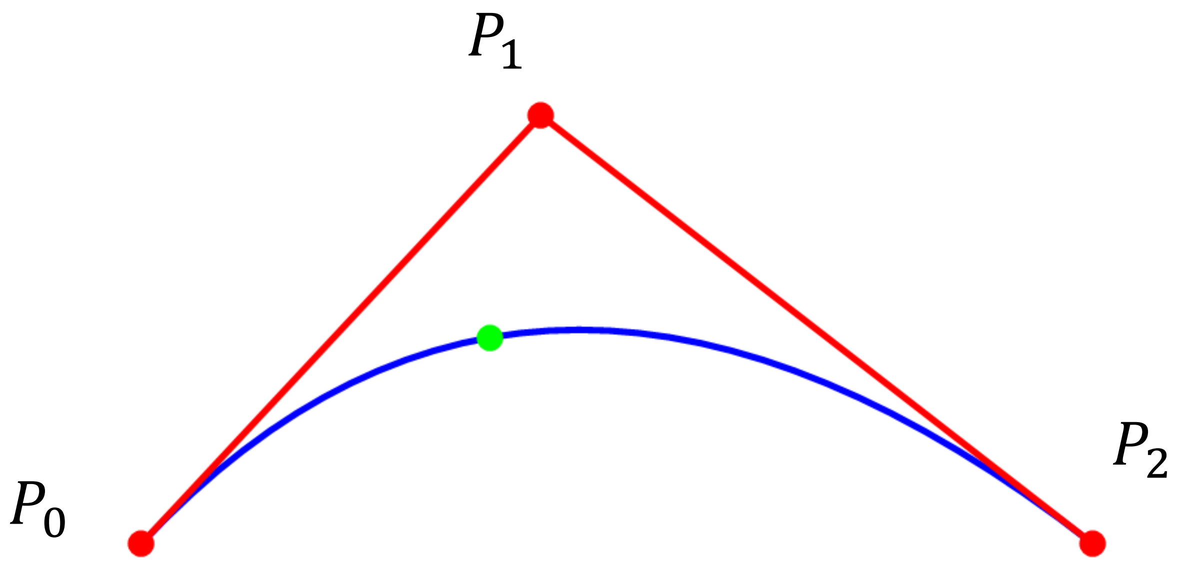
Not continuous with motion of control points

Quadratic spline



Quadratic Bezier Curve

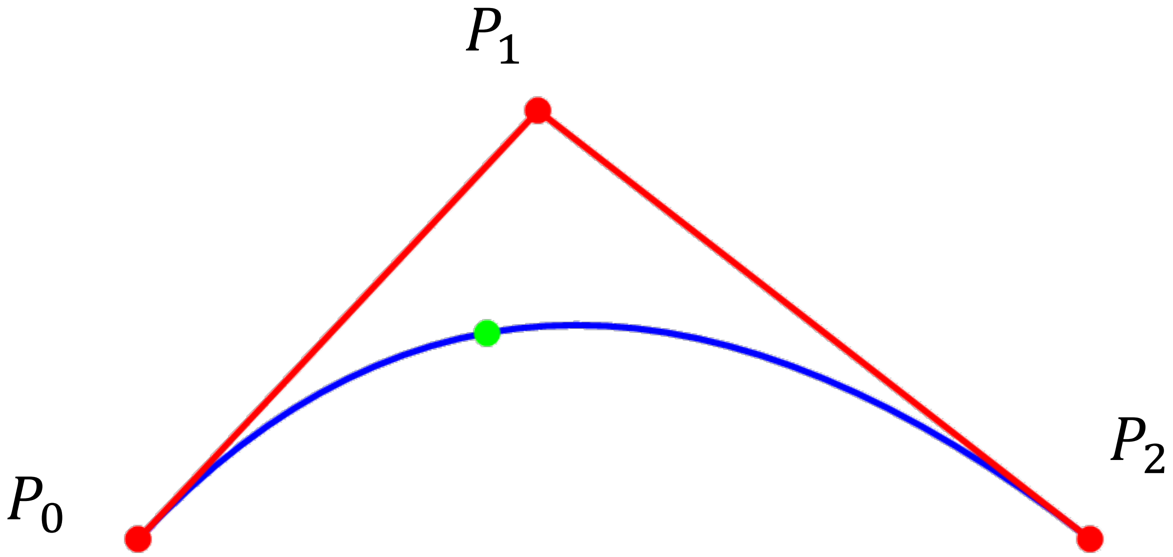
$$c(t) = (1-t)^2 P_0 + 2(1-t)tP_1 + t^2 P_2$$



Quadratic Bezier Curvature

$$\kappa(t) = \frac{\det(c'(t), c''(t))}{||c'(t)||^3}$$

$$= \frac{\Delta(P_0,P_1,P_2)}{||(1-t)(P_1-P_0)+t(P_2-P_1)||^3}$$



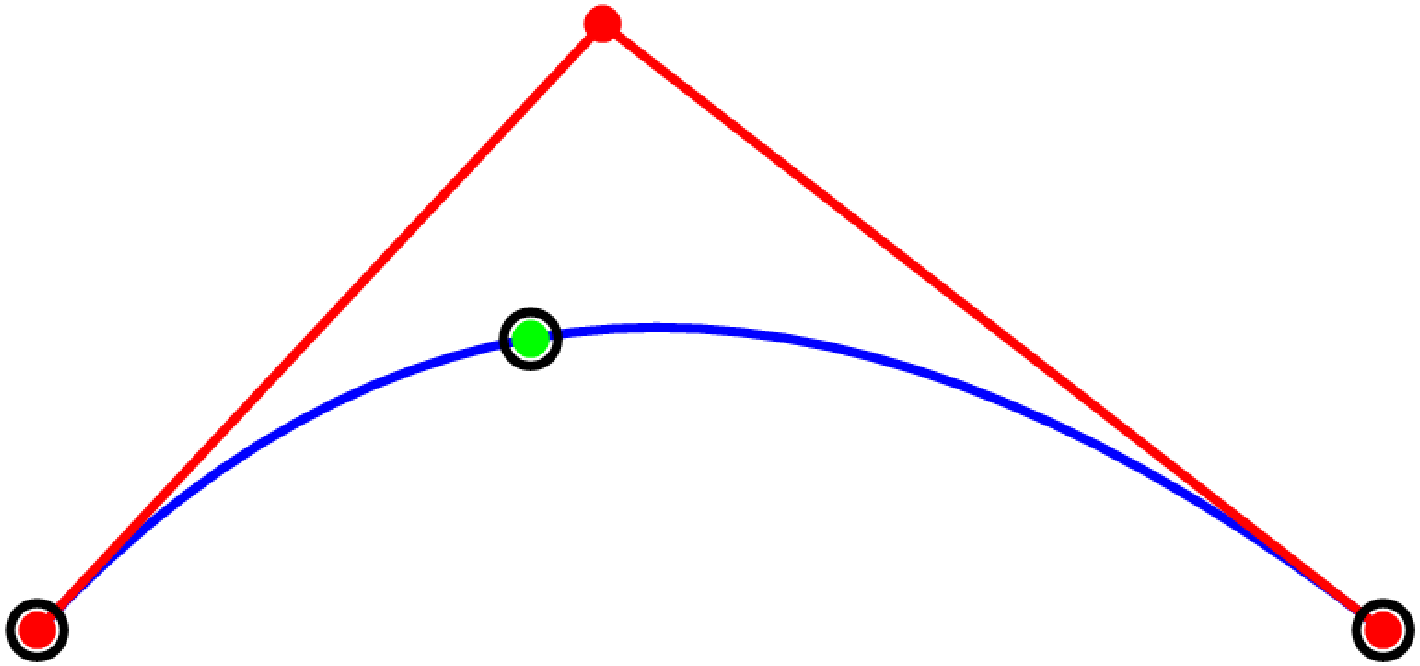
$$\kappa'(t) = 0 \Rightarrow$$

$$t_{max} = \frac{(P_0-P_1)\cdot(P_0-2P_1+P_2)}{(P_0-2P_1+P_2)\cdot(P_0-2P_1+P_2)}$$

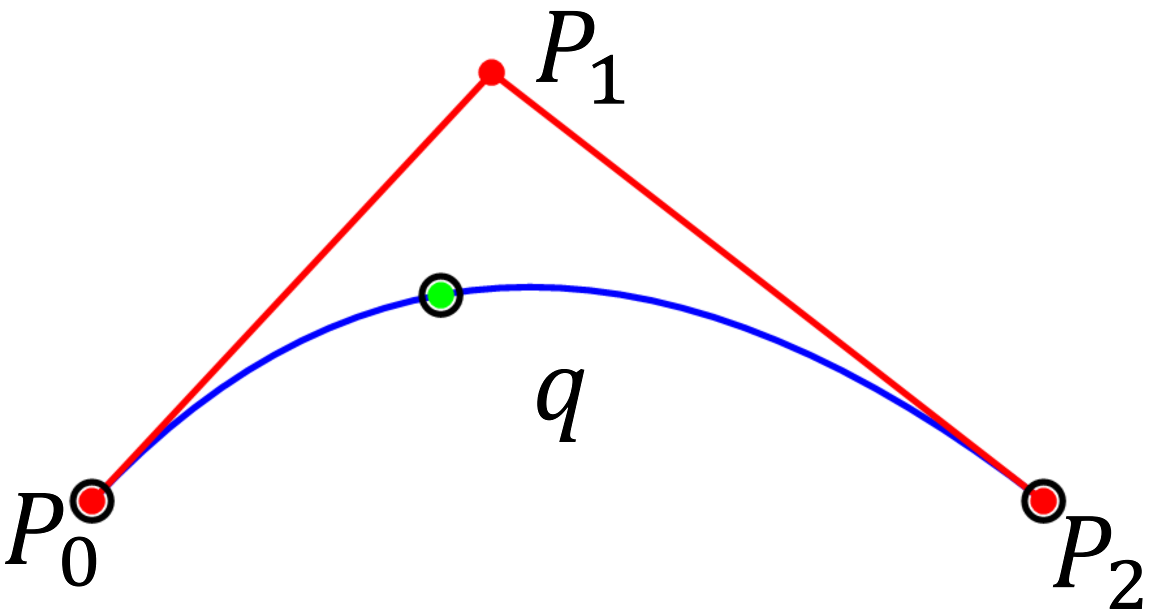
$$\kappa(t_{max}) = \max \kappa(t)$$

Interpolation at Max Curvature

Interpolation at Max Curvature

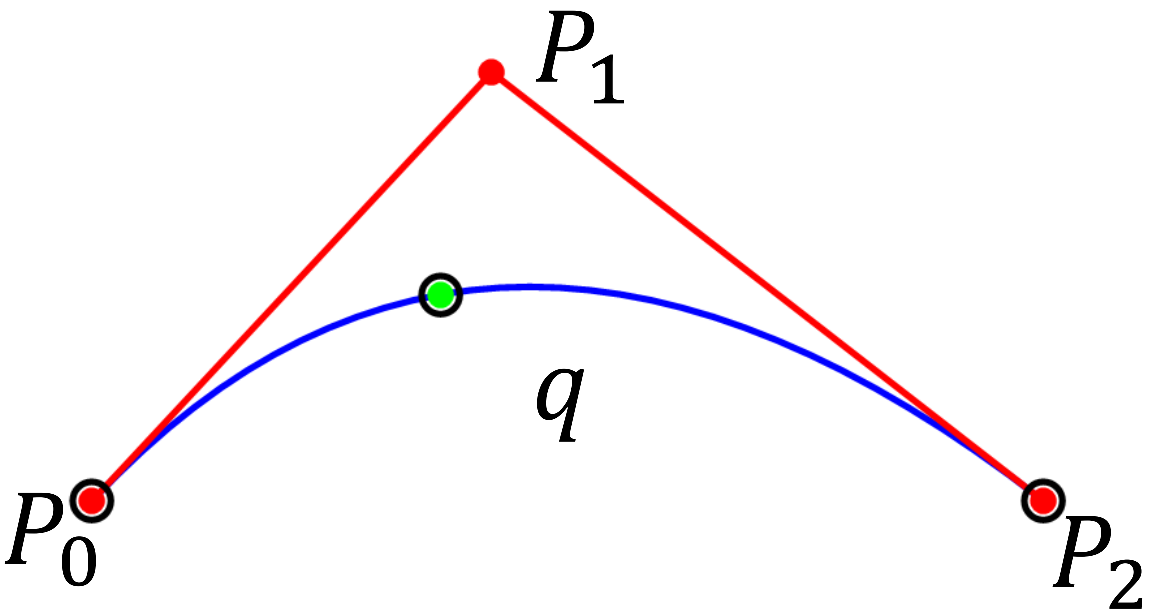


Quadratic Time Parameter



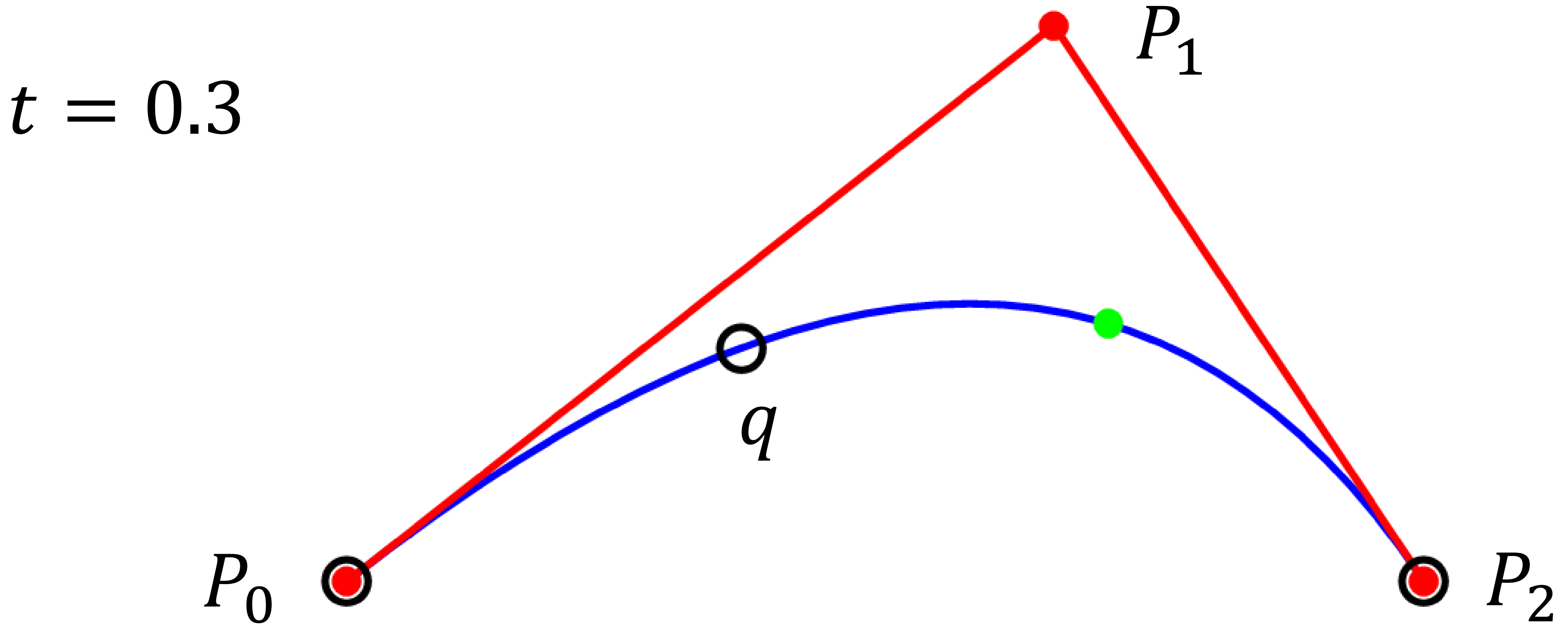
$$q = (1-t)^2 P_0 + 2(1-t)t P_1 + t^2 P_2$$

Quadratic Time Parameter



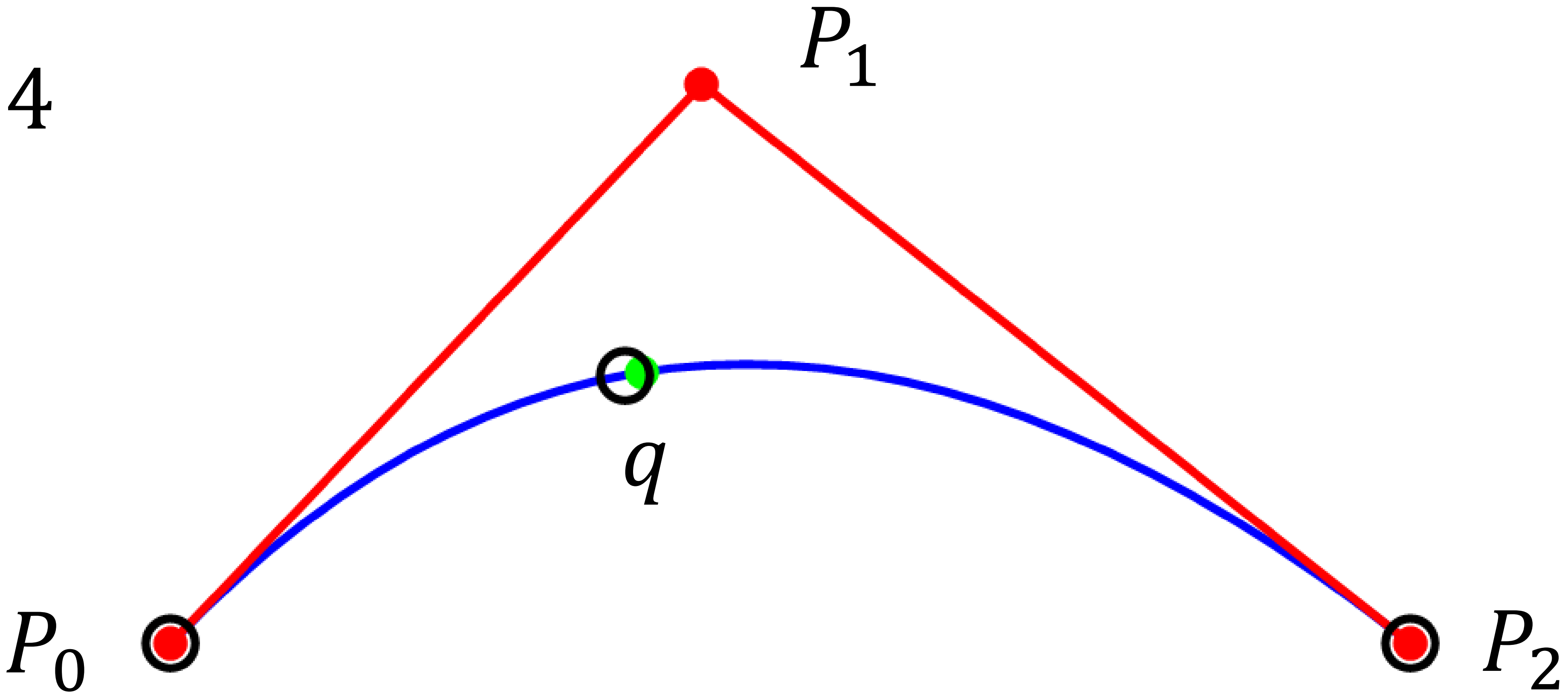
$$P_1 = \frac{q - (1-t)^2 P_0 - t^2 P_2}{2(1-t)t}$$

Quadratic Time Parameter



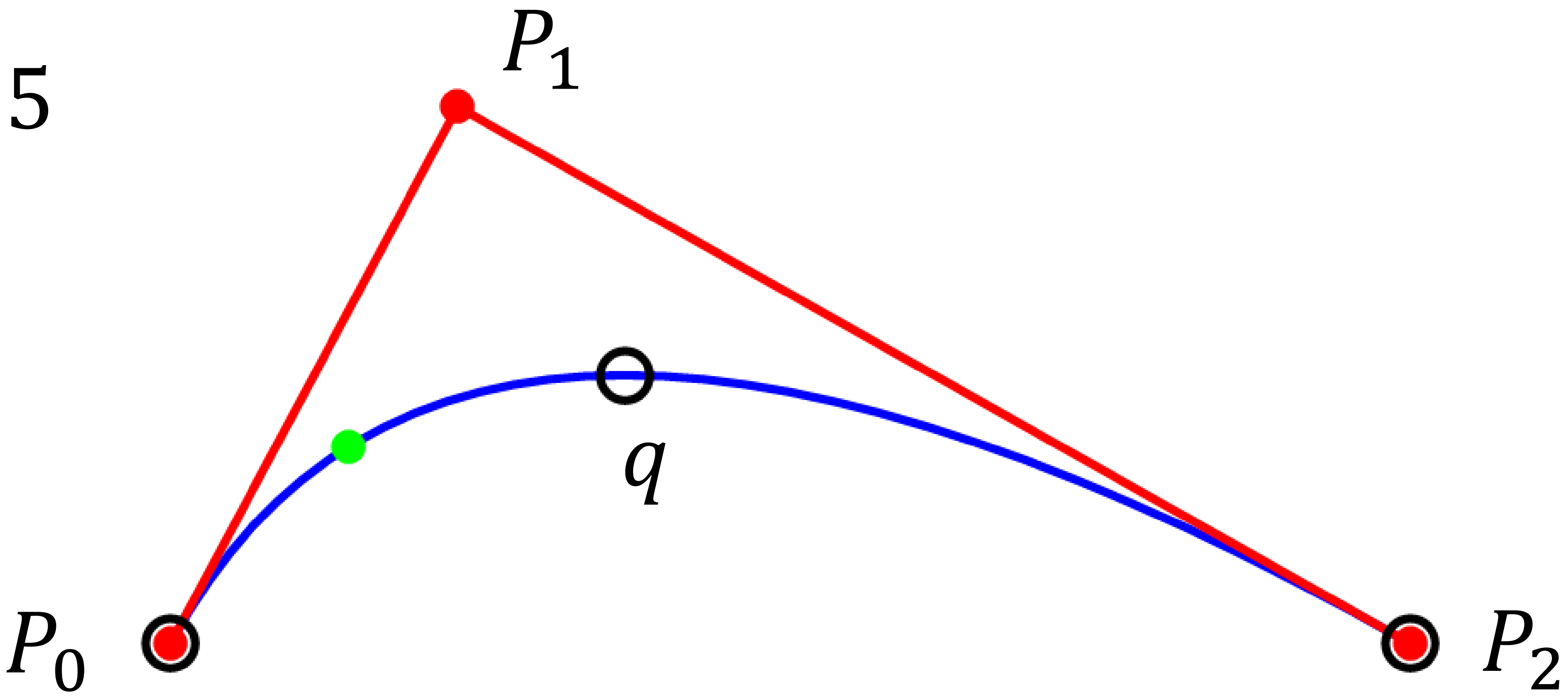
Quadratic Time Parameter

$t = 0.4$

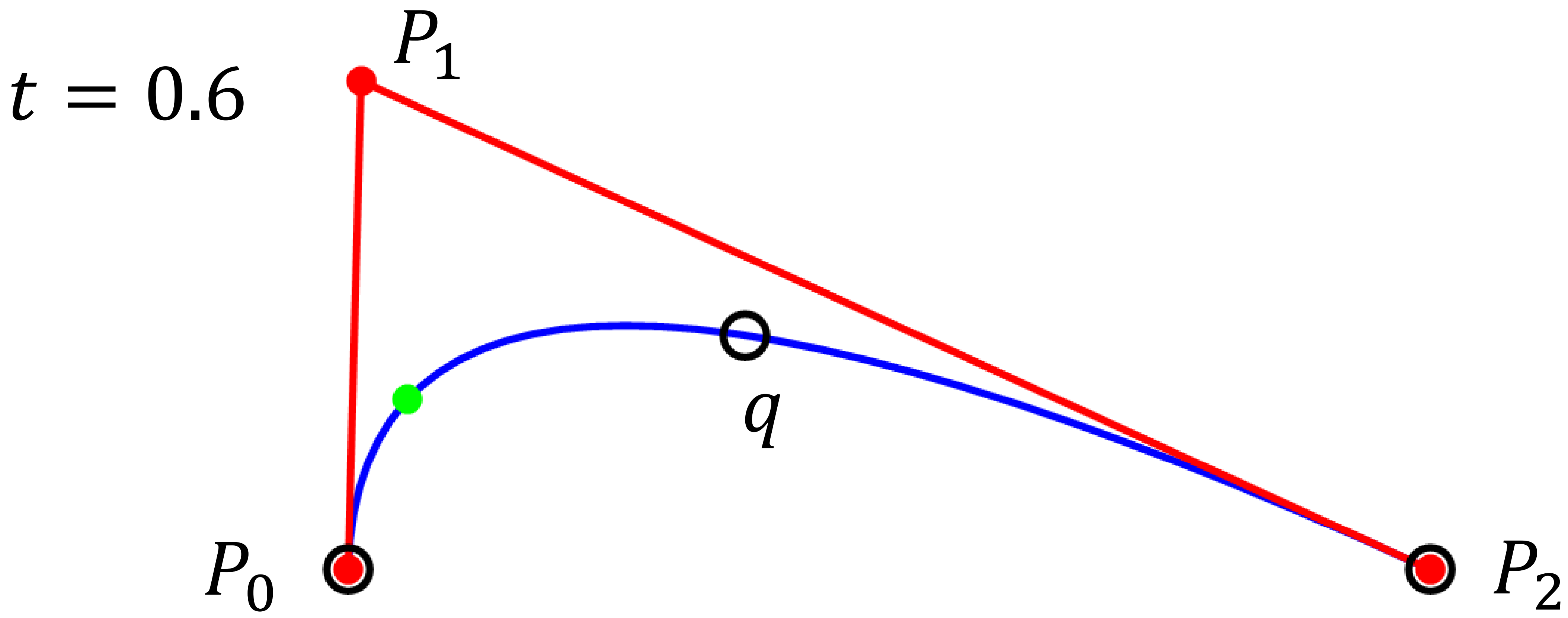


Quadratic Time Parameter

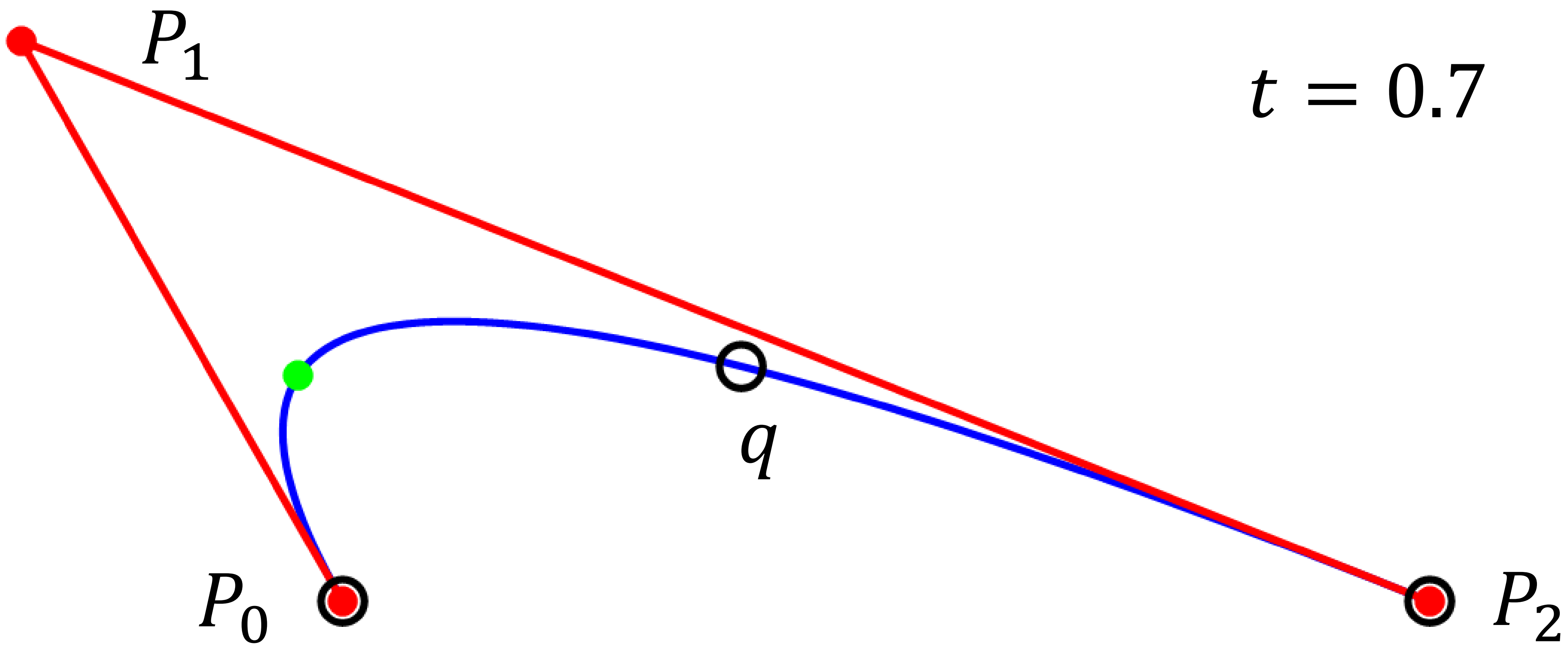
$t = 0.5$



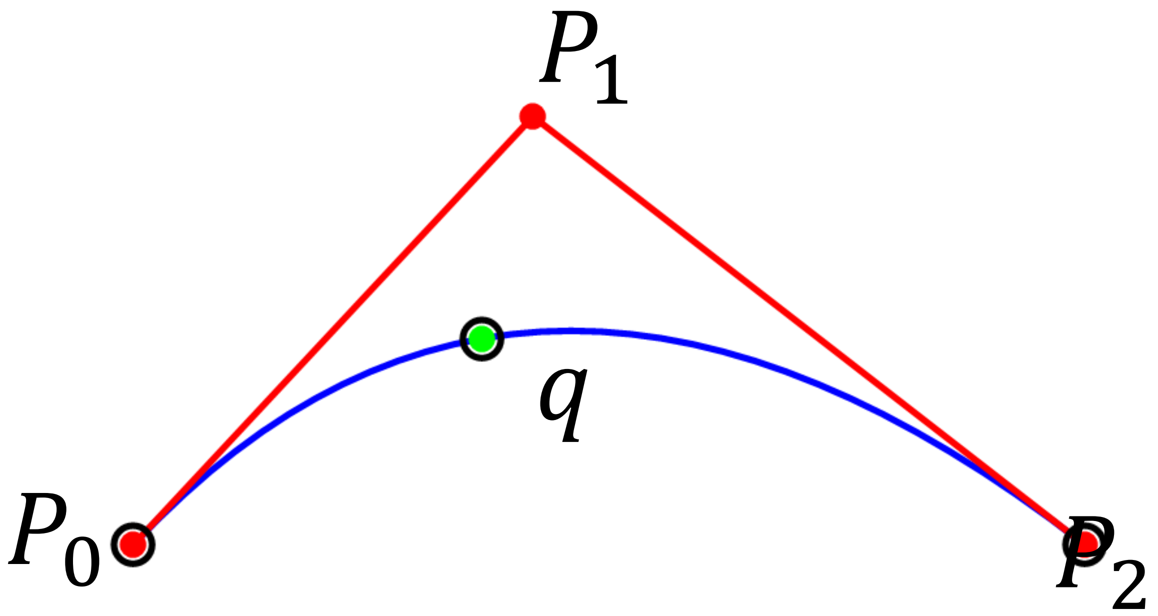
Quadratic Time Parameter



Quadratic Time Parameter



Quadratic Time Parameter



Unique $t \in (0,1)$ from P_0, q, and P_2

Computation of Max Curvature Time

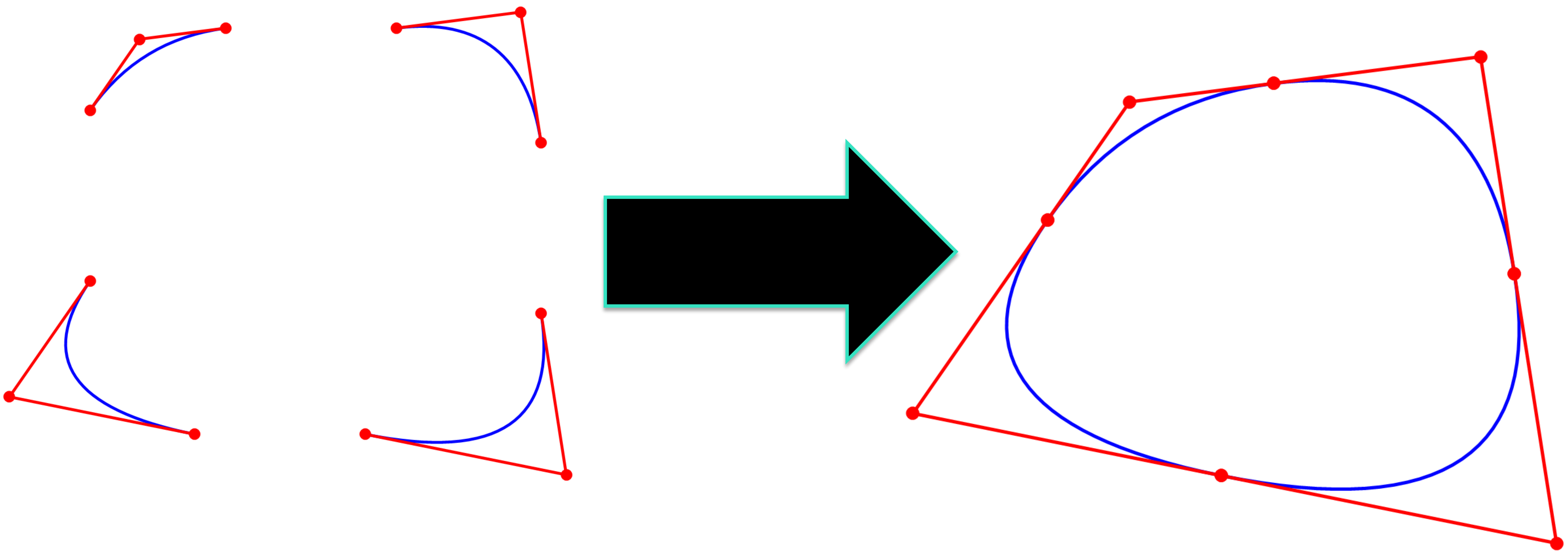
$$P_1 = \frac{q-(1-t_{max})^2 P_0 - {t_{max}}^2 P_2}{2(1-t_{max})t_{max}} \qquad (1)$$

$$t_{max} = \frac{(P_0-P_1).(P_0-2P_1+P_2)}{(P_0-2P_1+P_2).(P_0-2P_1+P_2)} \qquad (2)$$

Substitute (1) into (2)

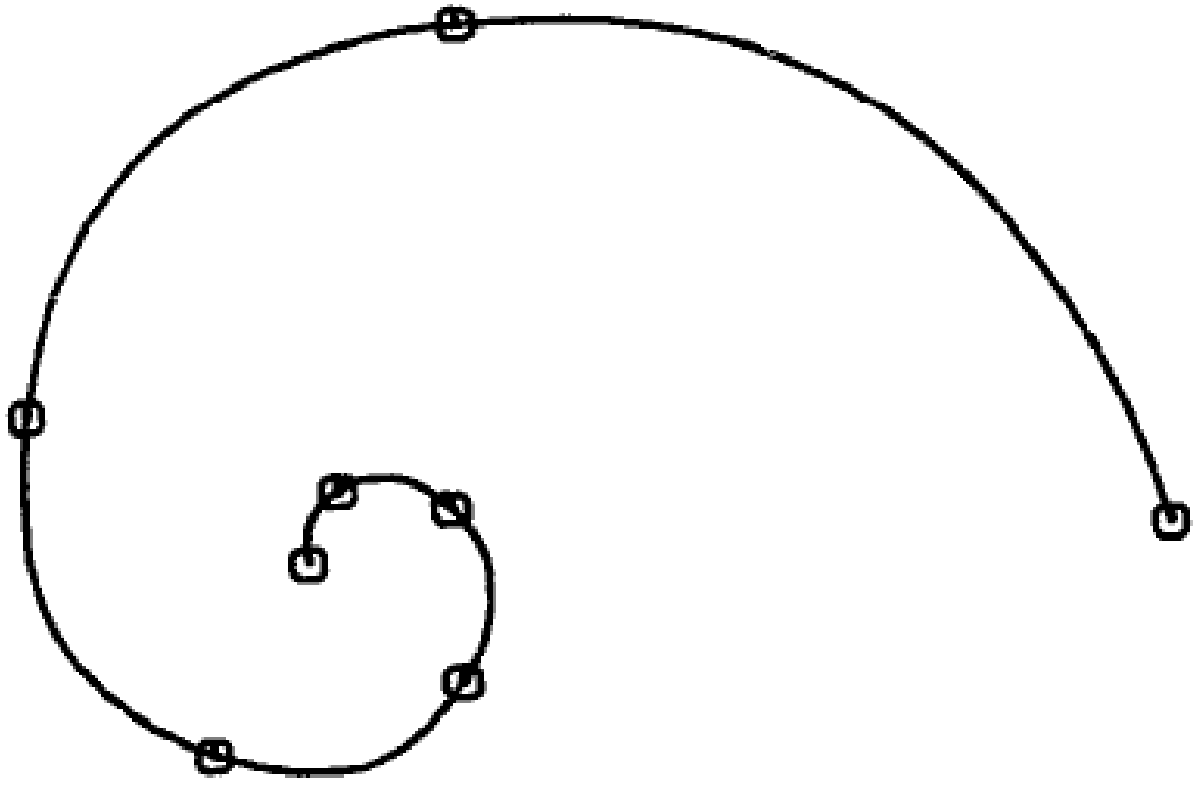
a${t_{max}}^3 + b{t_{max}}^2 + ct_{max} + d = 0$

Unique $t_{max} \in (0,1)$ as a function of P_0, q, and P_2



Related works

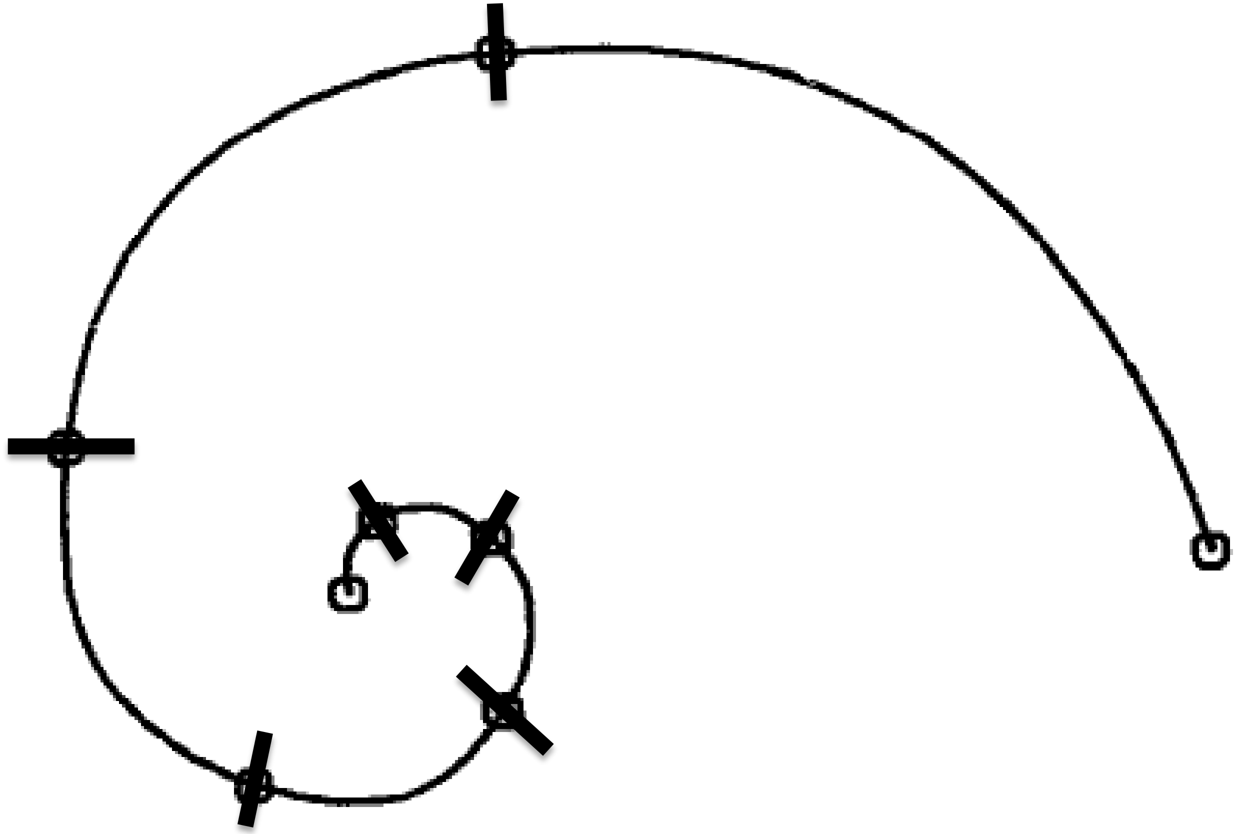
[Schaback 1989]



Related works

[Schaback 1989]

Use input points as splitting points

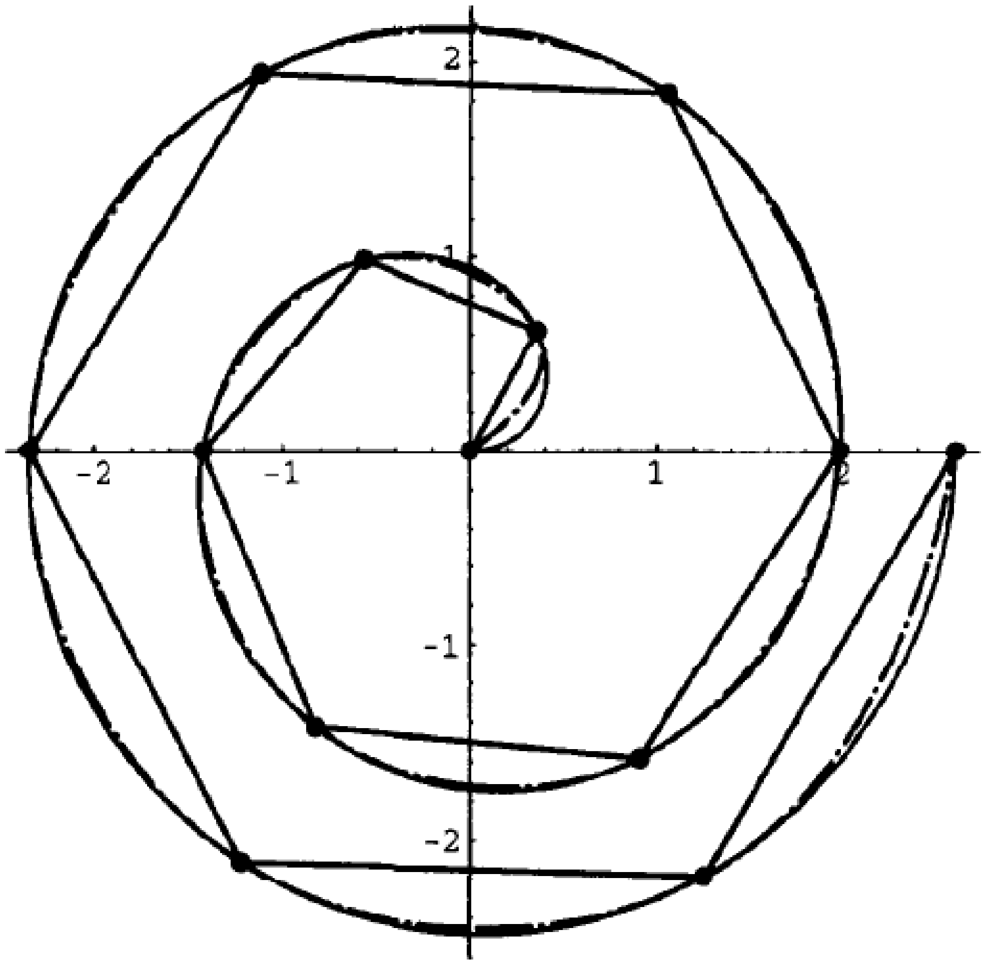


Related works

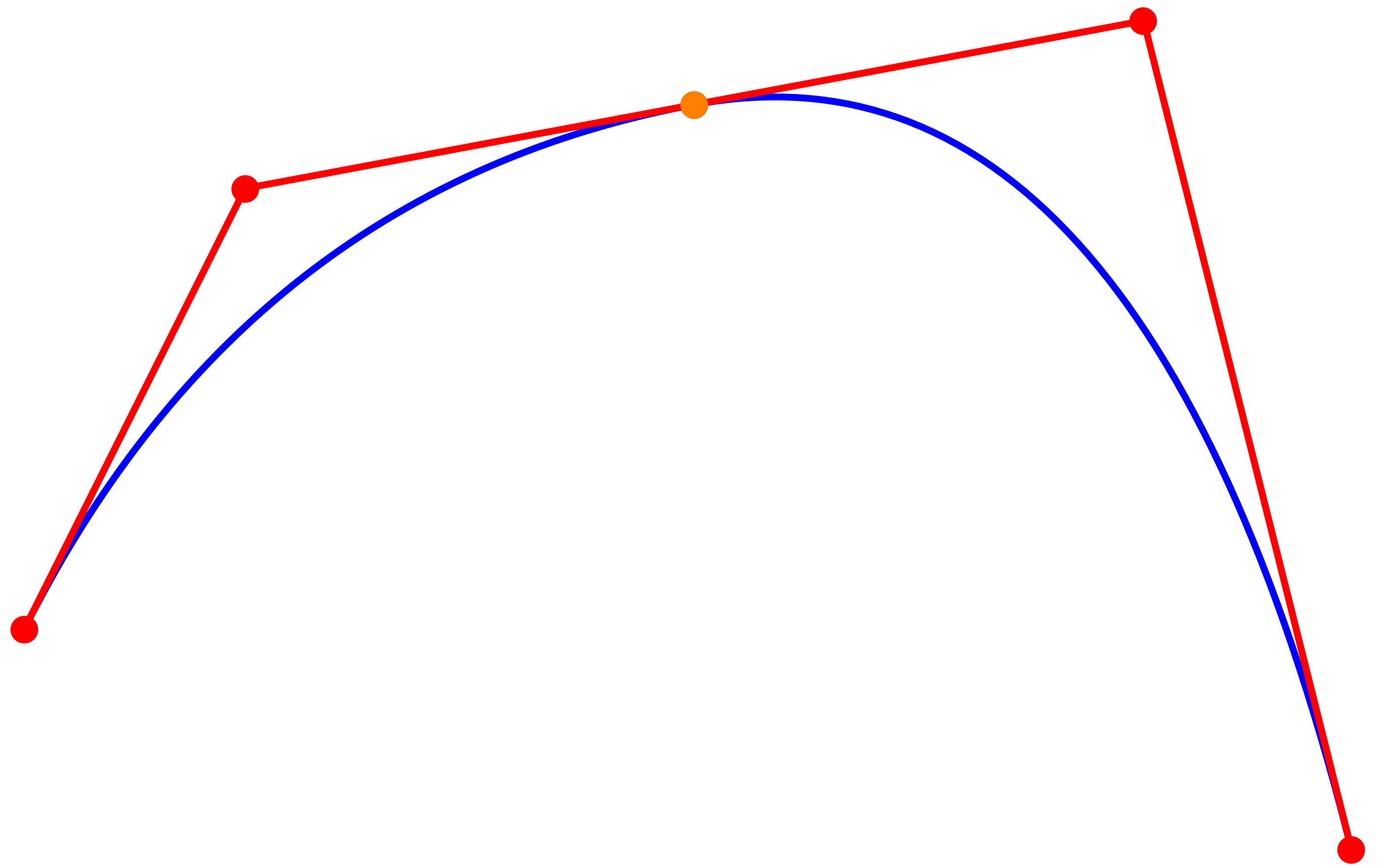
[Feng et al. 1995]

Interpolate tangent vectors at control points

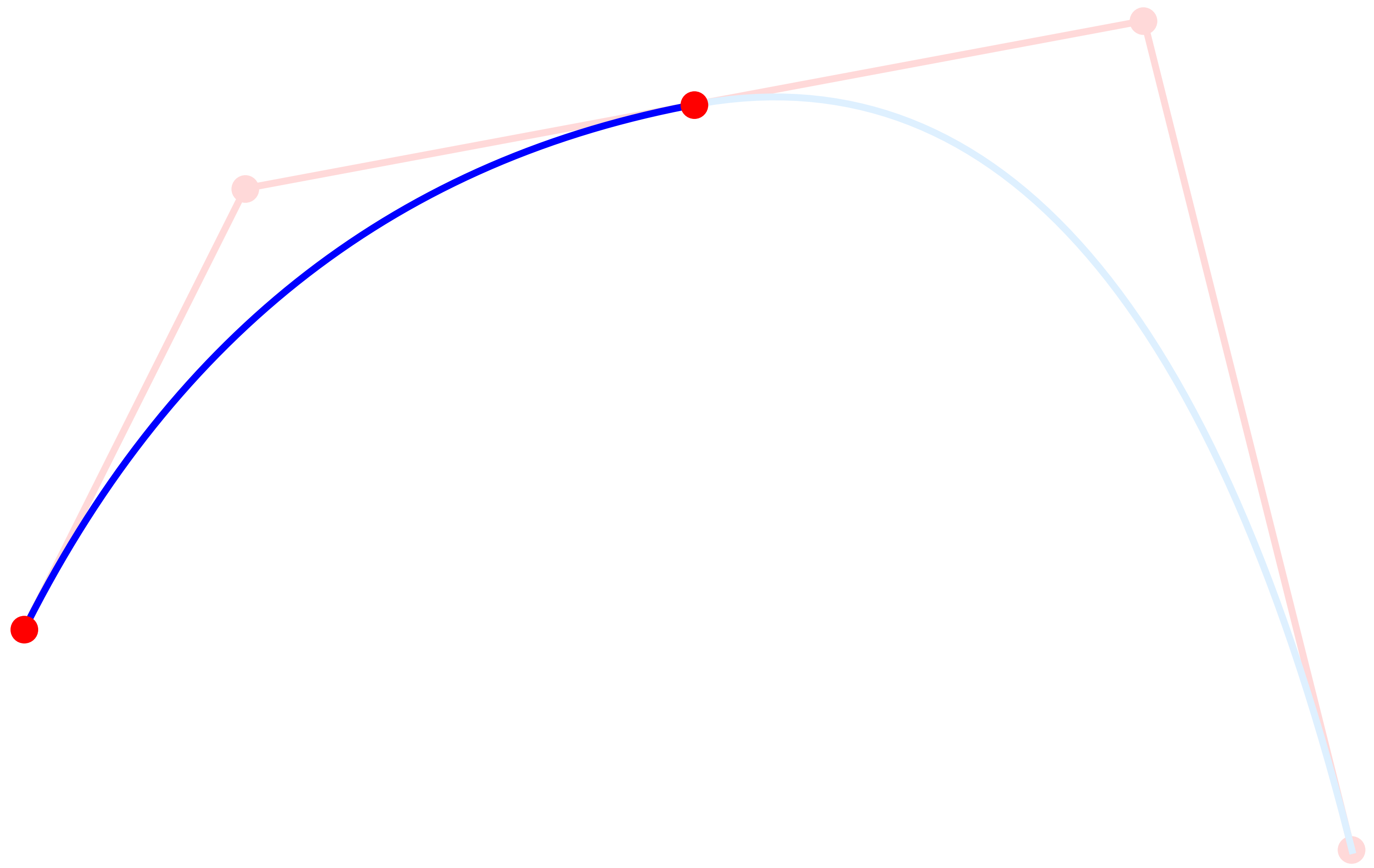
No curvature control.



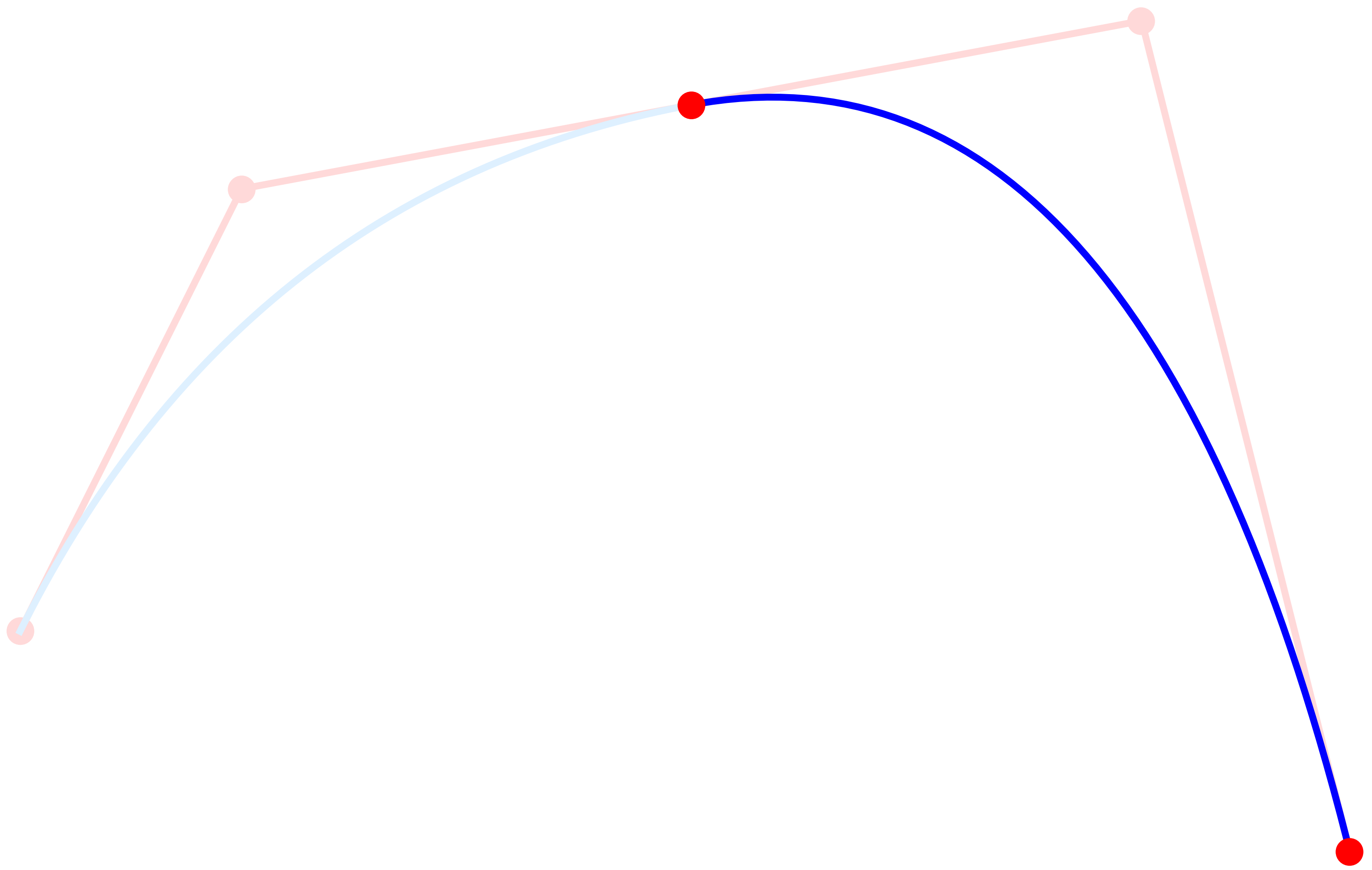
G^2 Condition



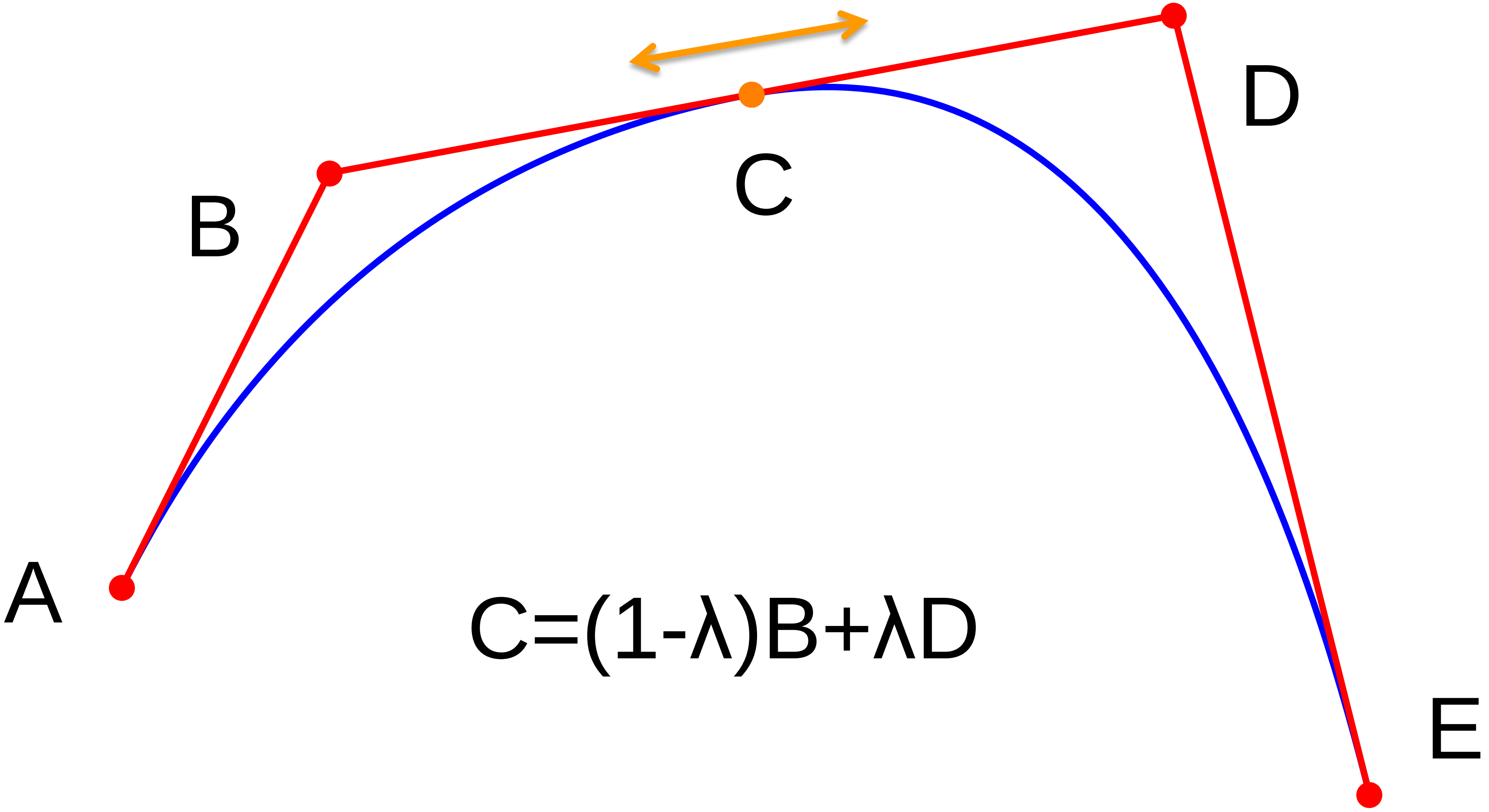
G^2 Condition



G^2 Condition



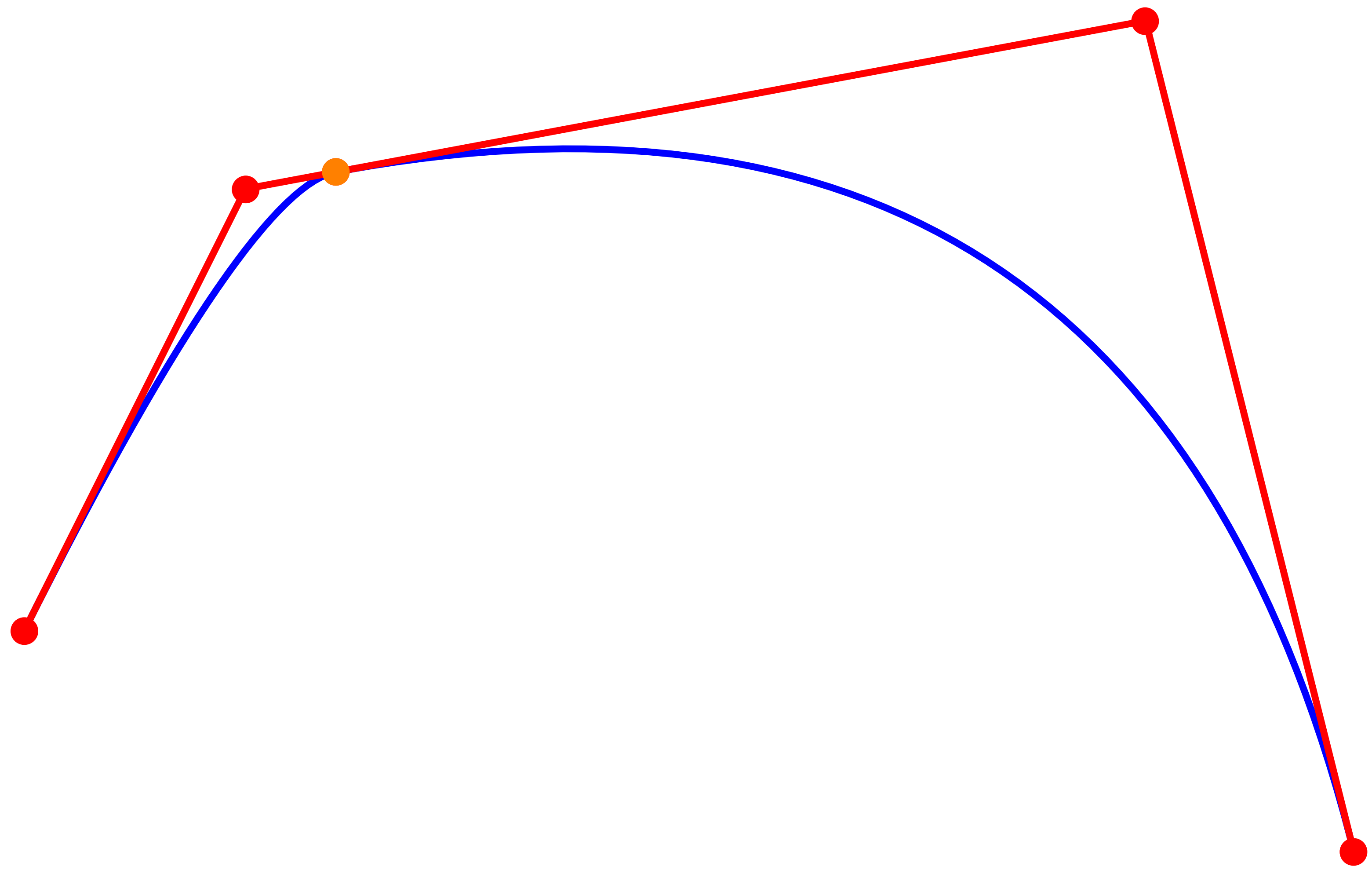
G^2 Condition



$$C=(1-\lambda)B+\lambda D$$

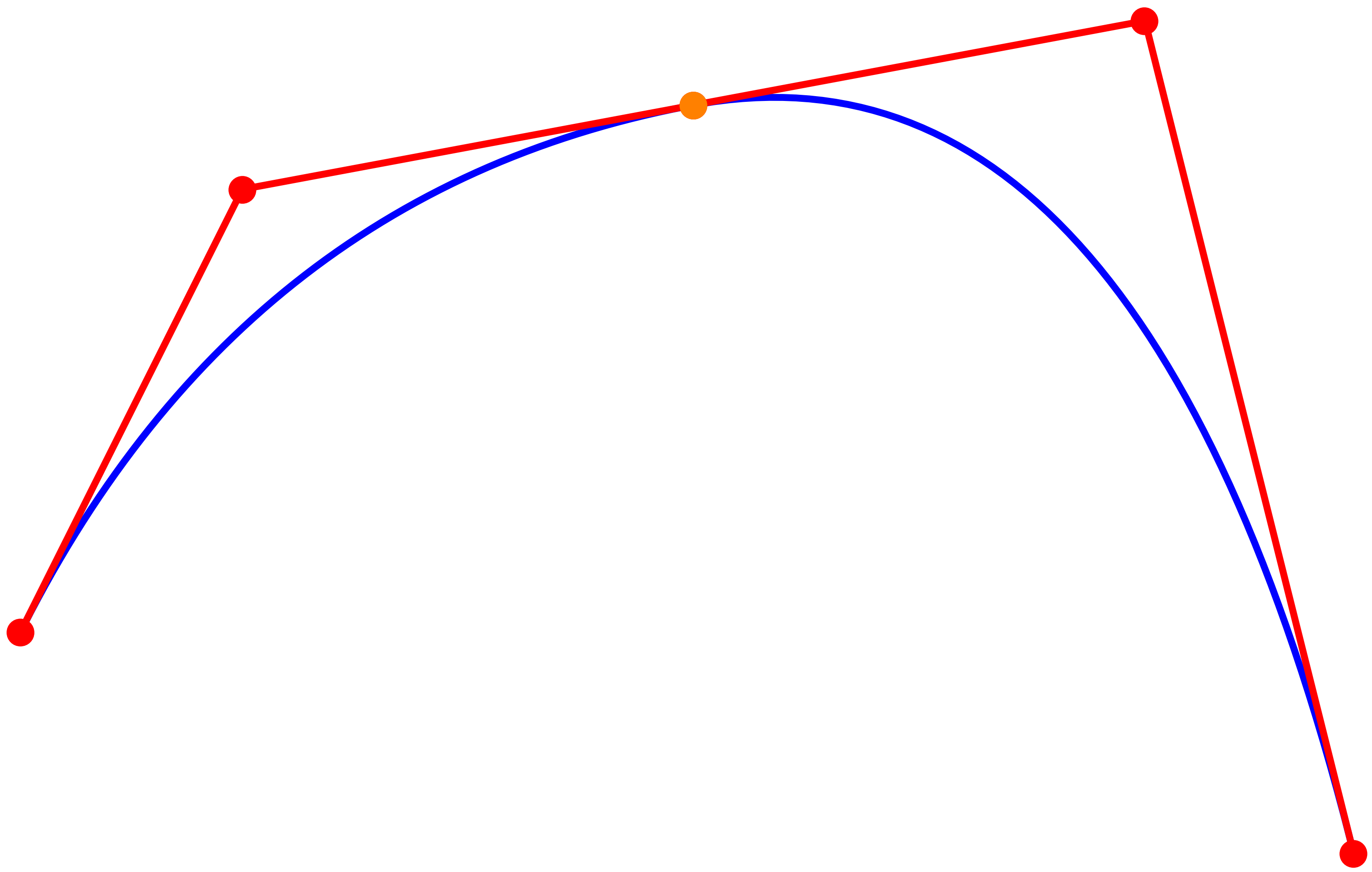
G^2 Condition

$\lambda = 0.3$



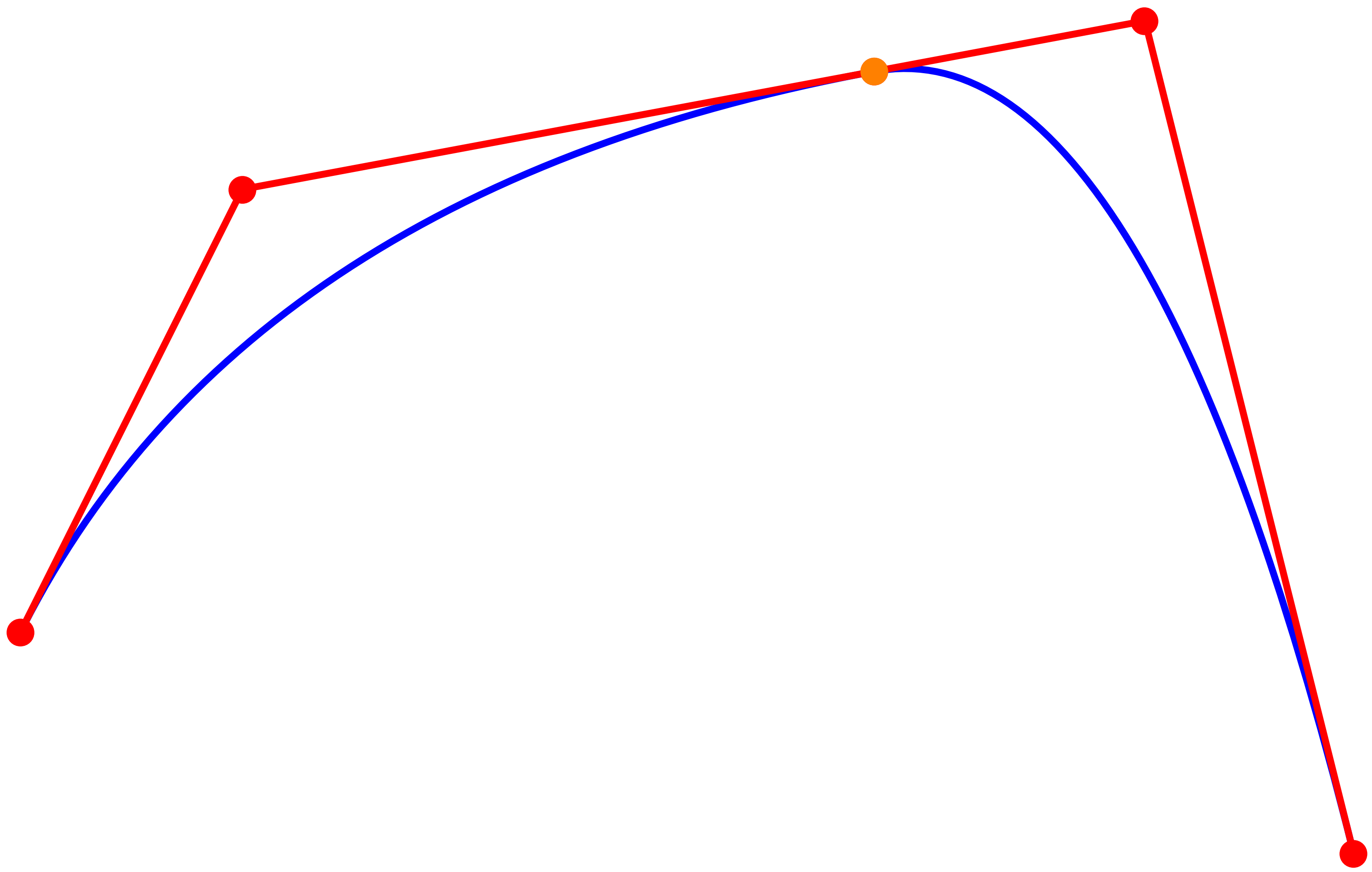
G^2 Condition

$\lambda = 0.5$

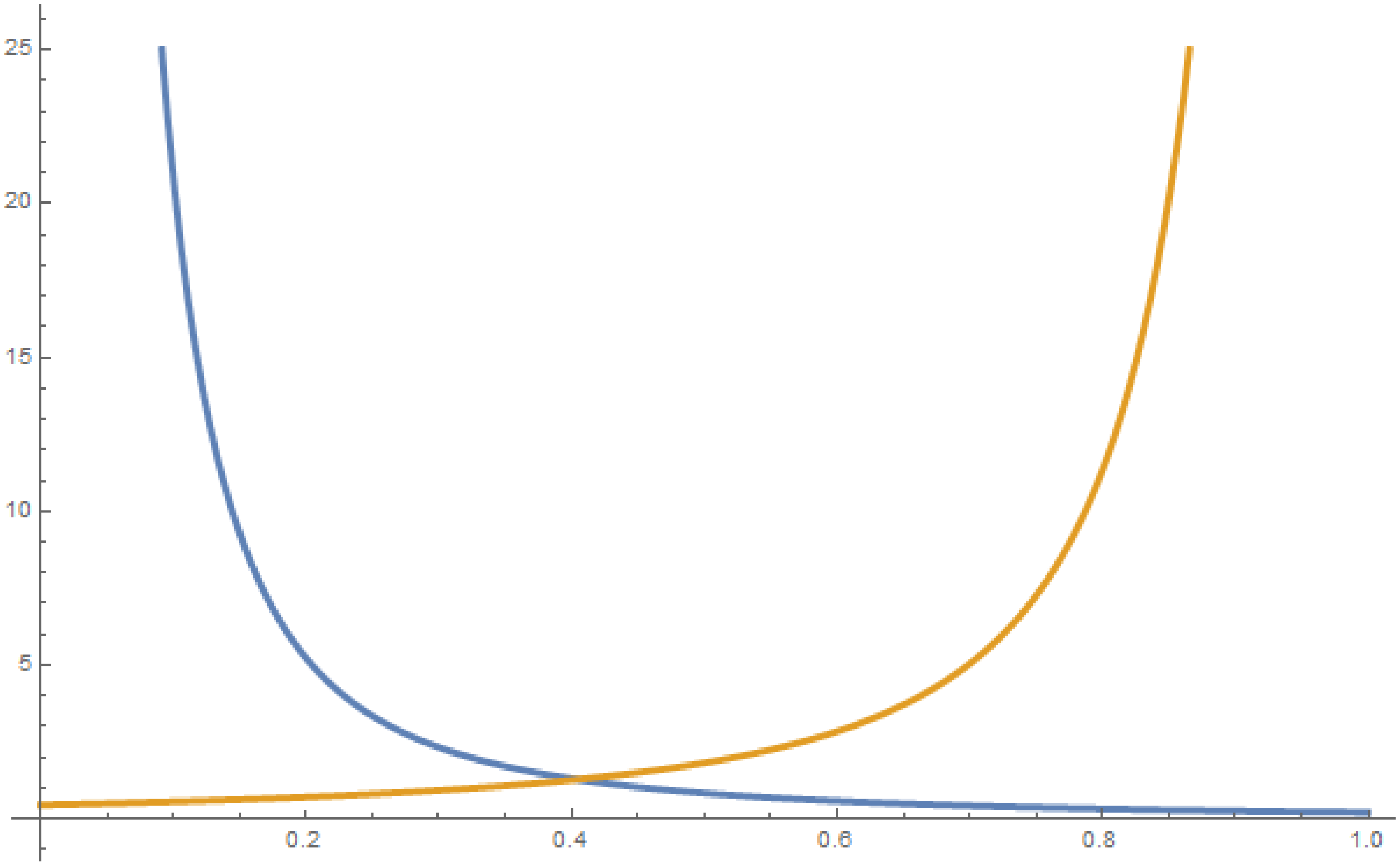


G^2 Condition

$\lambda = 0.7$

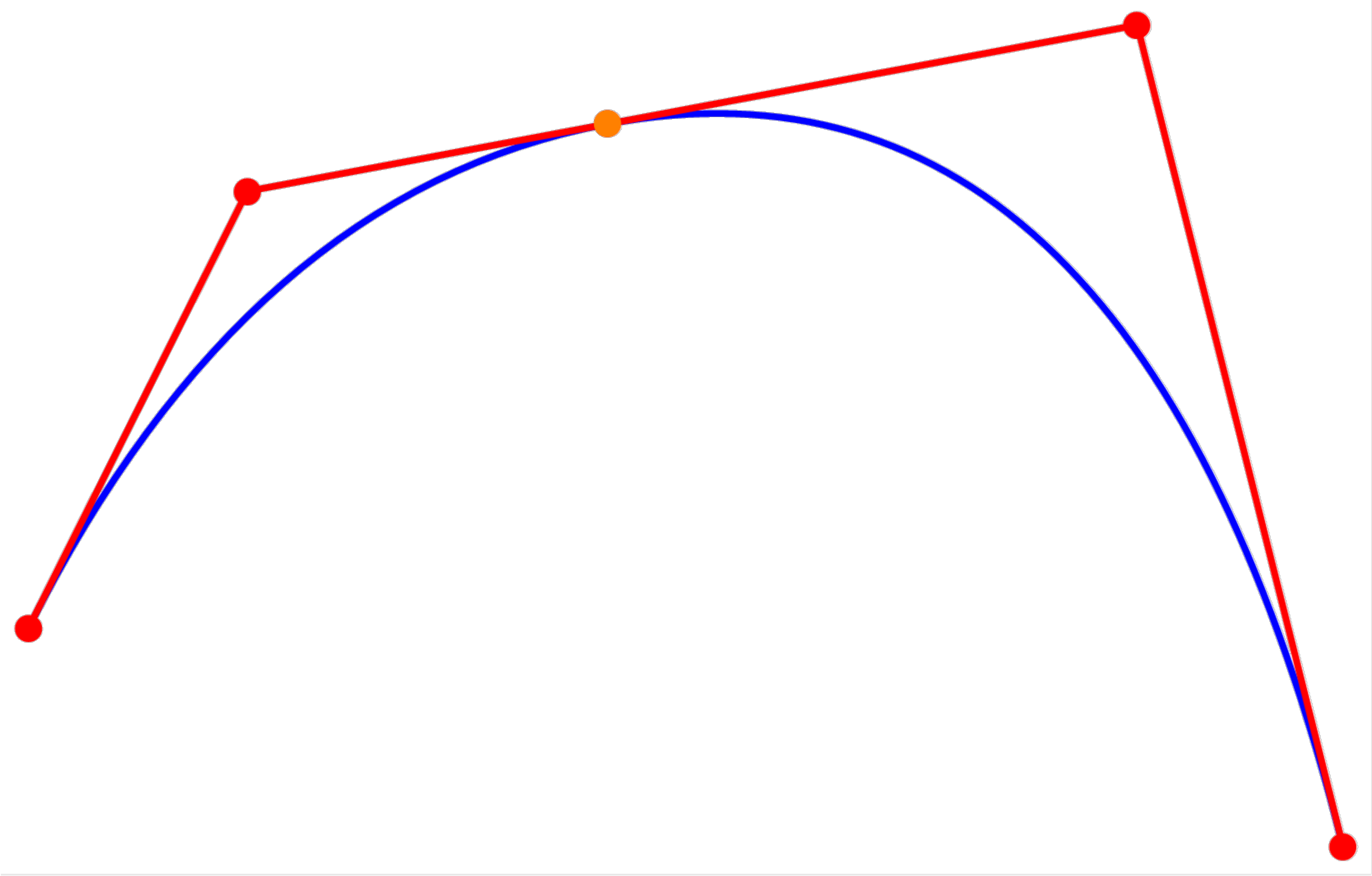


Curvature



G^2 Condition

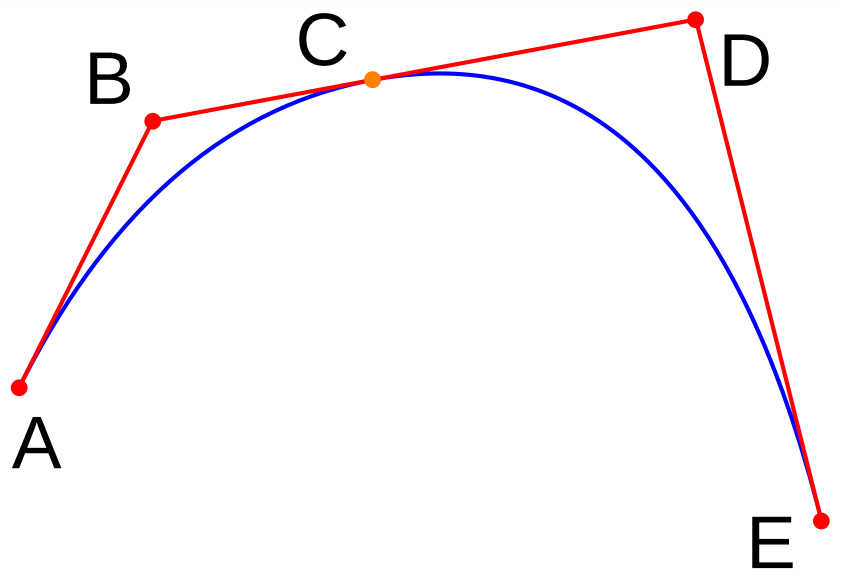
$\lambda = 0.40$



G^2 Joint

$$C = (1 - \lambda)B + \lambda D$$

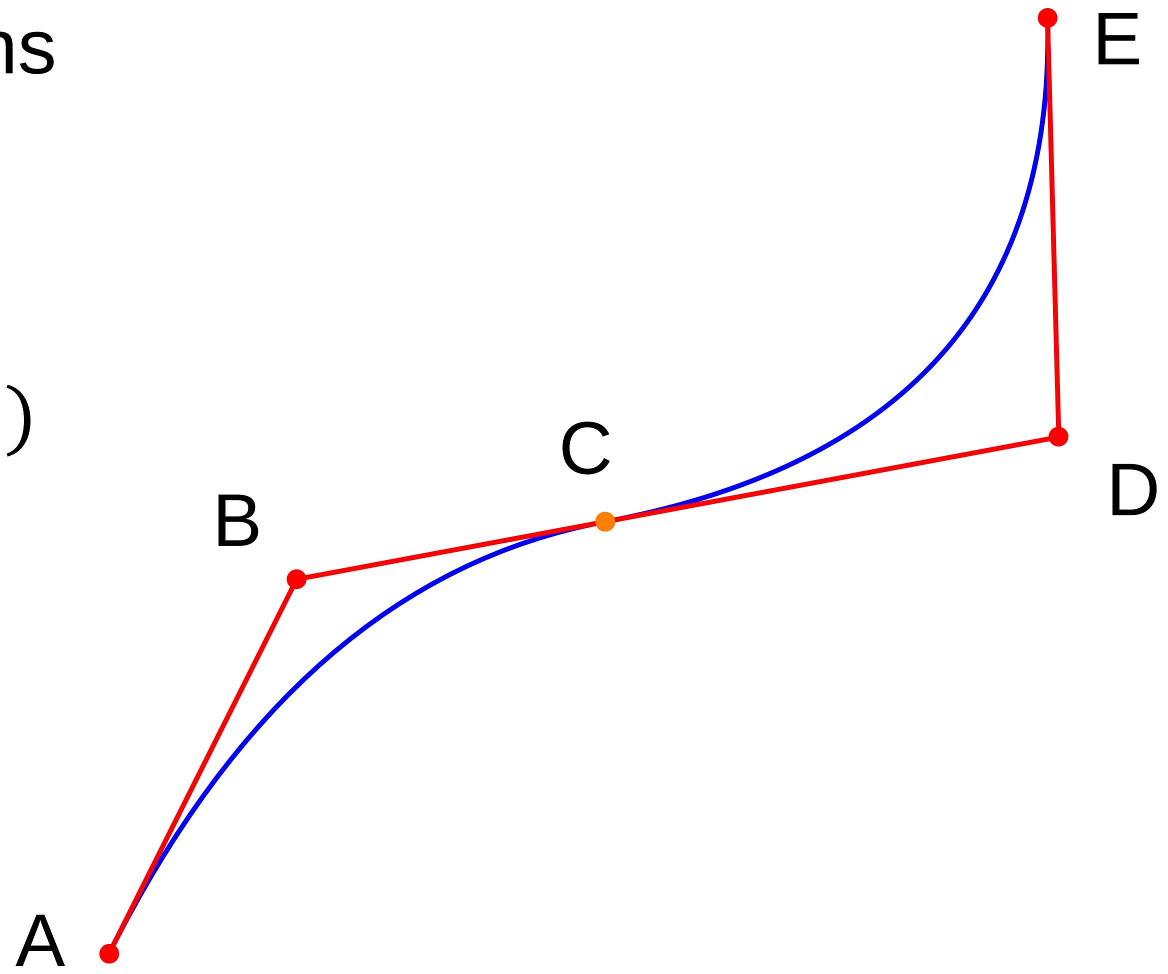
$$\lambda = \frac{\sqrt{\Delta(A, B, D)}}{\sqrt{\Delta(A, B, D)} + \sqrt{\Delta(B, D, E)}} \in (0,1)$$



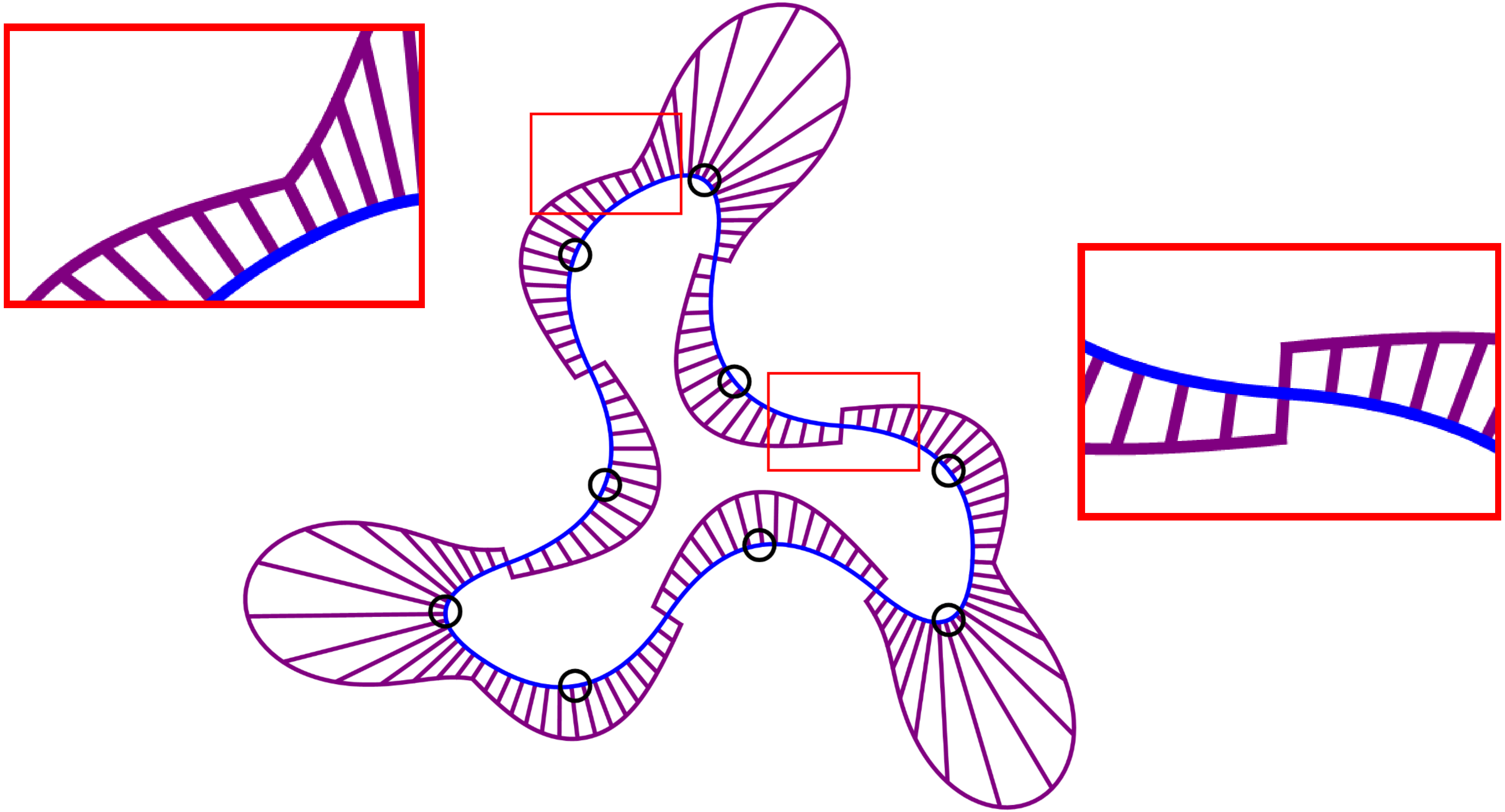
Inflection Point

- Curvatures have different signs

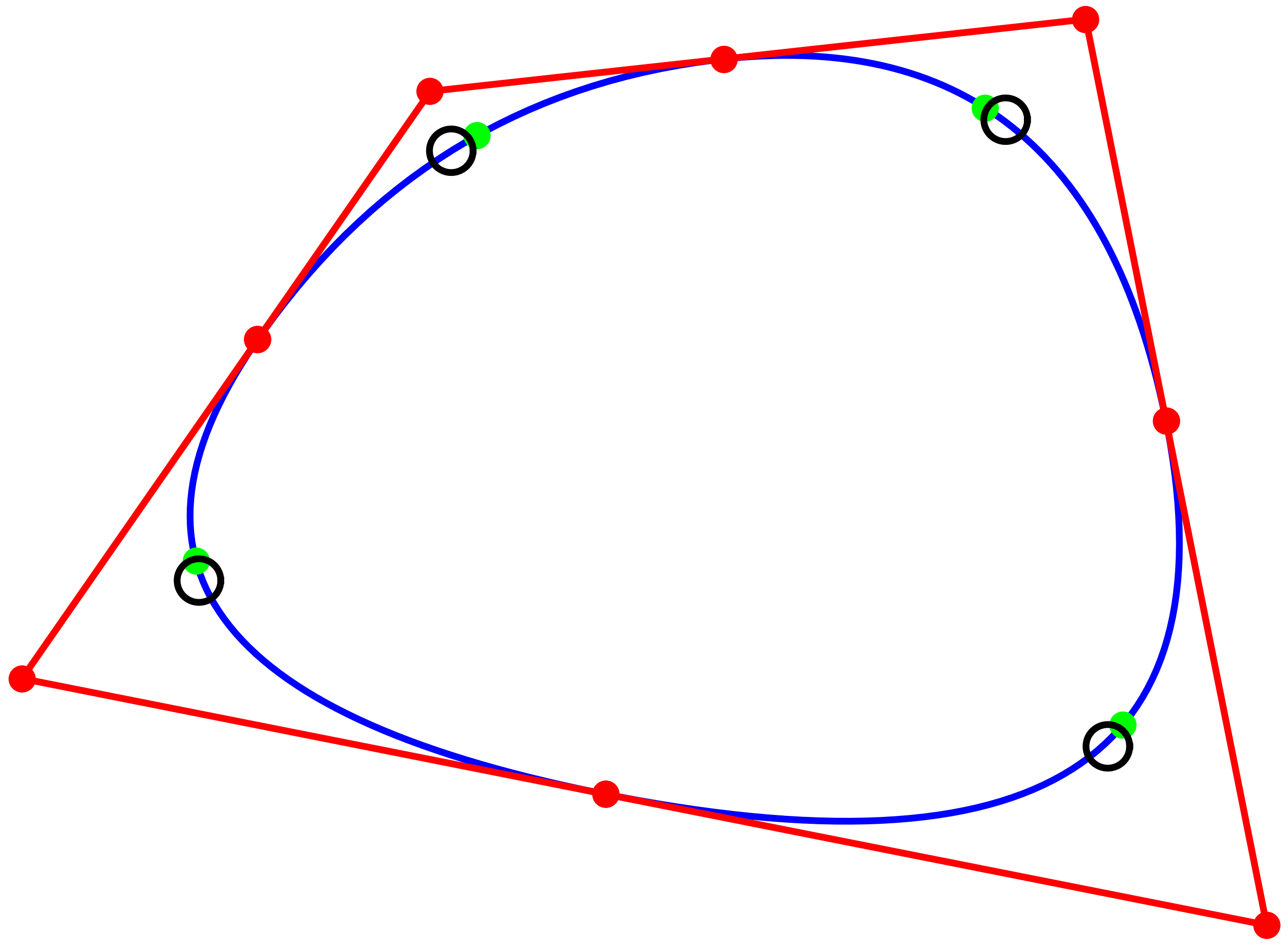
- $\lambda = \frac{\sqrt{|\Delta(A,B,D)|}}{\sqrt{|\Delta(A,B,D)|}+\sqrt{|\Delta(B,D,E)|}} \in (0,1)$



Inflection Point

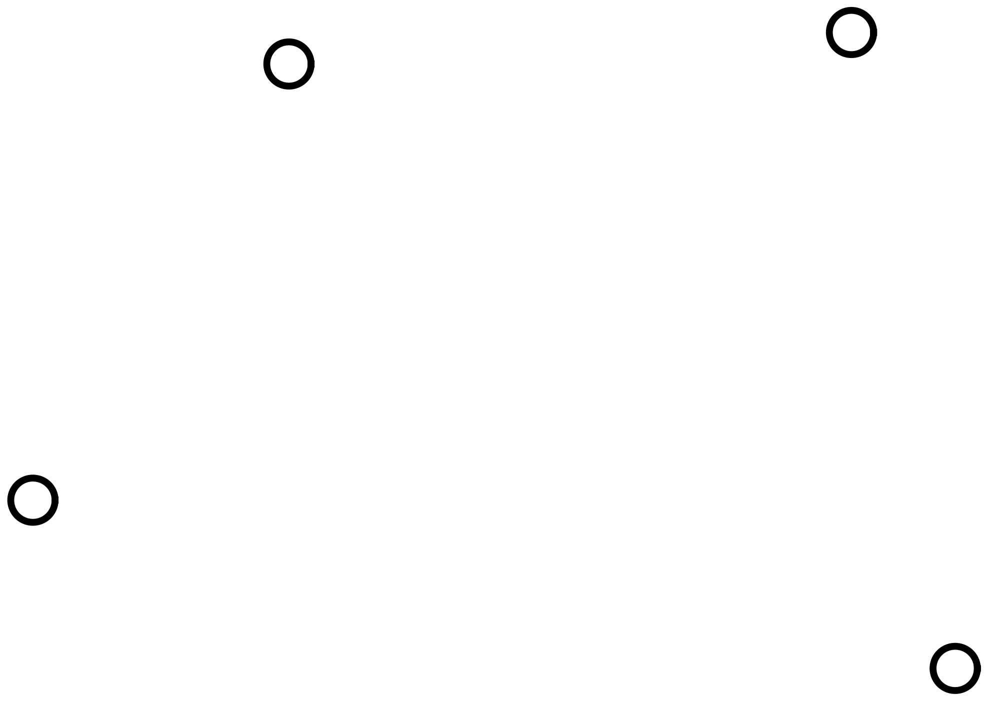


Optimization



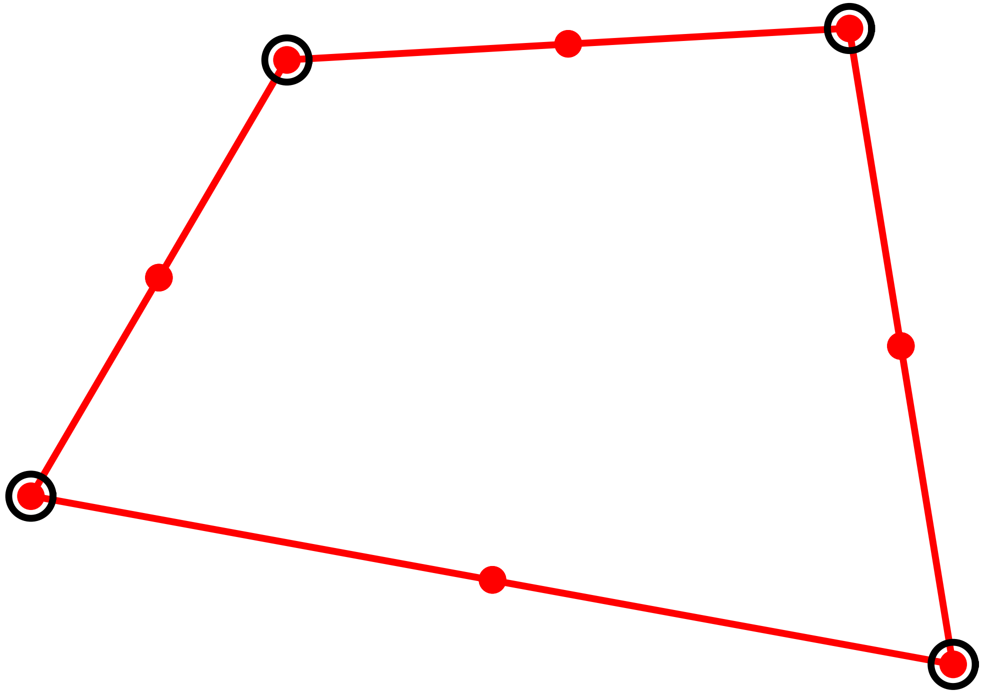
Optimization

Input

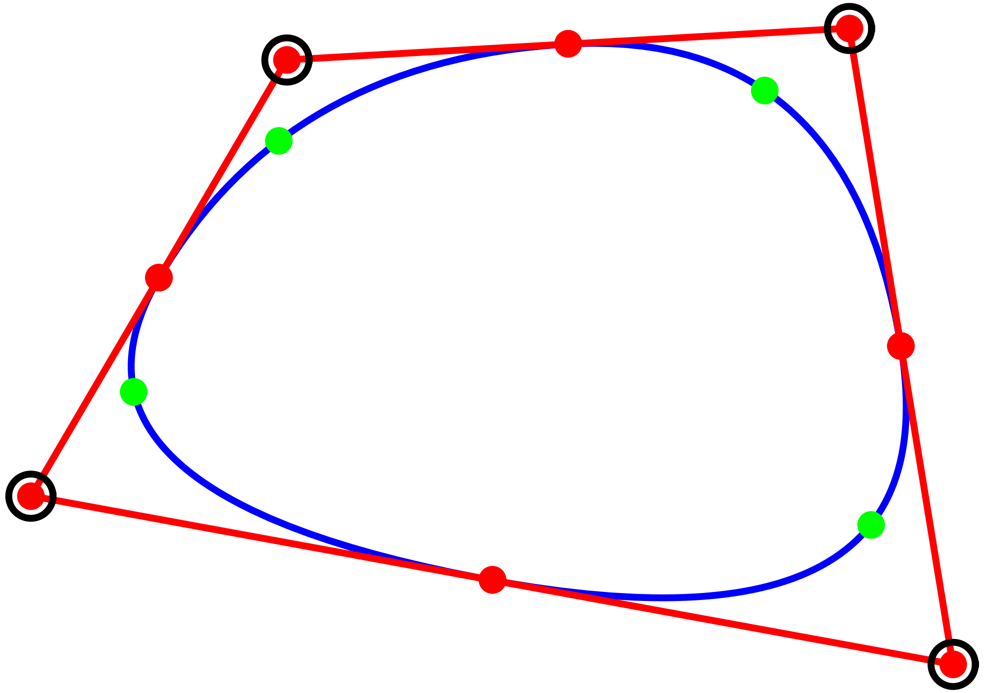


Optimization

Initialization

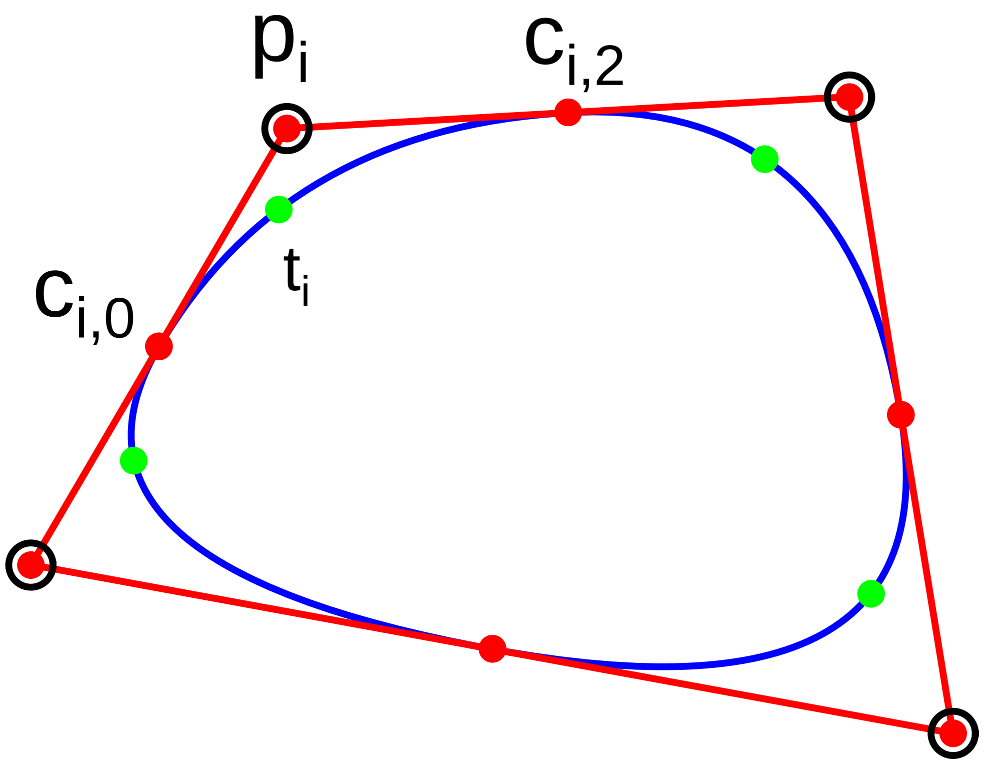


Optimization



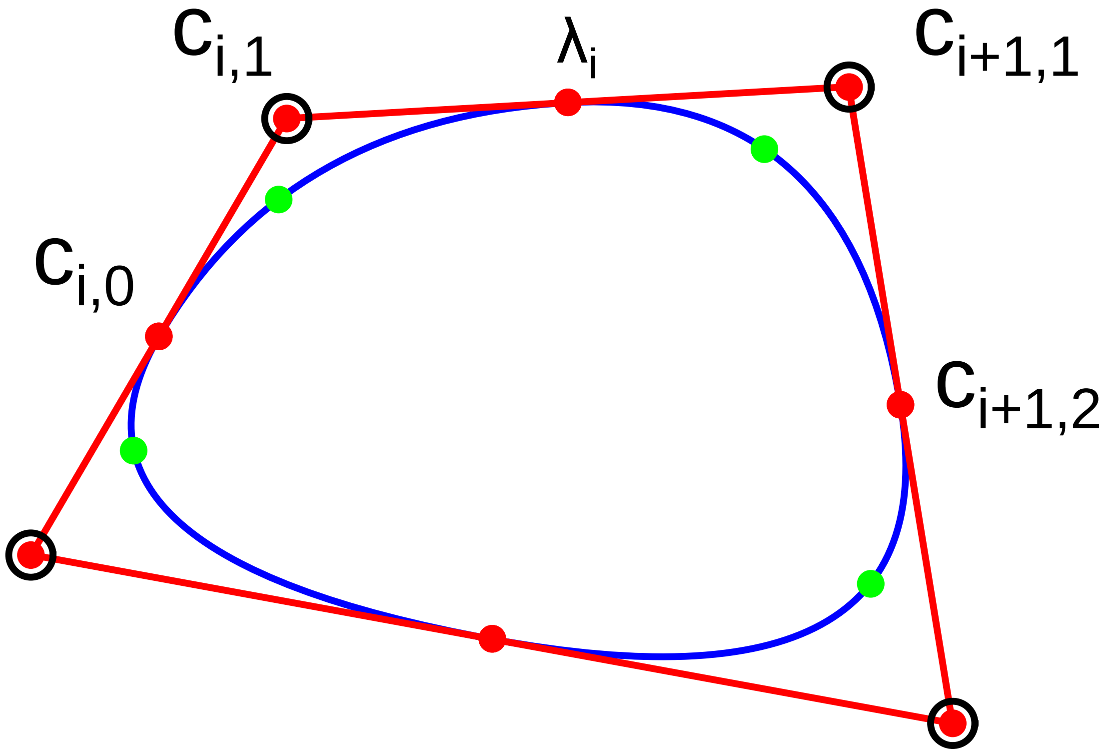
Optimization

Iteration 0



Optimization

Iteration 0



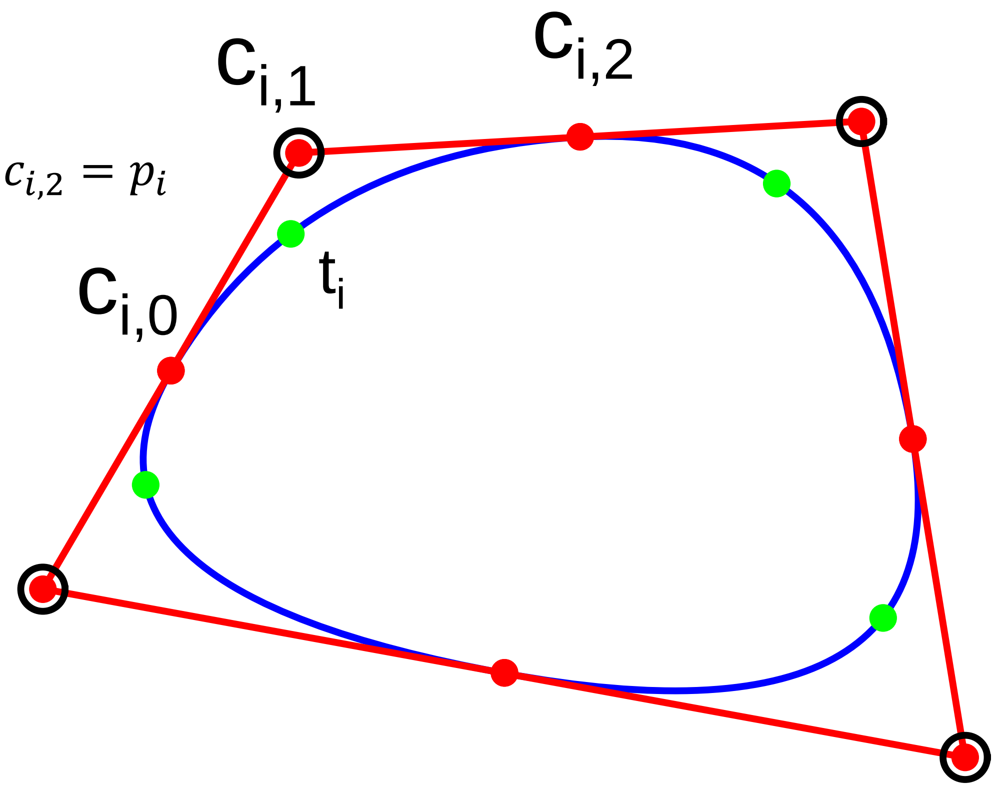
Optimization

Iteration 0

$$(1-t_i)^2 c_{i,0} + 2(1-t_i)c_{i,1} + t^2 c_{i,2} = p_i$$

$$c_{i,2} = (1-\lambda_i)c_{i,1} + \lambda_i c_{i+1,1}$$

Only $c_{i,1}$ is variable.

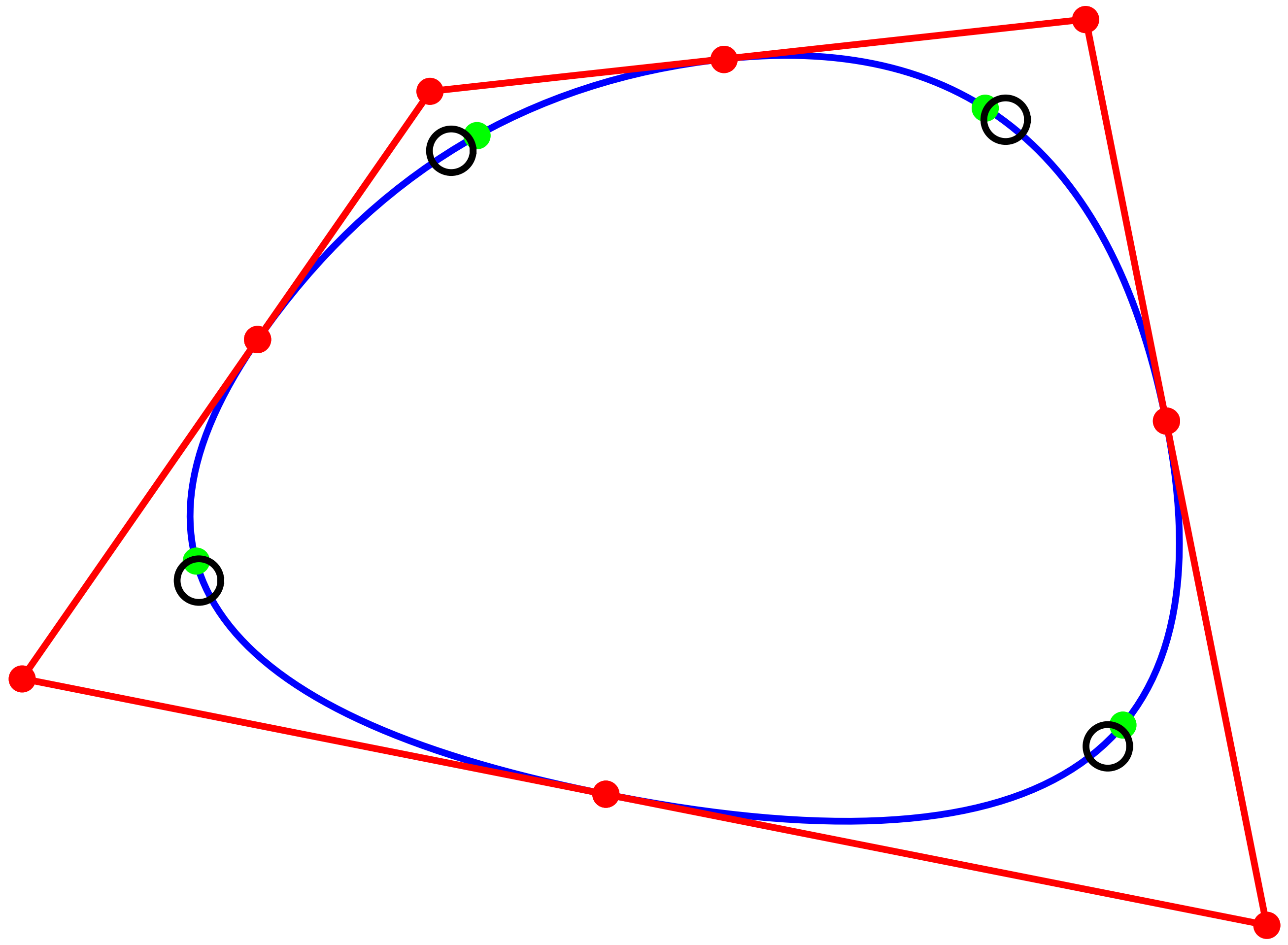


Step 2

$$\begin{bmatrix} a_{11} & a_{12} & & & & a_{1n} \\ a_{21} & a_{22} & a_{23} & & & \\ & a_{32} & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & a_{n-1,n} \\ a_{n1} & & & & a_{n,n-1} & a_{nn} \end{bmatrix} \begin{bmatrix} c_{1,1} \\ c_{2,1} \\ c_{3,1} \\ \vdots \\ \vdots \\ c_{n,1} \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ \vdots \\ p_n \end{bmatrix}$$

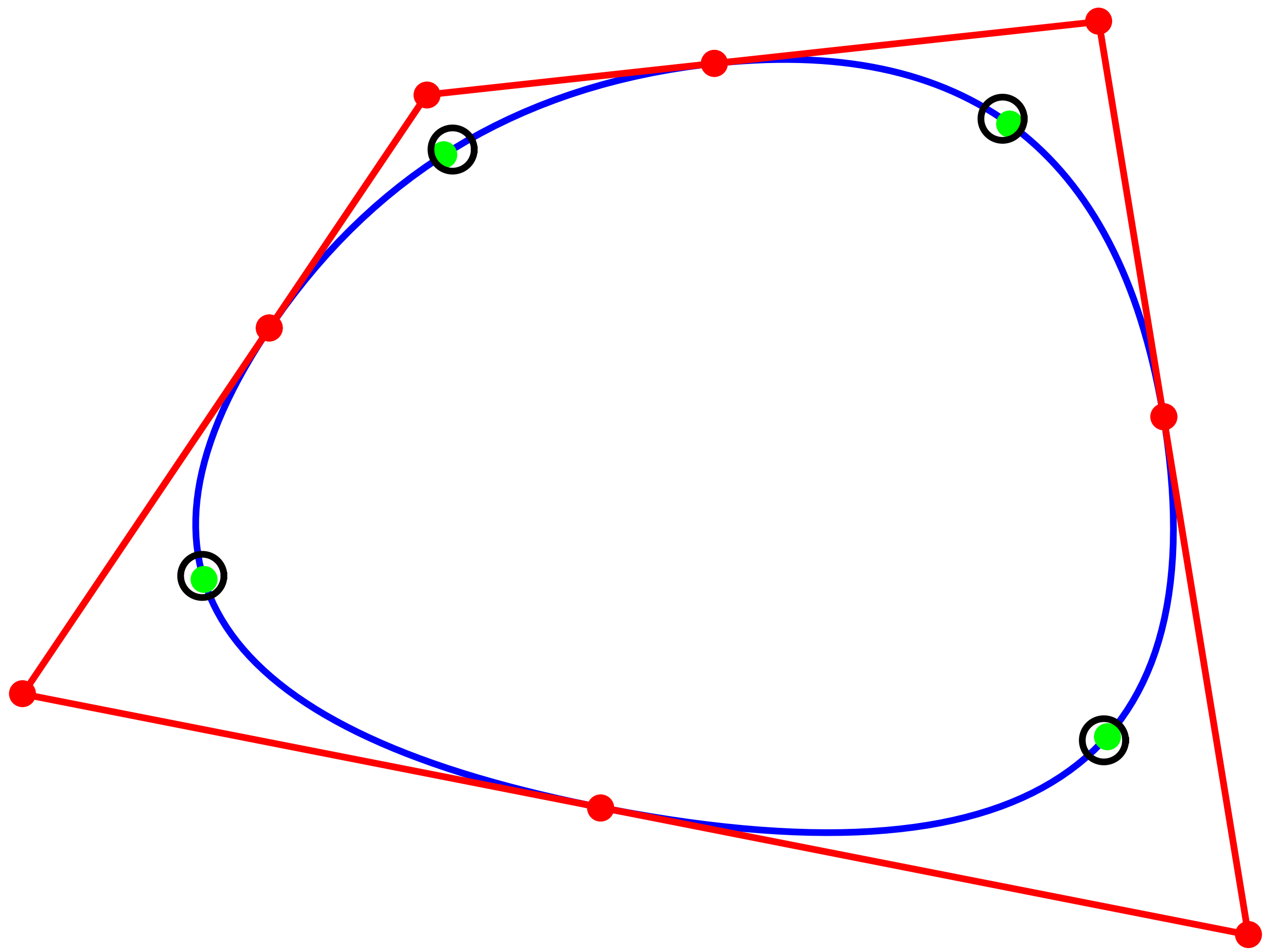
Optimization

Iteration 1



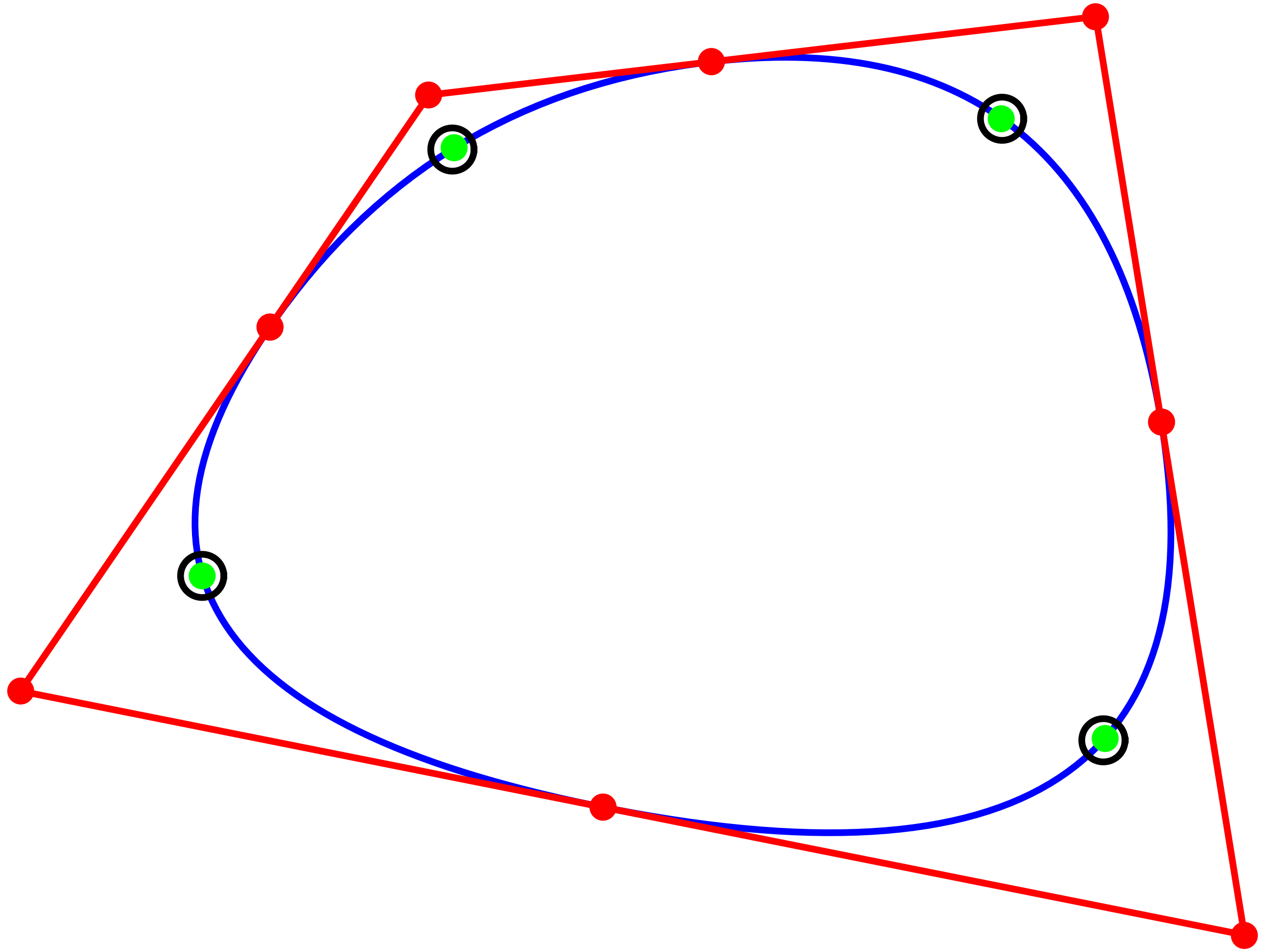
Optimization

Iteration 2



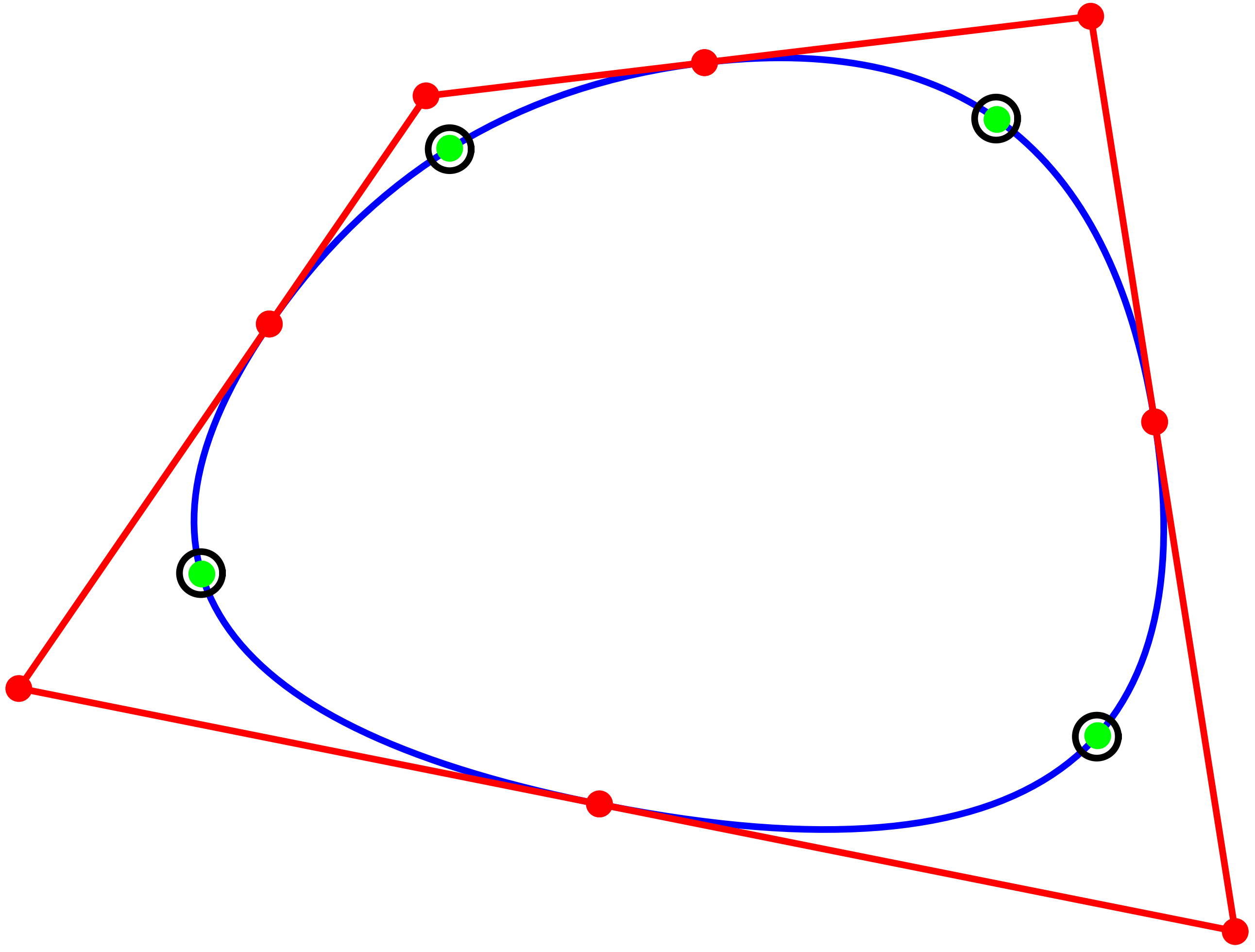
Optimization

Iteration 3



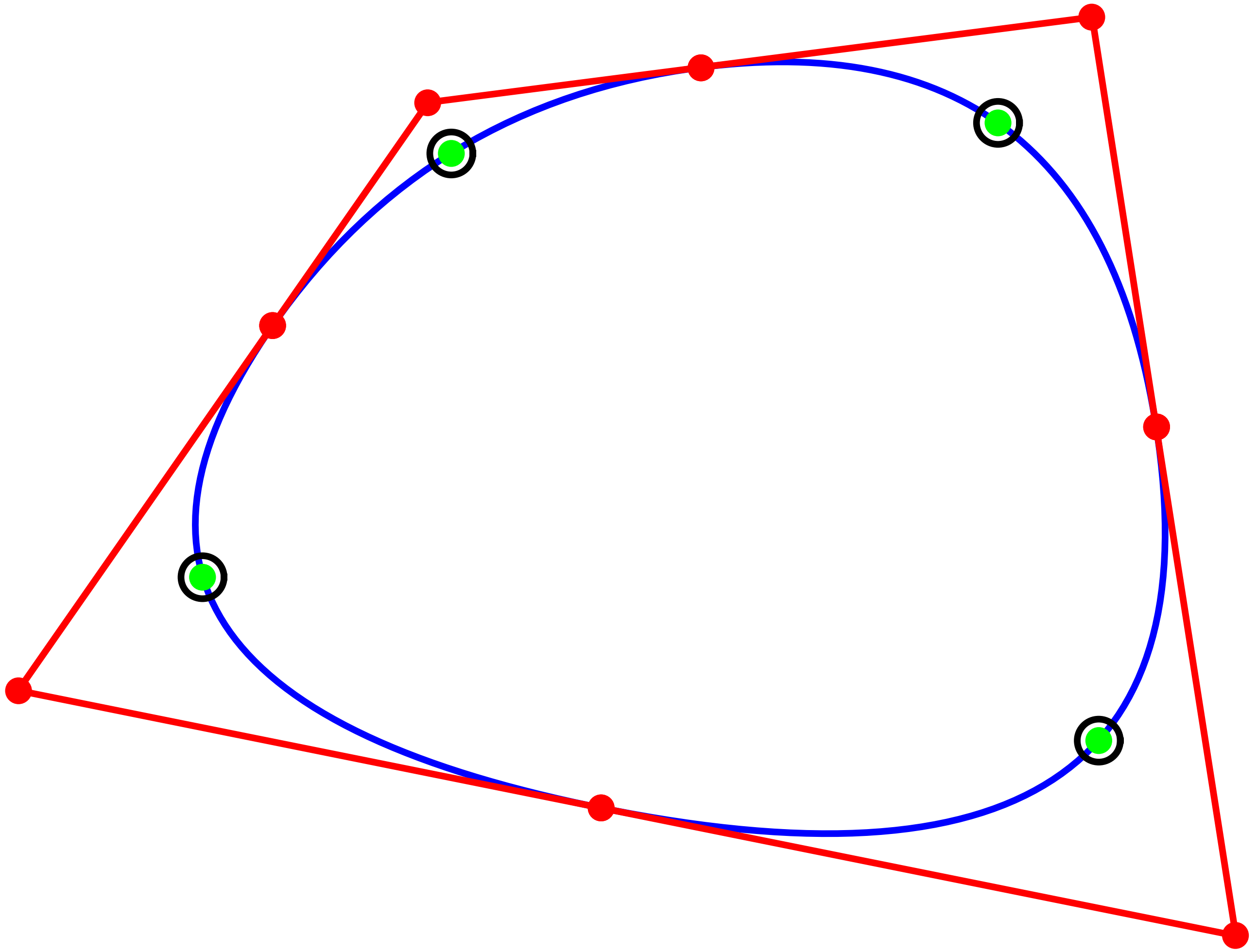
Optimization

Iteration 4



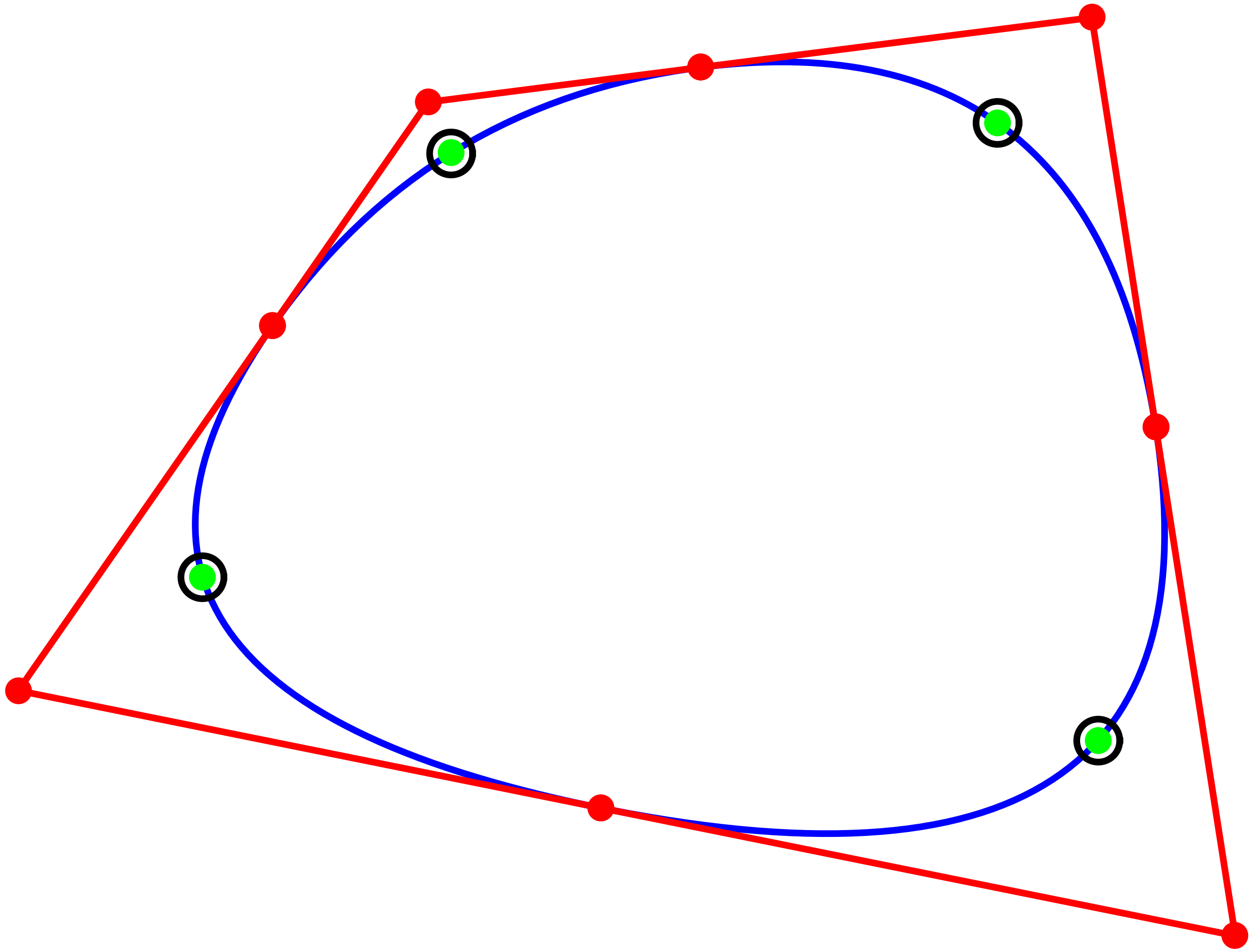
Optimization

Iteration 5



Optimization

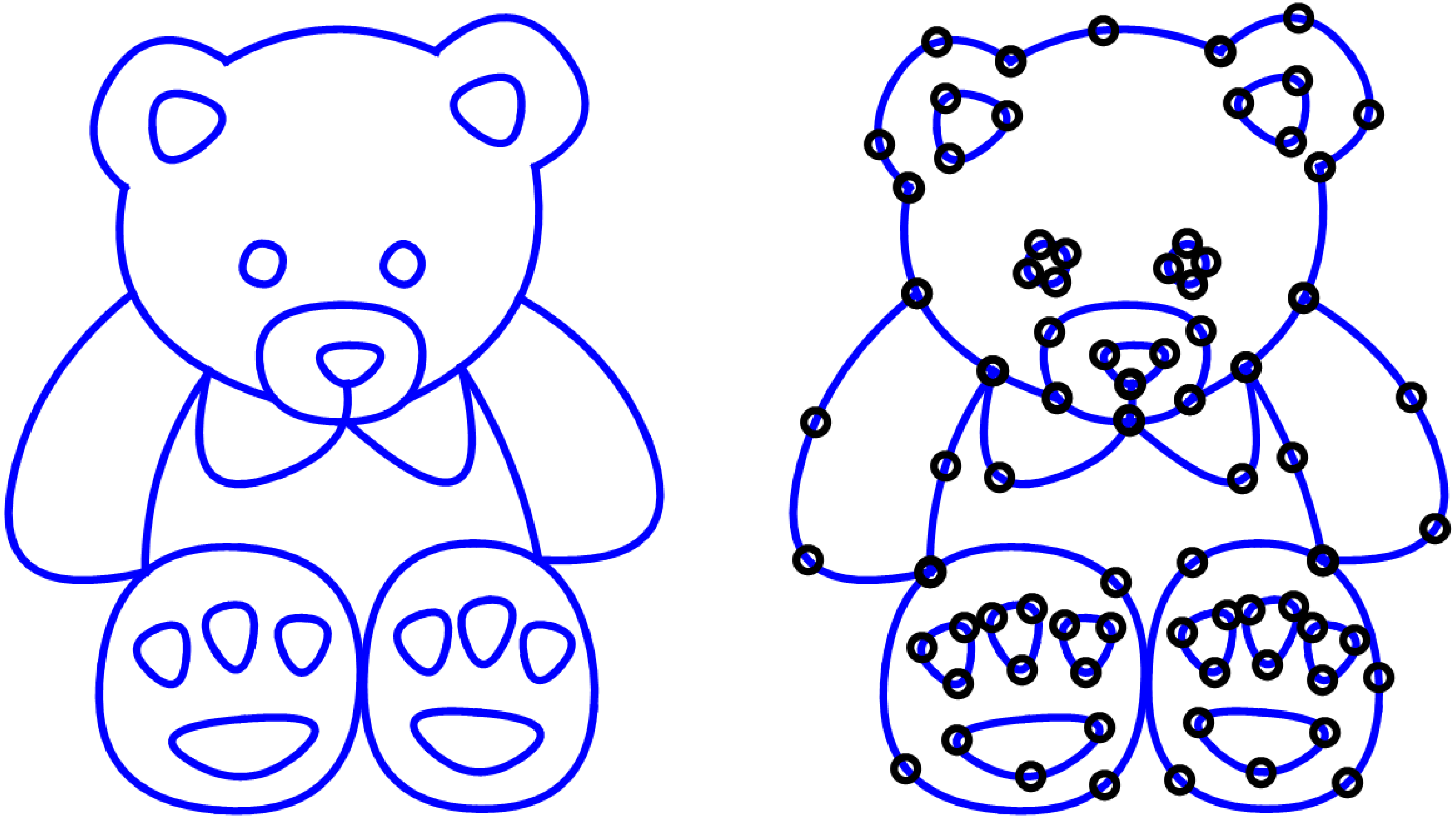
Iteration 10



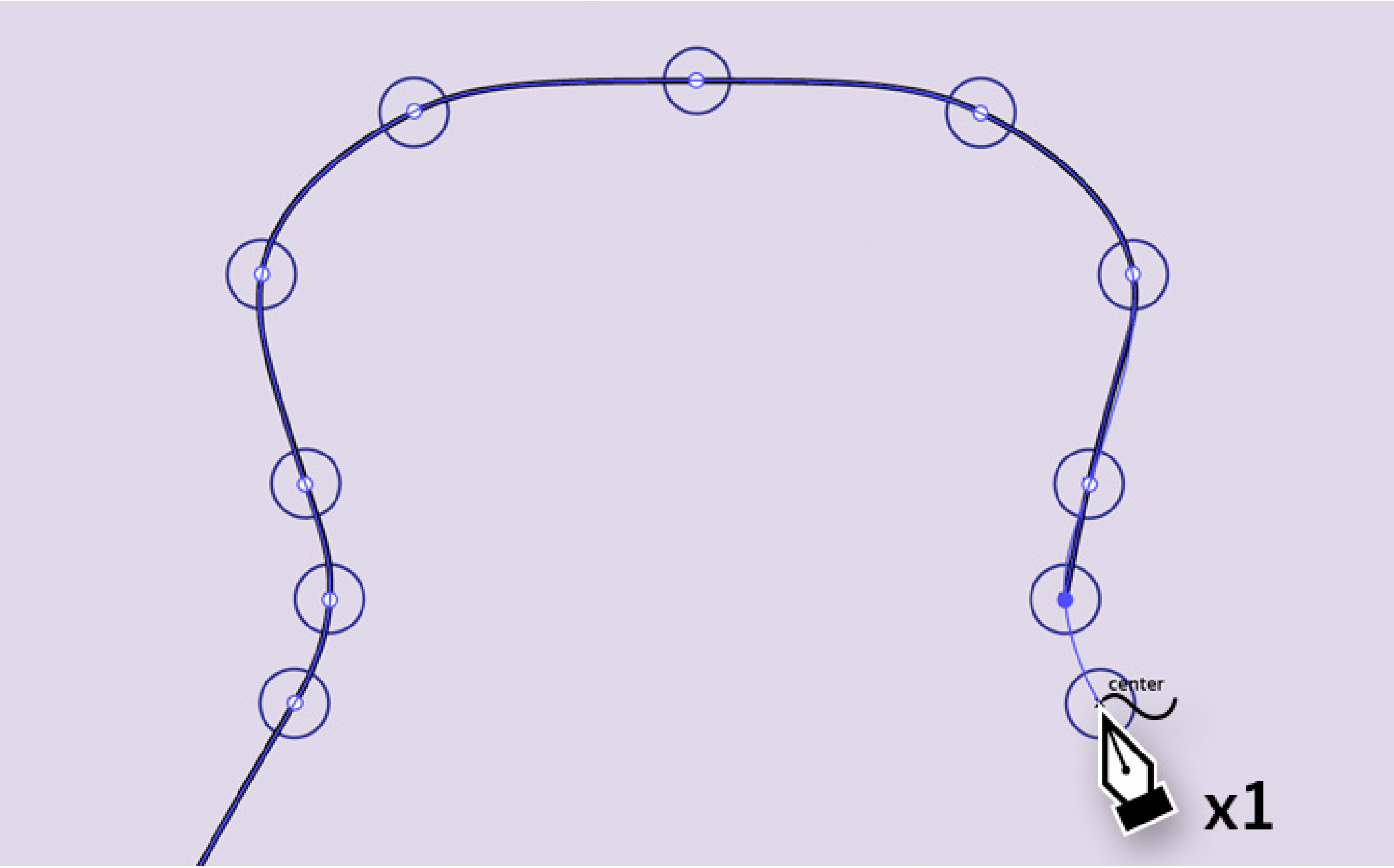
Curves with Boundaries

Local support in practice

Results and Examples



Adobe Illustrator



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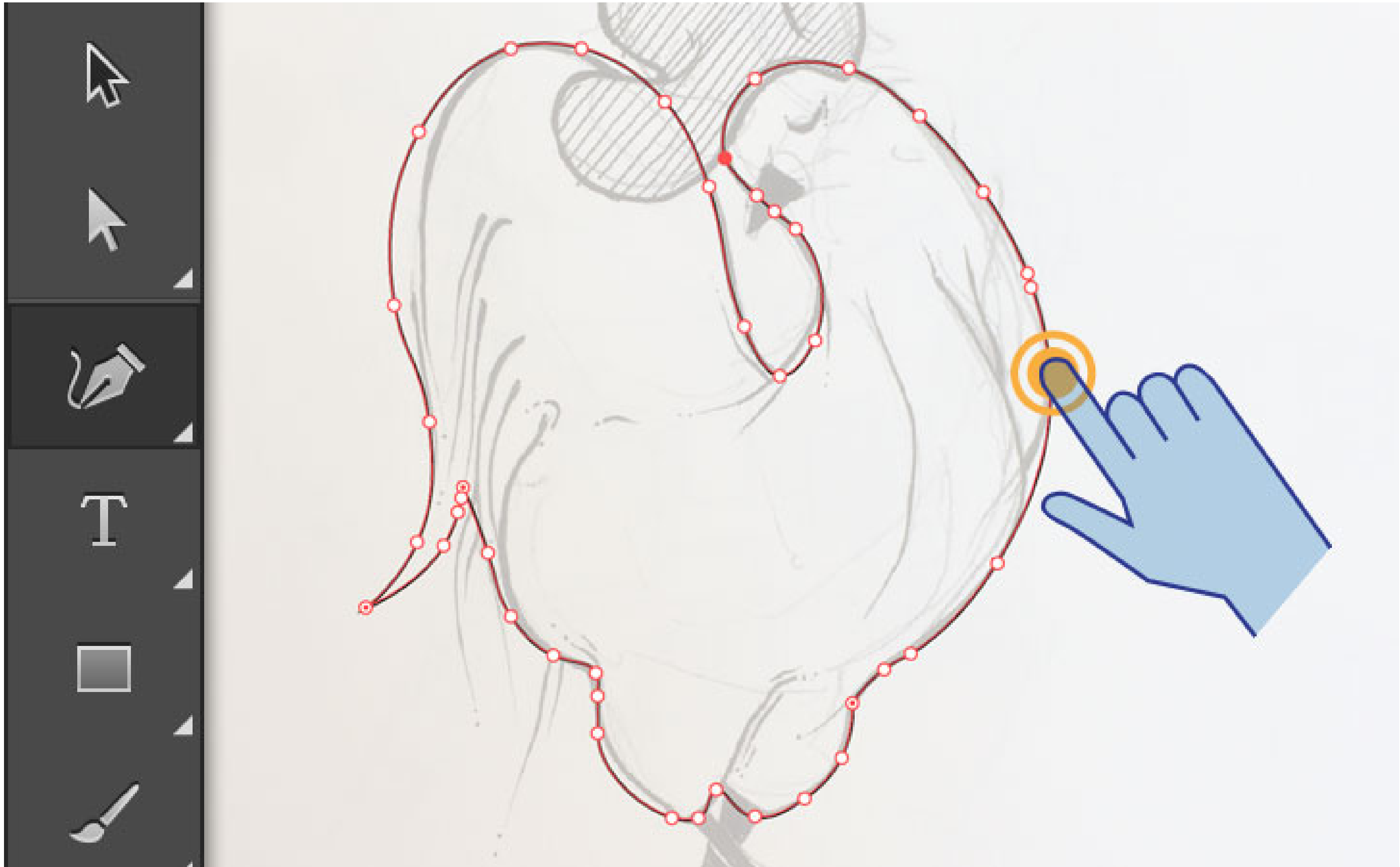
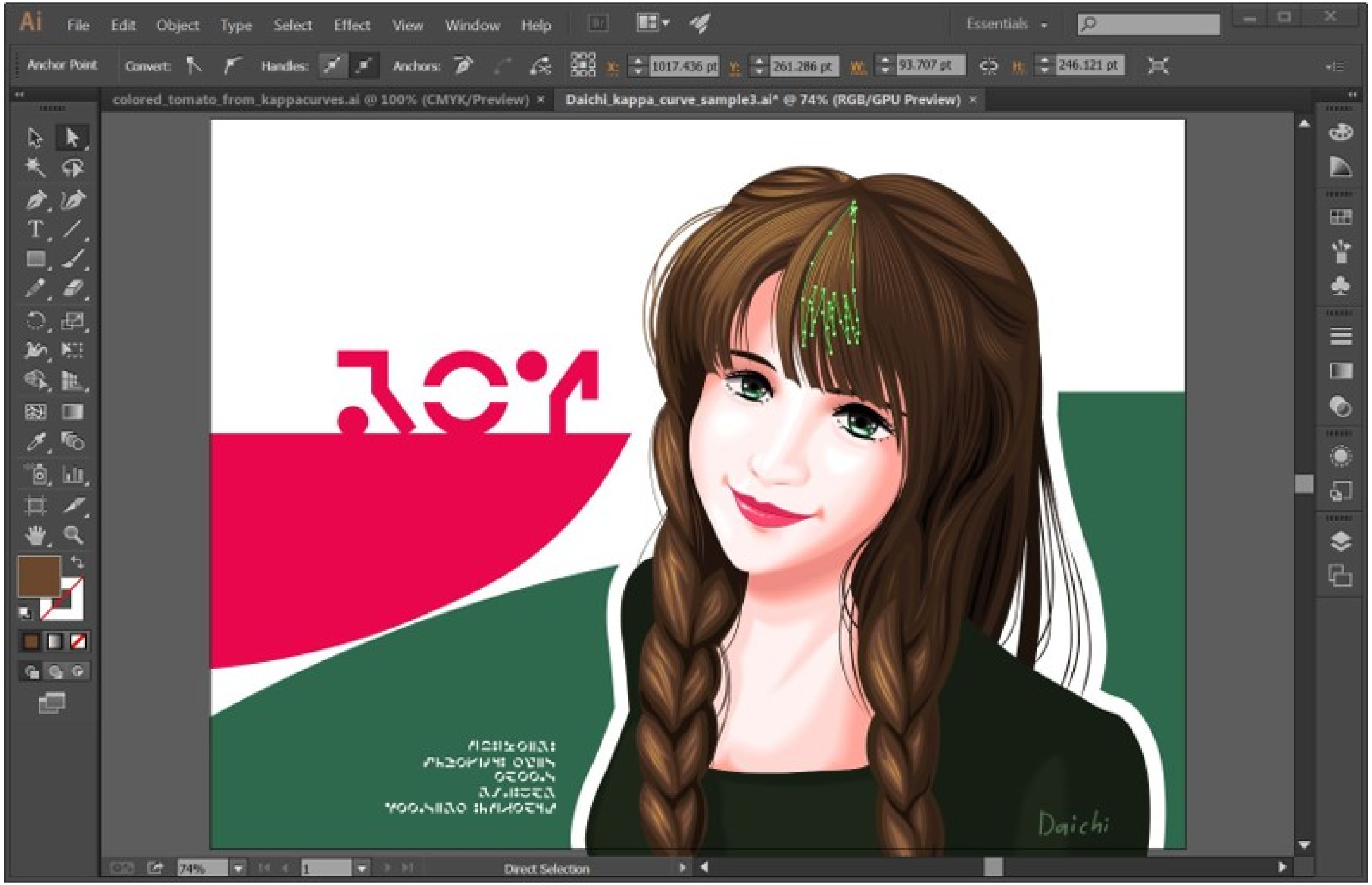


Adobe

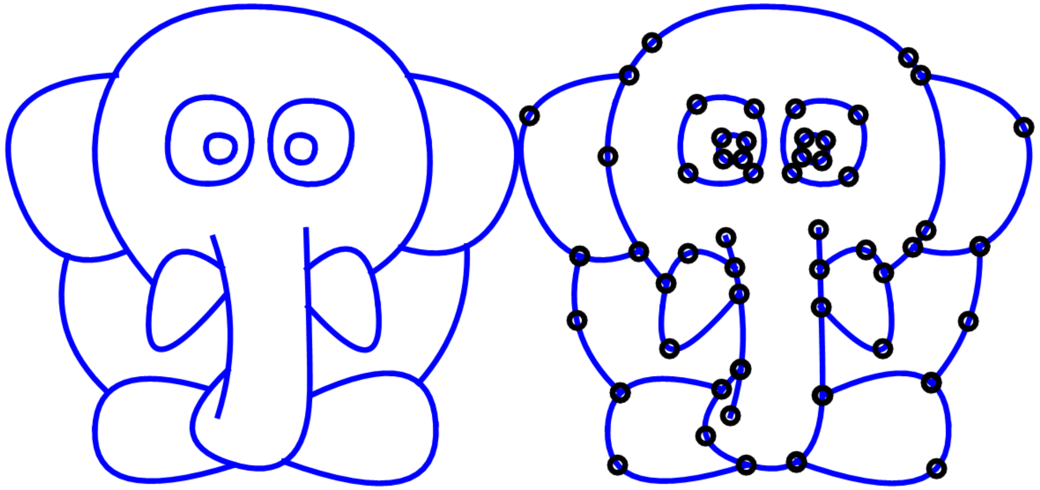
Adobe Illustrator



Adobe

Conclusions

- Quadratic spline
- G^2 continuous (G^1 at inflection points)
- Max curvatures only appear at control points
- Cusps only appear at control points
- Local support in practice



Future work

- Kappa curve can suit any curve primitives which have unique max curvature point
  - High degree like cubic curve
  - Rational curves
  - Integral curves
- Curves in 3D