30 JULY – 3 AUGUST Los Angeles
SIGGRAPH2017

Wasserstein Blue Noise Sampling

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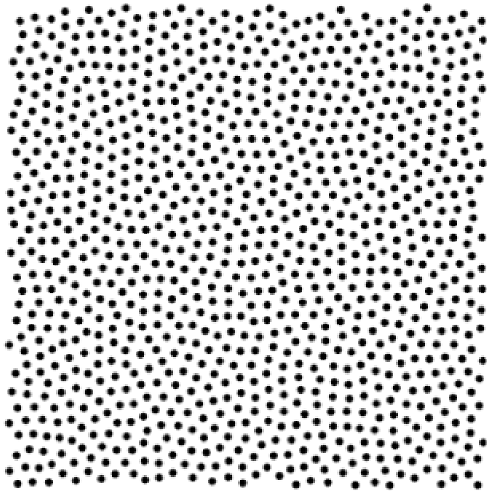
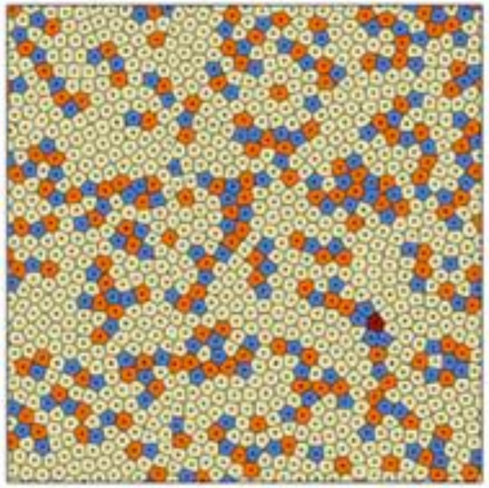
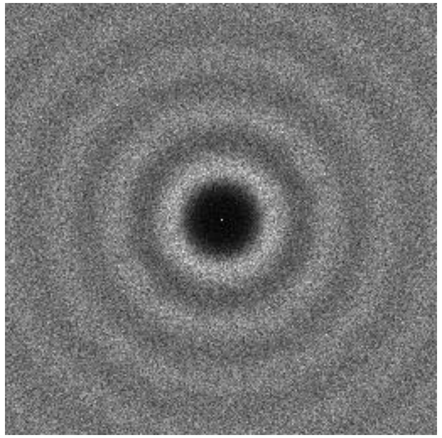
Blue noise distribution: random & uniform

Sampling

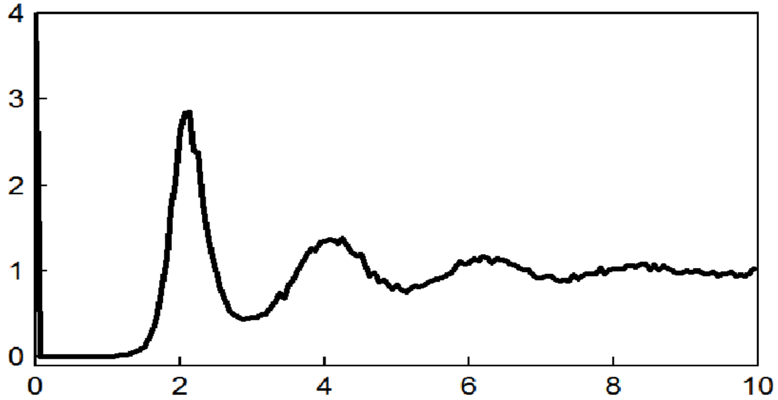


Number of Neighbors

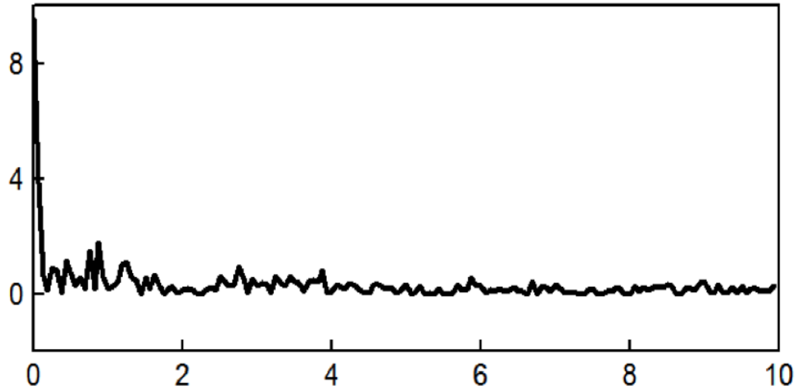




Power Spectrum



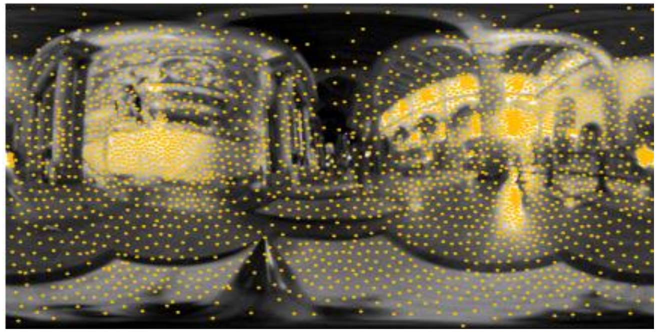
Radial Mean



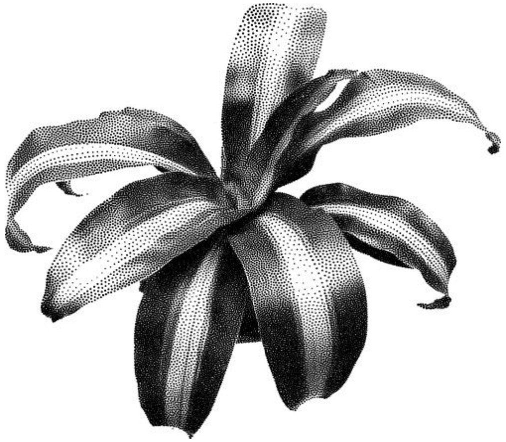
Anisotropy

Blue noise sampling application

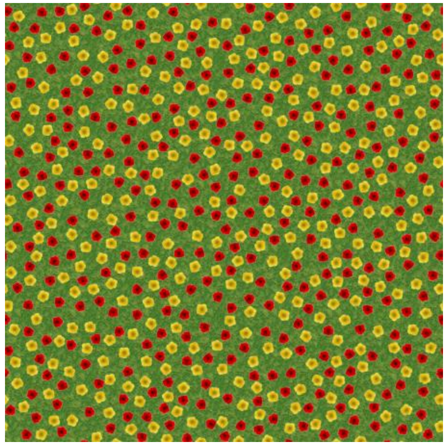
Ray tracing
[Cook 1984]



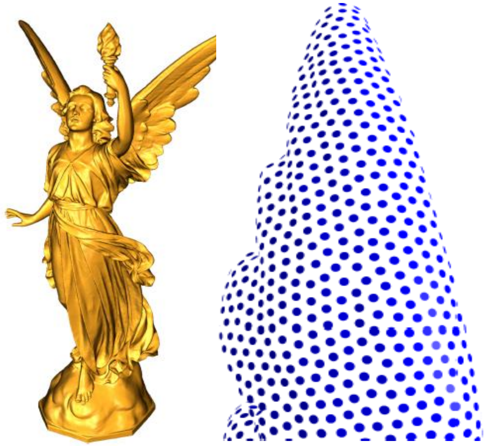
Important sampling
[Ostromoukhov et al. 2007]



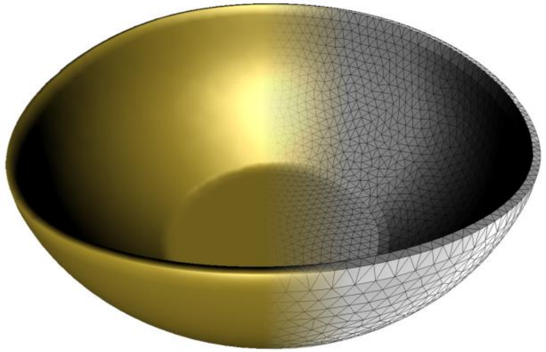
Stippling
[Balzer et al. 2009]



Object displacement
[Wei et al. 2010]



Point-based Modeling
[Öztireli et al. 2010]

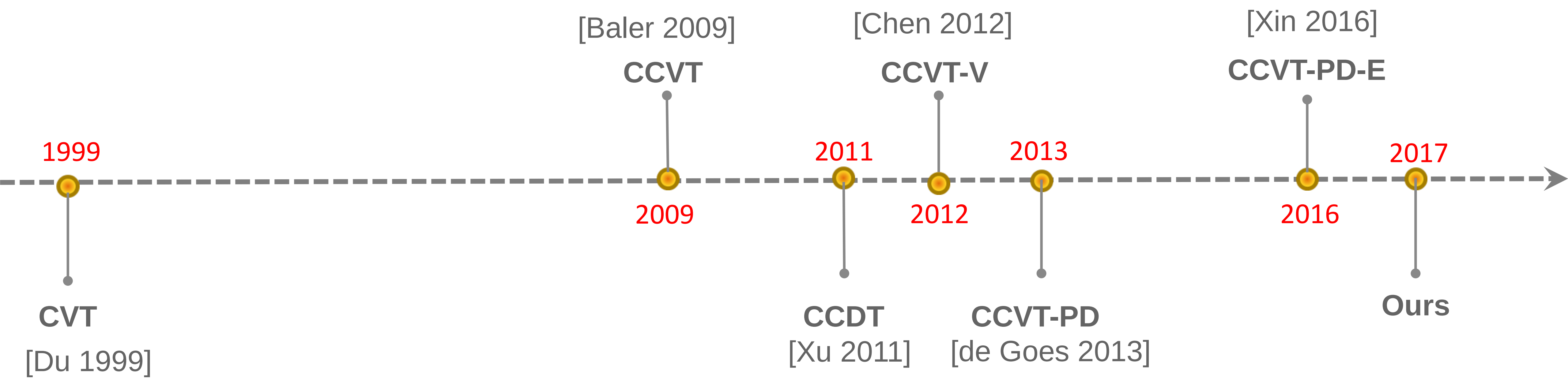


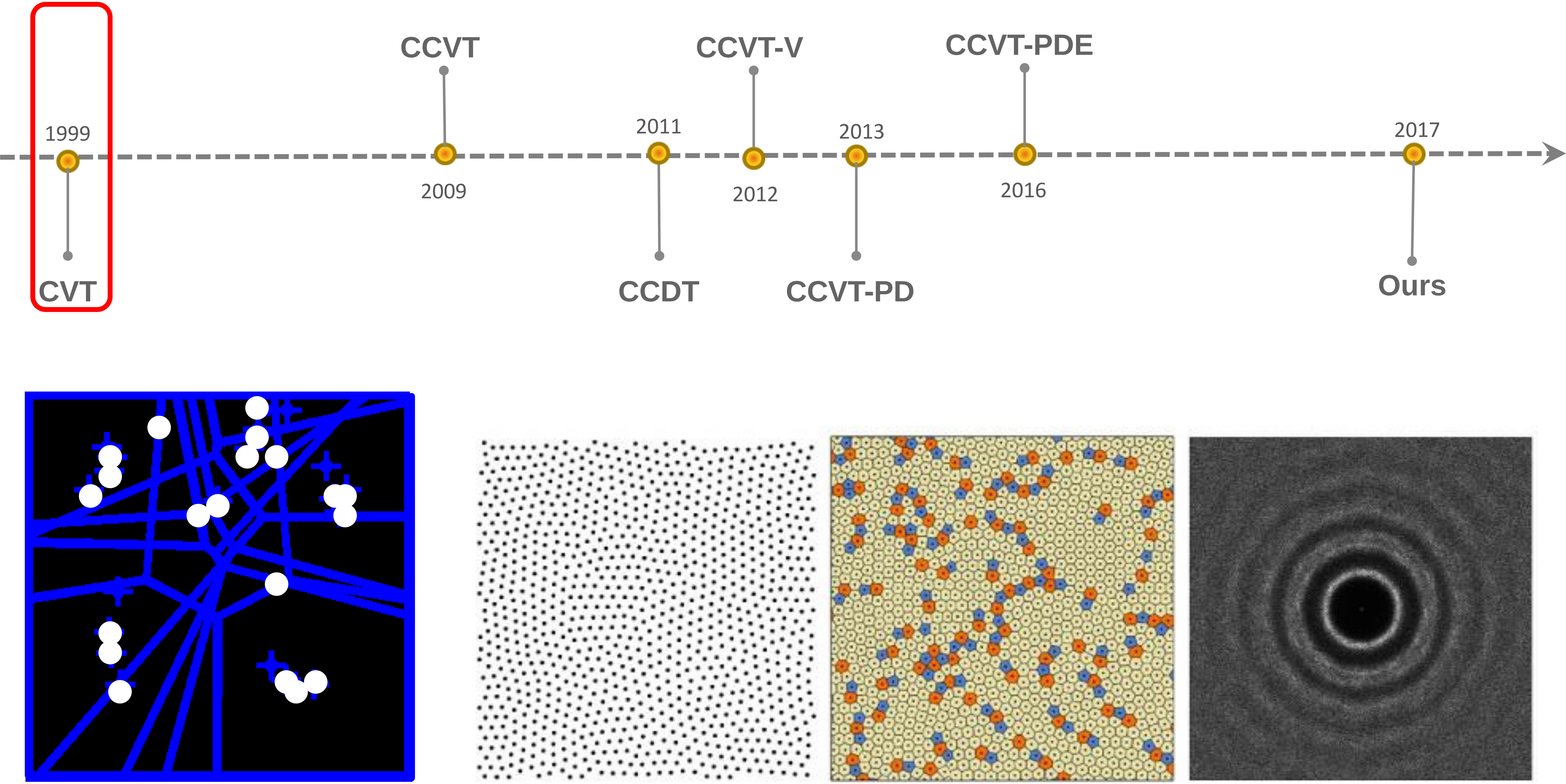
Remeshing
[Jiang et al. 2015]

Previous work

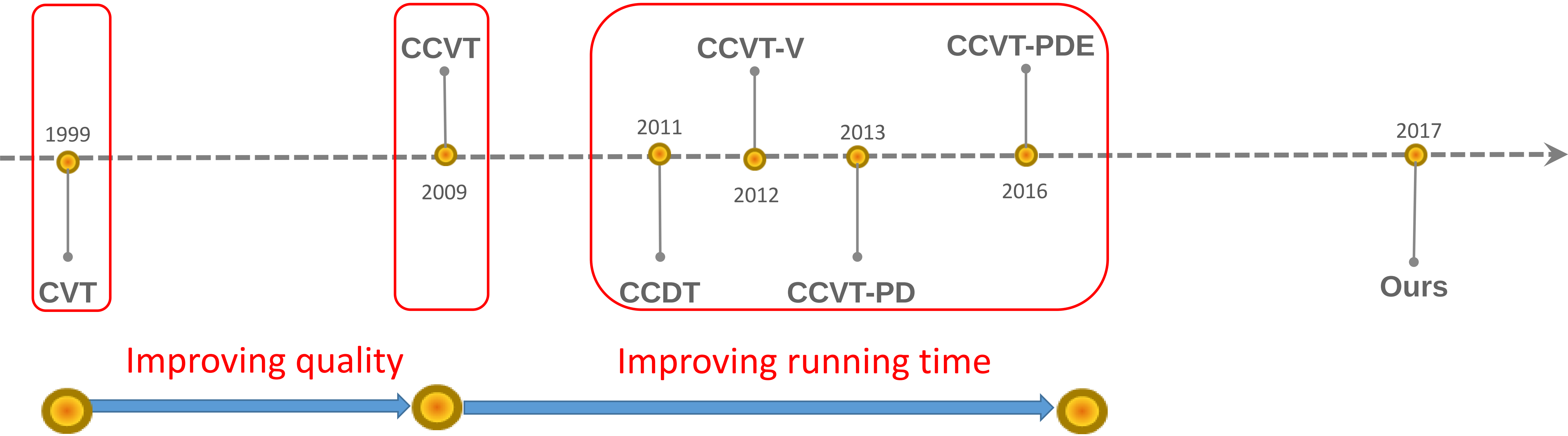
- Stochastic sampling
  - Dart throwing method and its variations [Cook 86, Mitchell 87, Jones 06, White et al. 07, Ebeida et al. 12, Yuksel 15]
- Title-based sampling [Kopf 06, Wachte 14]
- Optimal sampling
  - Lloyd relaxation method [Lloyd 82]
  - Kernel density model [Fattal 11, Jiang et al. 15]

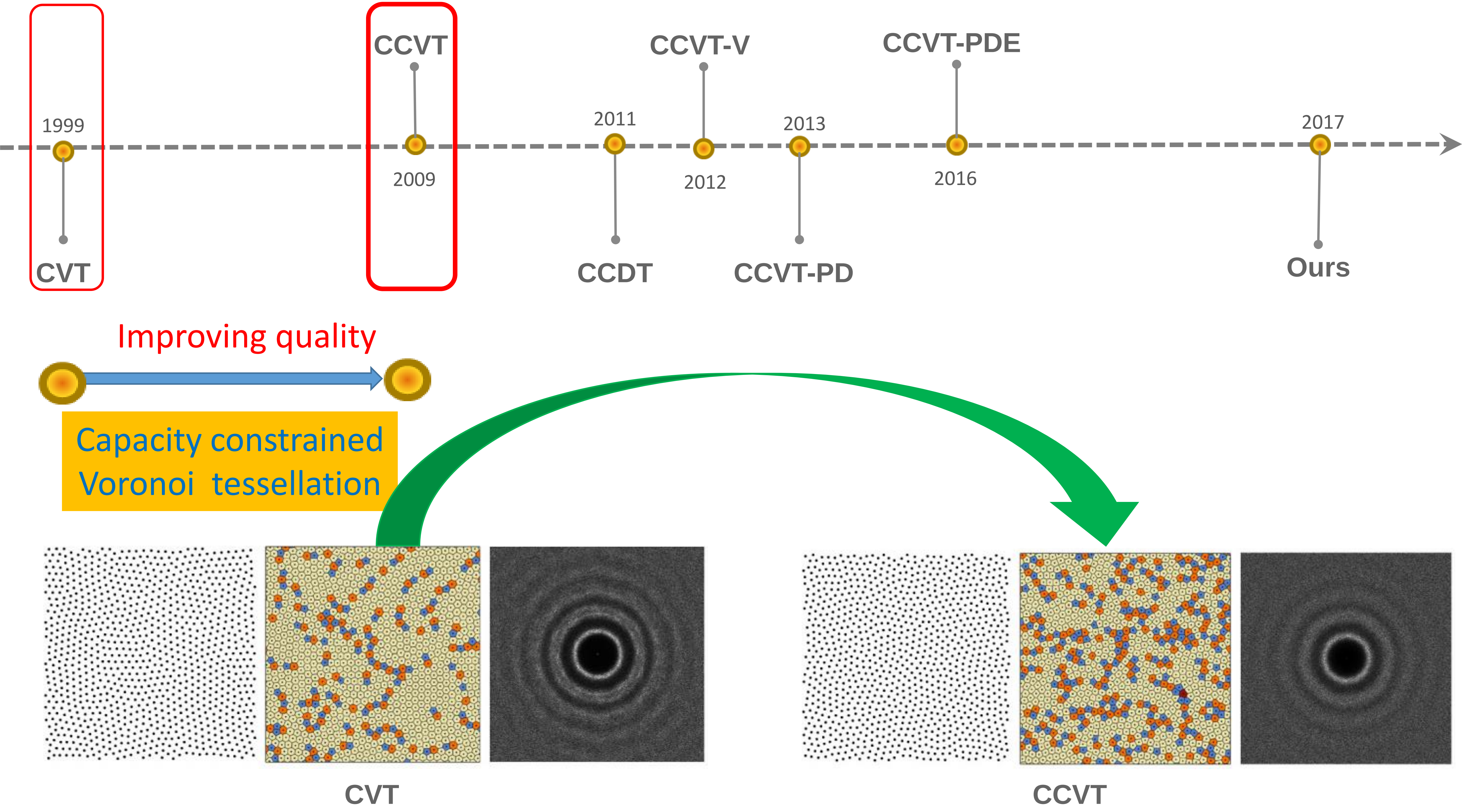
Lloyd relaxation method

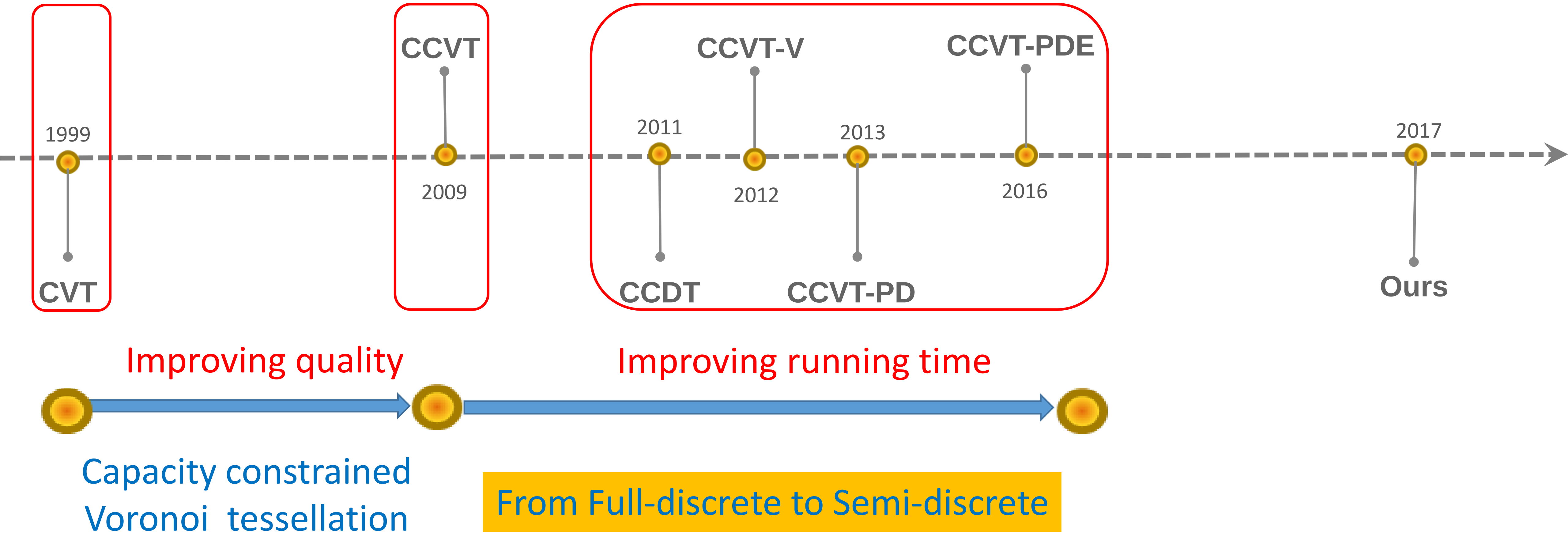


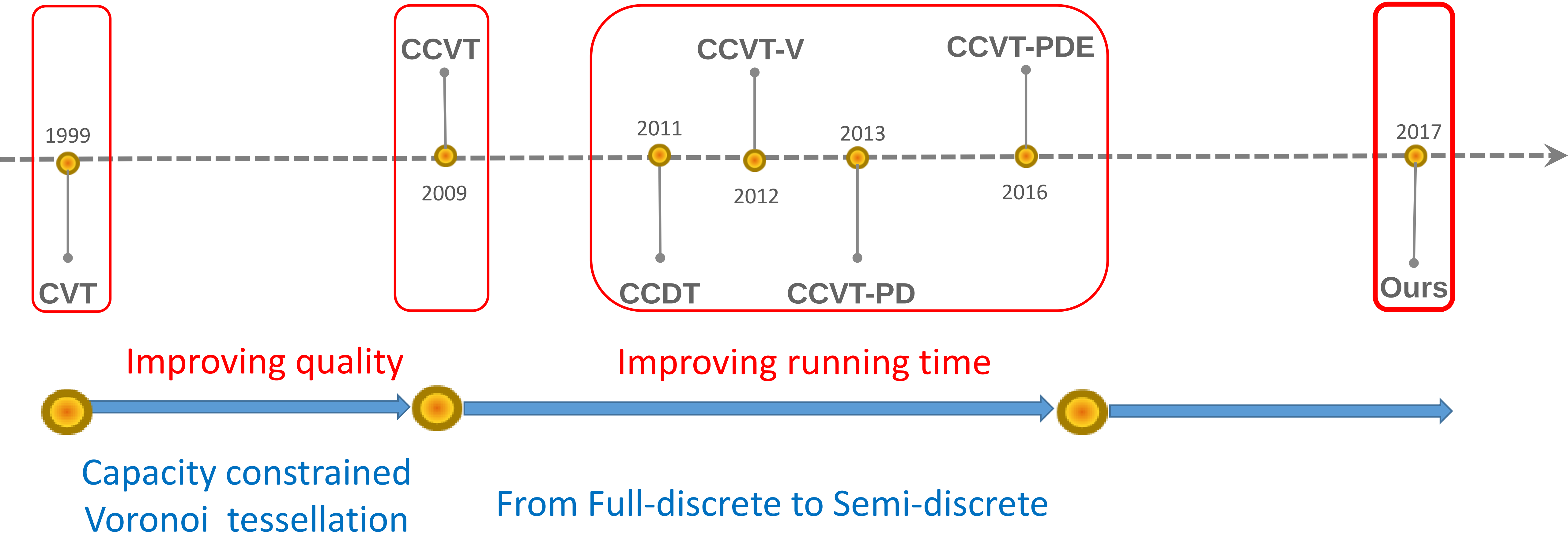


CCVT
CCVT-V
CCVT-PDE
1999
2011
2013
2017
2009
2012
2016
CVT
CCDT
CCVT-PD
Ours


CCVT
CCVT-V
CCVT-PDE
1999
2011
2013
2017
2009
2012
2016
CVT
CCDT
CCVT-PD
Ours
Improving quality
Improving running time


CCVT
CCVT-V
CCVT-PDE
1999
2011
2013
2017
2009
2012
2016
CVT
CCDT
CCVT-PD
Ours
Improving quality
Capacity constrained Voronoi tessellation
CVT
CCVT

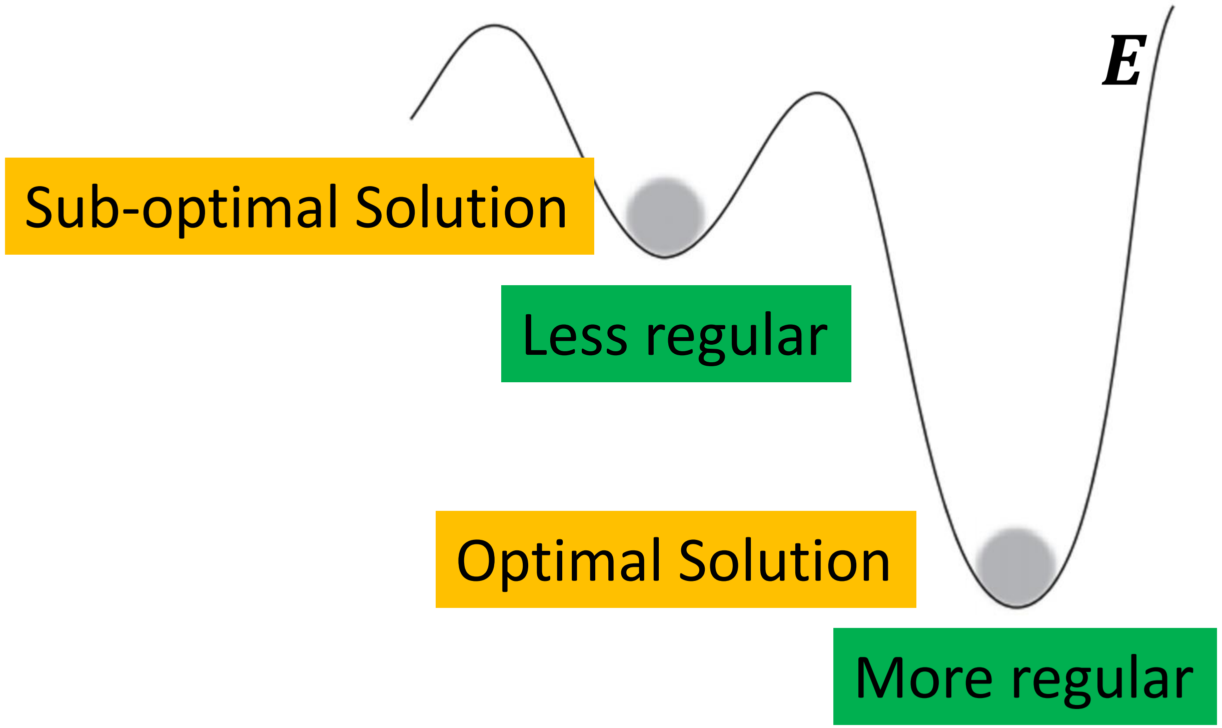

CCVT
CCVT-V
CCVT-PDE
1999
2011
2013
2017
2009
2012
2016
CVT
CCDT
CCVT-PD
Ours
Improving quality
Improving running time
Capacity constrained
Voronoi tessellation
From Full-discrete to Semi-discrete



Improving controllability of point distribution quality !

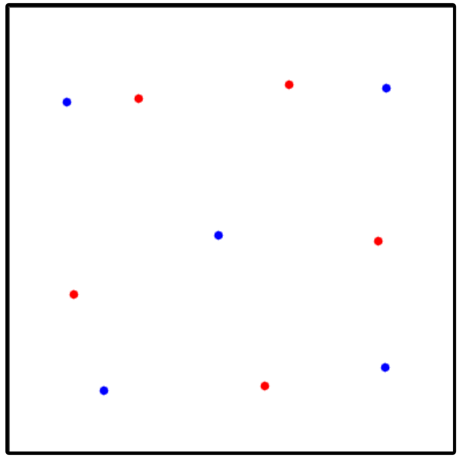
Limitations of previous Lloyd relaxation method

- Uncontrollability of the solution

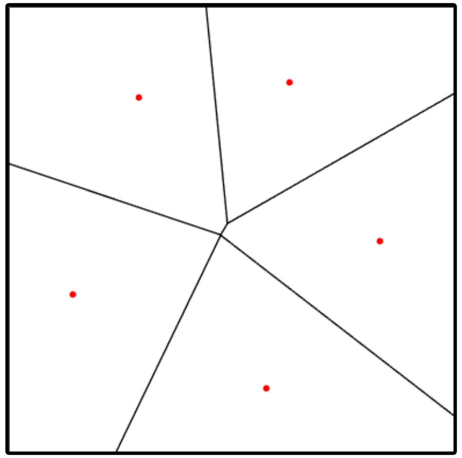


Limitations of previous Lloyd relaxation method

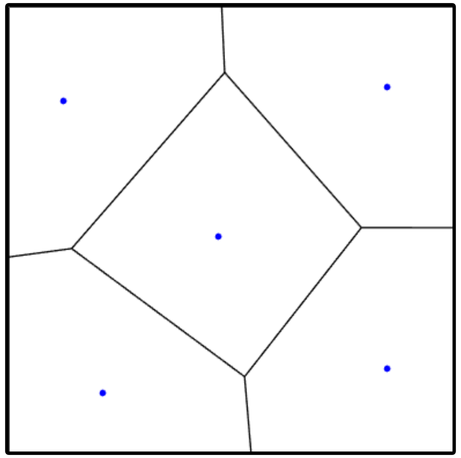
- Conflicts on multi-class sampling



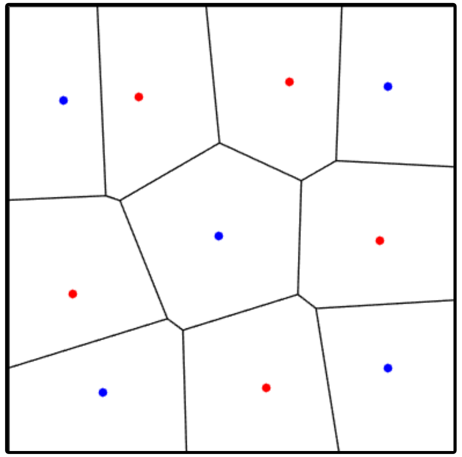
Initial points



Red points



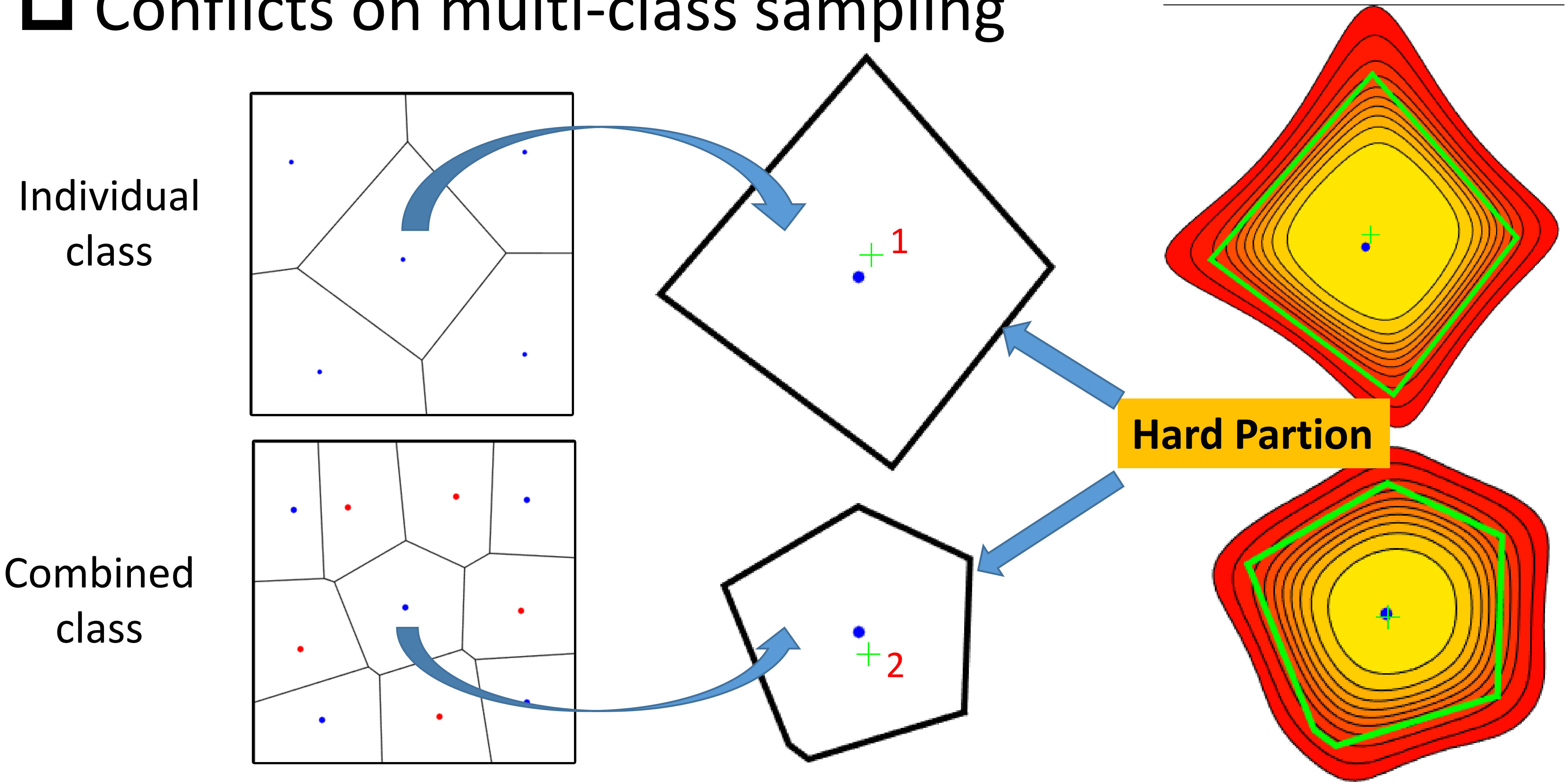
Blue points



Red and blue points

Limitations of previous Lloyd relaxation method

- Conflicts on multi-class sampling



Core idea

- Relaxing the optimal transport problem
  - Controlling spatial regularity of the point distribution
  - Relaxing the conflict of multi-class sampling

General optimal transport problem

Sampling

Density function $\varrho(x)$ $\longrightarrow$ **Points** $\{x_i\}$

$$\boxed{\nu(U) = \int_U \varrho(x)dx, U \subseteq \Omega}$$

Probability measure

$$\boxed{\mu = \sum_i \rho_i \delta_{x_i} \quad s.t. \sum_i \rho_i = 1, \delta_{x_i} = \begin{cases} 1 & x_i \in \Omega \\ 0 & \text{others} \end{cases}}$$

Discrete Probability measure

Constrained Wasserstein barycenter

$$\mu = arg\min_{\mu} W_p^p(\mu, \nu) = arg\min_{\mu} \inf_{\pi} \int_{X \times \Omega} d(x_i, y)^p \, d\pi(x_i, y)$$

$$s.t. \int_{\Omega} \pi(x_i, y) = \rho_i, \sum_i \int_{U \subseteq Q} \pi(x_i, y)dy = \nu(U), \ \pi(x_i, y) \geq 0$$

General optimal transport problem

Constrained Wasserstein barycenter

Wasserstein distance

Distance function

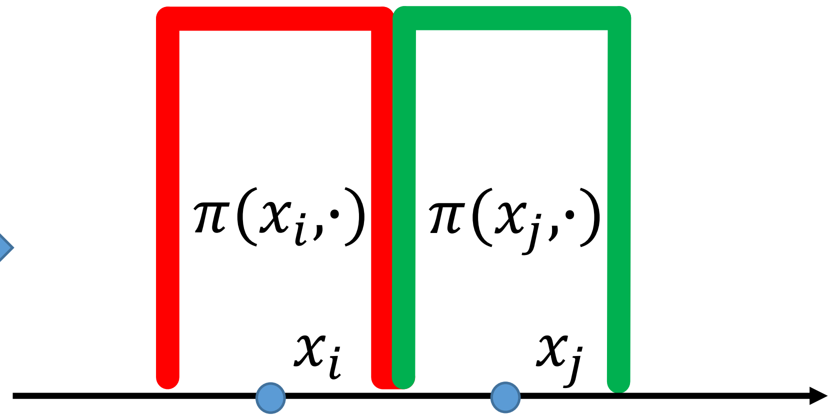
Transport plan

$$\mu = \arg\min_{\mu} \boxed{W_p^p(\mu,\nu)} = \arg\min_{\mu} \boxed{\inf_{\pi} \int_{X\times\Omega} \boxed{d(x_i,y)}^p \, d\boxed{\pi(x_i,y)}}$$

Mass conservation law $s.t. \int_{\Omega} \pi(x_i,y) = \rho_i, \sum_i \int_{U\subseteq Q} \pi(x_i,y)dy = \nu(U),\ \pi(x_i,y) \geq 0$

$$\boxed{\pi(x_i,y)\cdot\pi(x_j,y) = 0}$$

Property



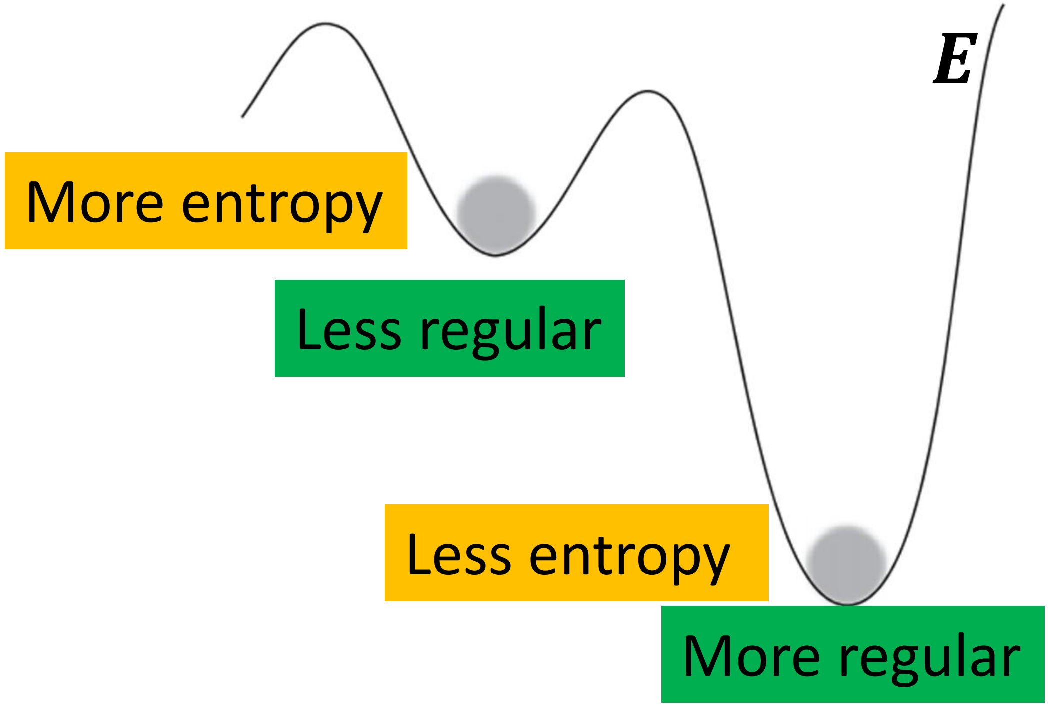
Transport Plan on 1D

Relaxed optimal transport problem

Entropic regularization term

$$\mu = \arg\min_{\mu} W_p^p(\mu, \nu) = \arg\min_{\mu}\inf_{\pi} \int_{X\times\Omega} d(x_i, y)^p \, d\pi(x_i, y) + \boxed{\epsilon H(\pi)}$$

$$s.t. \int_{\Omega} \pi(x_i, y) = \rho_i, \sum_i \int_{U\subseteq Q} \pi(x_i, y) dy = \nu(U),\ \pi(x_i, y) \geq 0$$



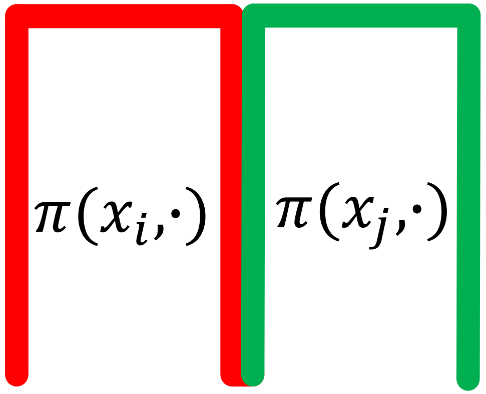
Core idea

- Relaxed optimal transport problem

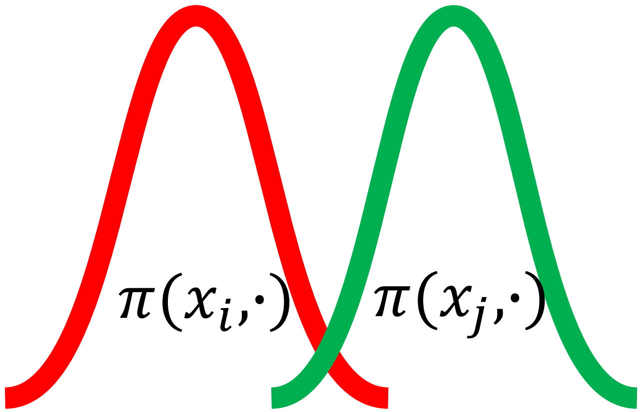
Entropic regularization term

$$\mu = \arg\min_{\mu} W_p^p(\mu, \nu) = \arg\min_{\mu} \inf_{\pi} \int_{X\times\Omega} d(x_i, y)^p \, d\pi(x_i, y) + \boxed{\epsilon H(\pi)}$$

$$s.t. \int_{\Omega} \pi(x_i, y) = \rho_i , \sum_i \int_{U\subseteq Q} \pi(x_i, y) dy = \nu(U),\ \pi(x_i, y) \geq 0$$



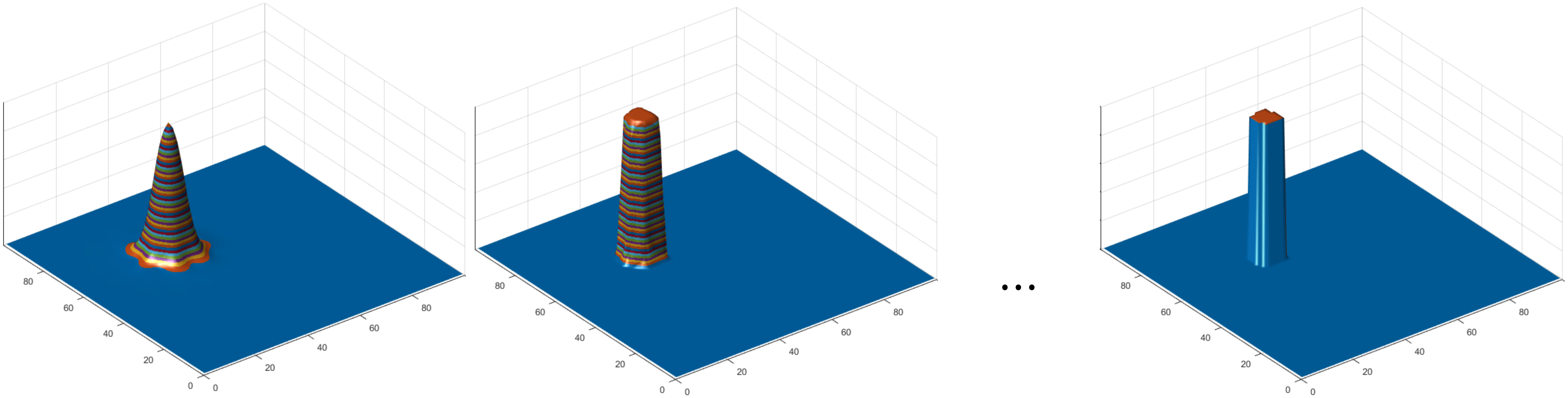
Less entropy



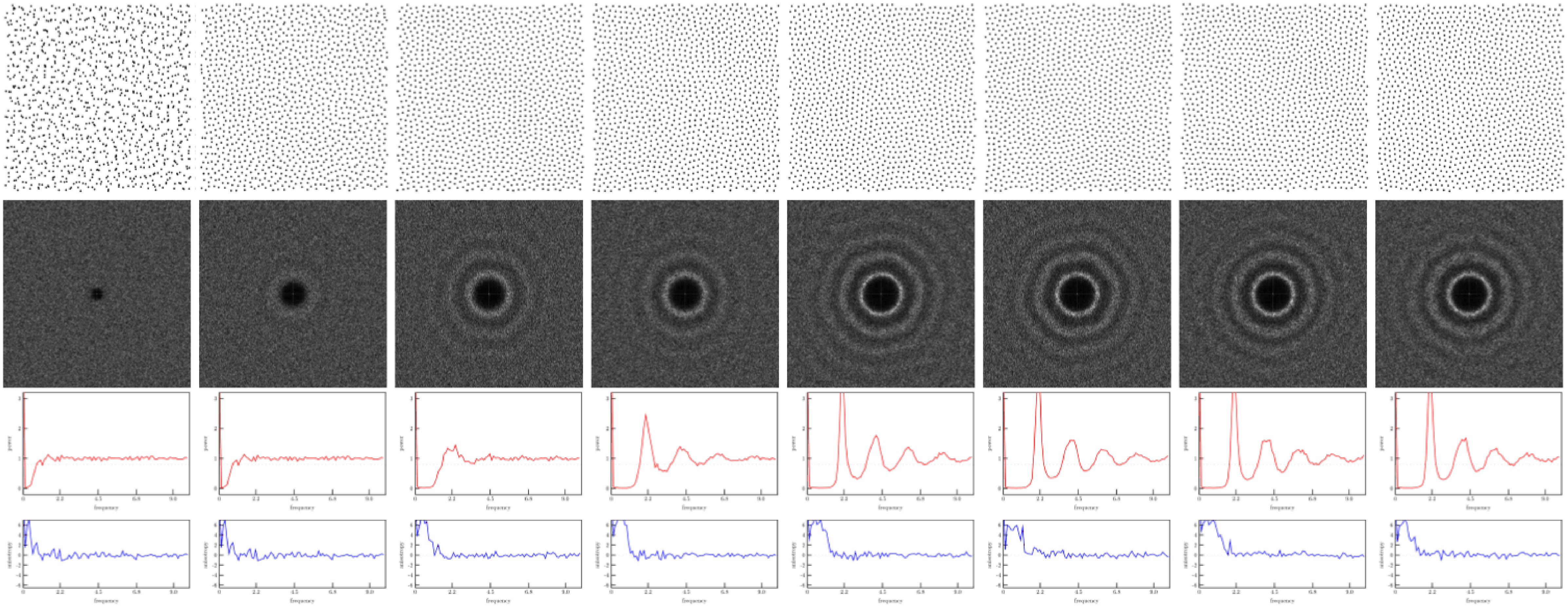
More entropy

Single class samplings on different $\boldsymbol{\varepsilon}$

$\epsilon = \mathbf{1/40}$ $\epsilon = \mathbf{1/280}$ … $\epsilon = \mathbf{0}$



Single class samplings on different $\boldsymbol{\varepsilon}$



	ϵ =1/40	ϵ =1/80	ϵ =1/120	ϵ =1/160	ϵ =1/200	ϵ =1/240	ϵ =1/280	ϵ =1/320
Ω_6	0.367	0.347	0.397	0.415	0.557	0.537	0.572	0.595
δ_{avg}	0.508	0.755	0.846	0.856	0.894	0.893	0.894	0.894

Multi-class sampling

$$\bar{\mu} = \arg\min_{\bar{\mu}} \sum_{i=1}^{K} \boxed{\lambda_i} W_p^p(\mu_i, \nu_i)$$

$$s.t. \sum_{i=1}^{K} \lambda_i = 1, (\lambda_i \geq 0)$$

Individual probability measures

$$\bar{\mu} = \{\boxed{\mu_1, \mu_2, \dots, \mu_N}, \boxed{\mu_{N+1}, \dots, \mu_K}\}$$

Combined probability measures

Discrete representation

Discrete probability density

Position

$$\mu = \sum_i \boxed{\rho_i}\delta_{\boxed{x_i}} \quad s.t. \sum_i \rho_i = 1, \delta_{x_i} = \begin{cases} 1 & x_i \in \Omega \\ 0 & \text{others} \end{cases}$$

$$\mathbf{X} = arg\min_{\boldsymbol{X}} \sum_{i=1}^{K} \lambda_i < \mathbf{D}_i, \prod_i >$$

$\mathbf{D}_i$: the distance matrix

$\prod_i$: the transport plan matrix

A loop iteration algorithm on GPU

- An iterative Bregman projection for $\prod_i$

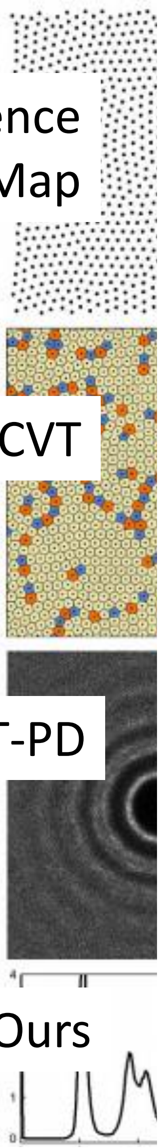
$$W_p^p(D, \Pi) = arg\, \underset{\boxed{\Pi}}{\min} < D, \Pi > + \varepsilon H(\Pi)$$

- A Newton iterative method for $\mathbf{X}$

$$\mathbf{X} = (1 - \theta)\mathbf{X} + \theta\mathbf{Y}\mathbf{\Pi}^{\mathrm{T}}\mathbf{diag}(\rho^{-1}), 0 < \theta < 1$$

[Cuturi 2013]

Comparison on single class sampling

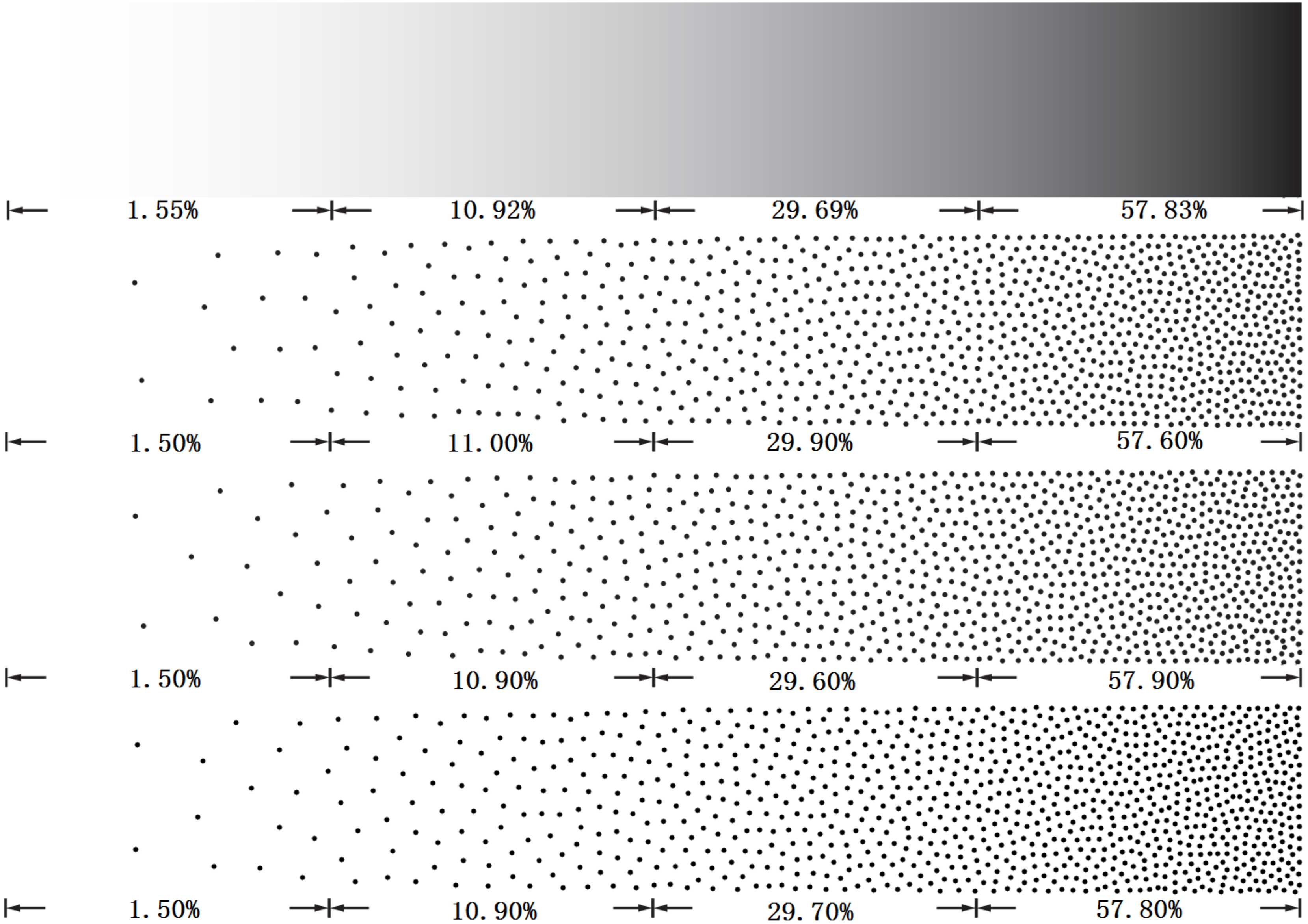


Reference Map

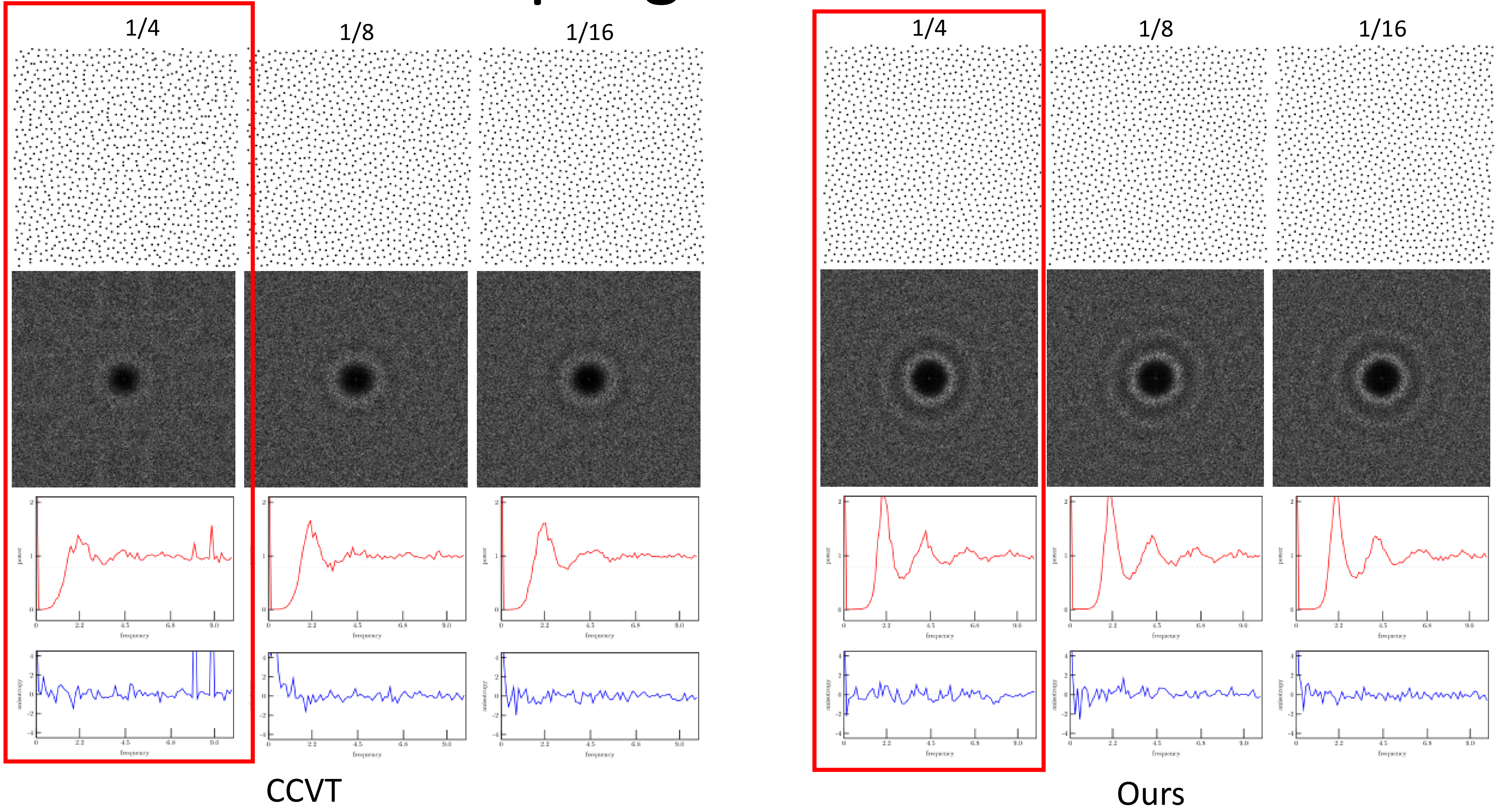
CCVT

CCVT-PD

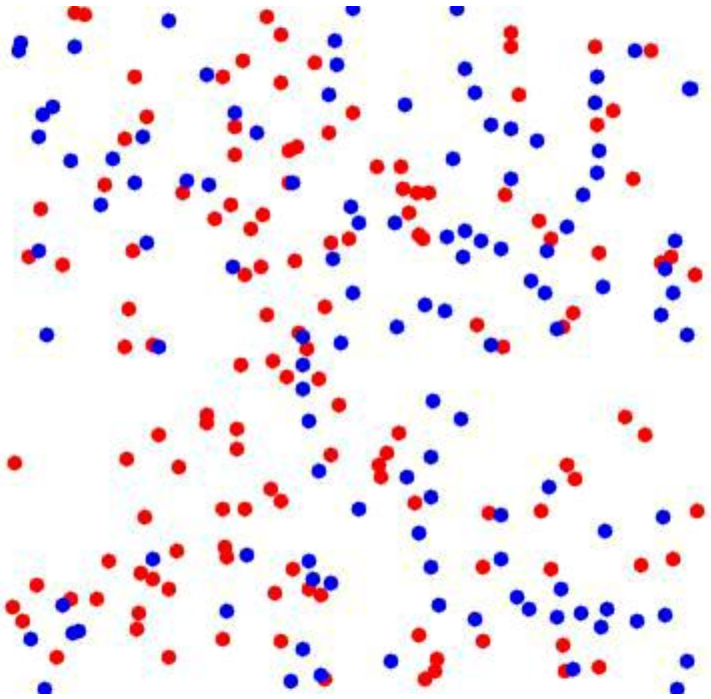
Ours



Evaluation of sampling ratio

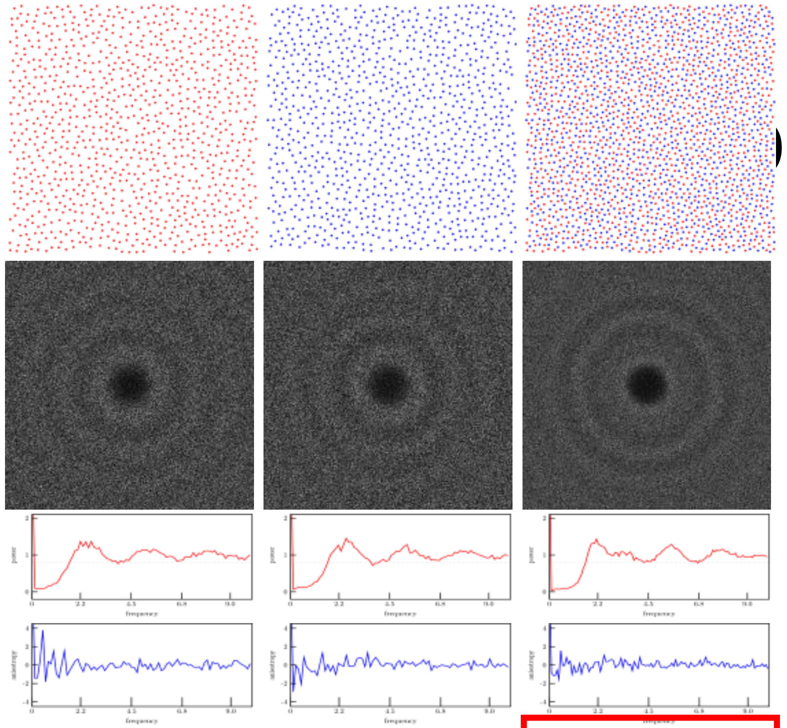


Two-class sampling

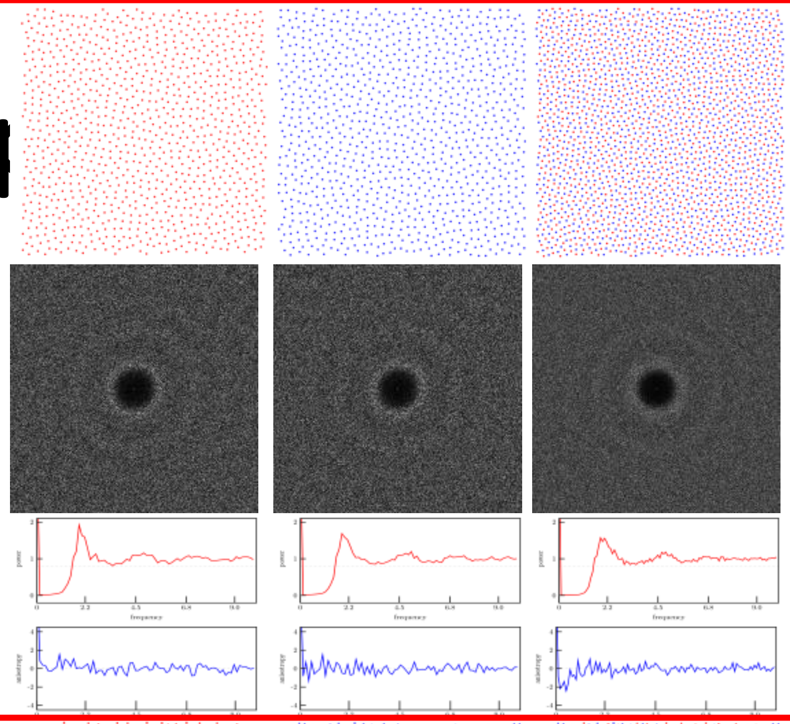


The co -class samp

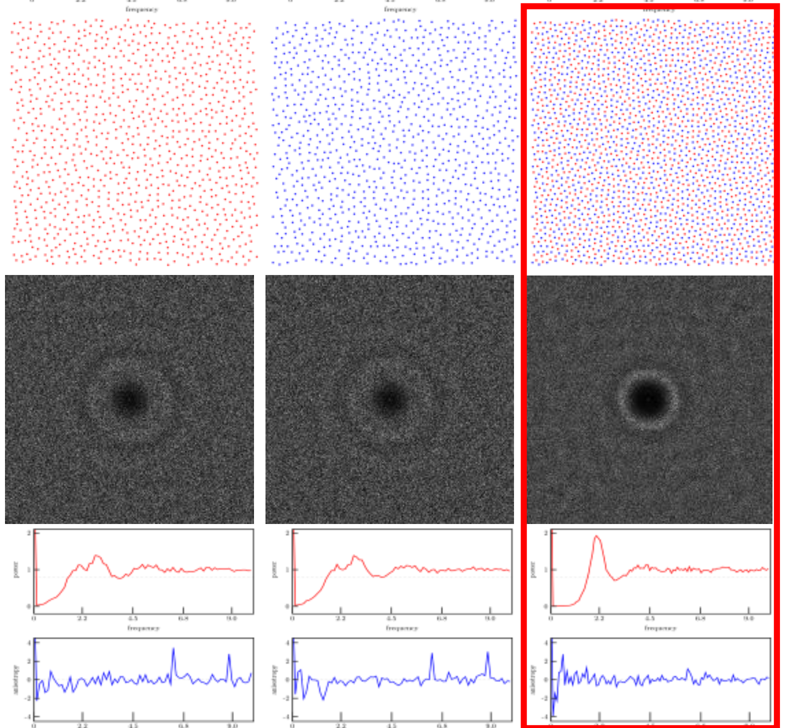
Dart throwing
[Wei 2010]



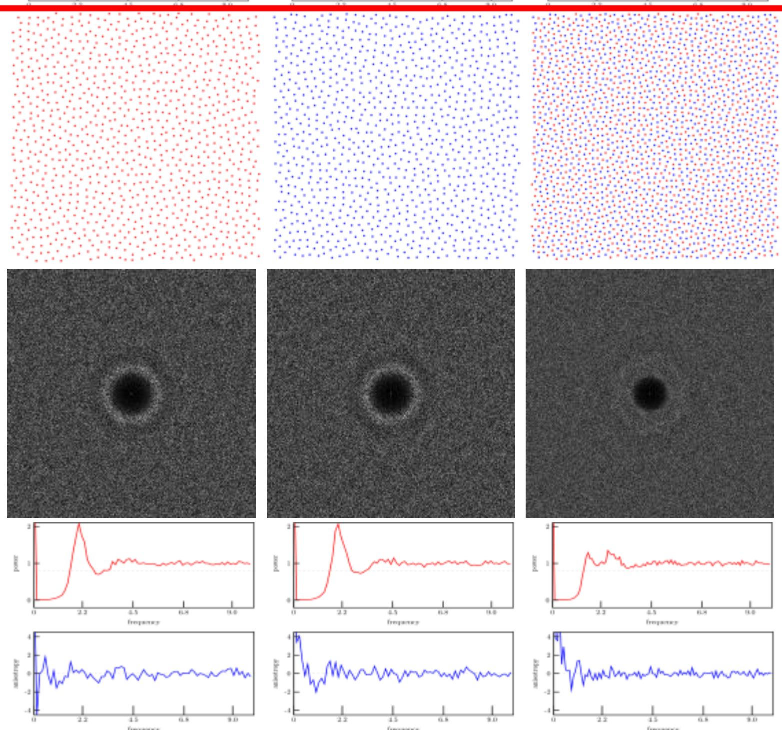
Ours
$\lambda_{1,2,3} = (1, 1, 2)/4$



Ours
$\lambda_{1,2,3} = (1, 1, \boxed{6)/8}$

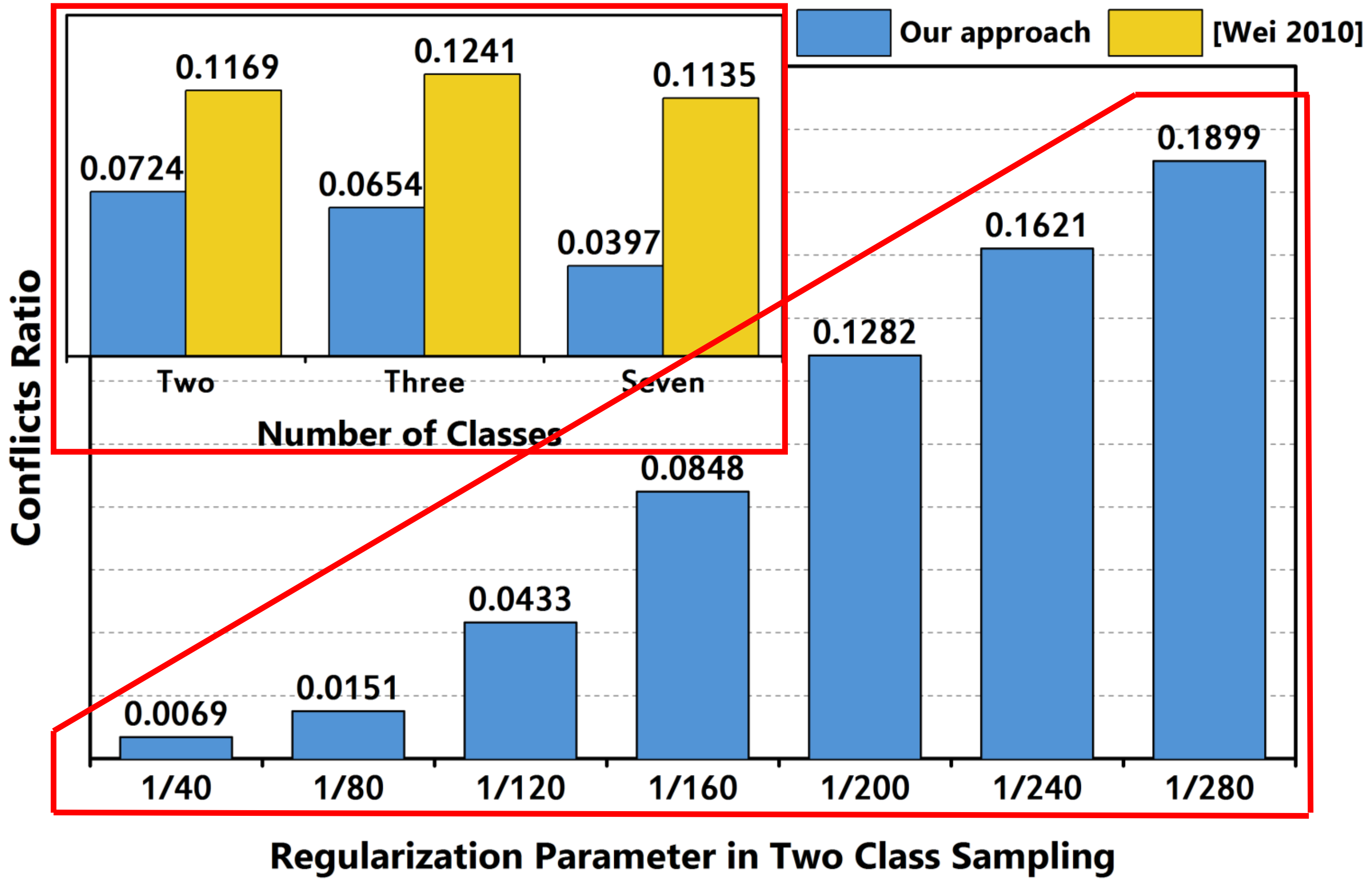


Ours
$\lambda_{1,2,3} = (1, 1, 1)/3$

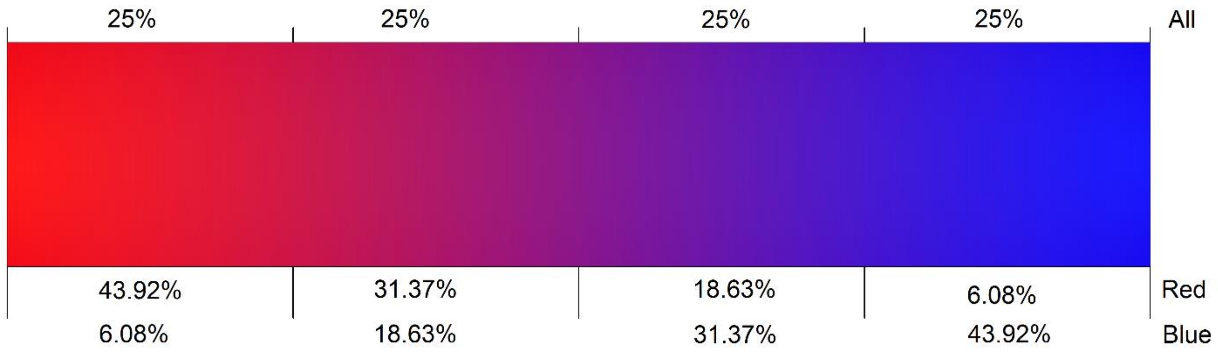


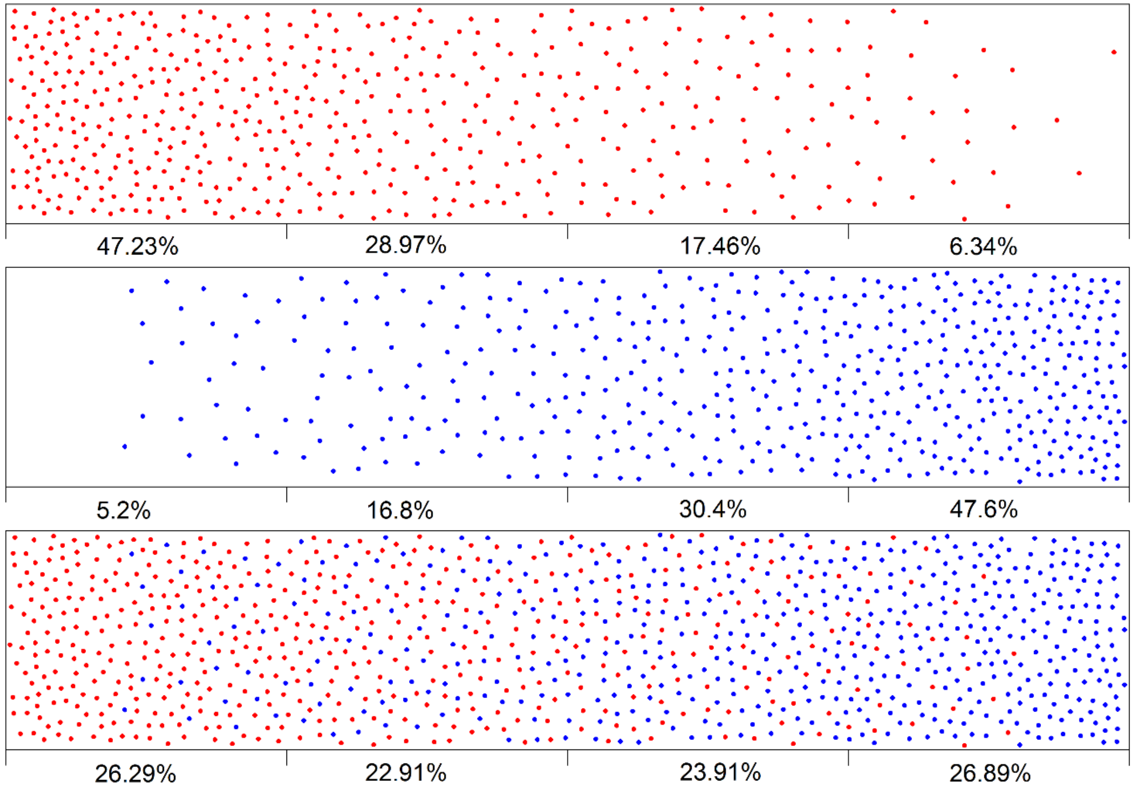
Conflicts ratio

$$R = \sum_{i=1}^{N}\sum_{j=1}^{n_i}\left\|C^{I}_{x_i^j} - C^{C}_{x_i^j}\right\| \Big/ \sum_{i=1}^{N}\sum_{j=1}^{n_i} D\left(x_i^j\right)$$

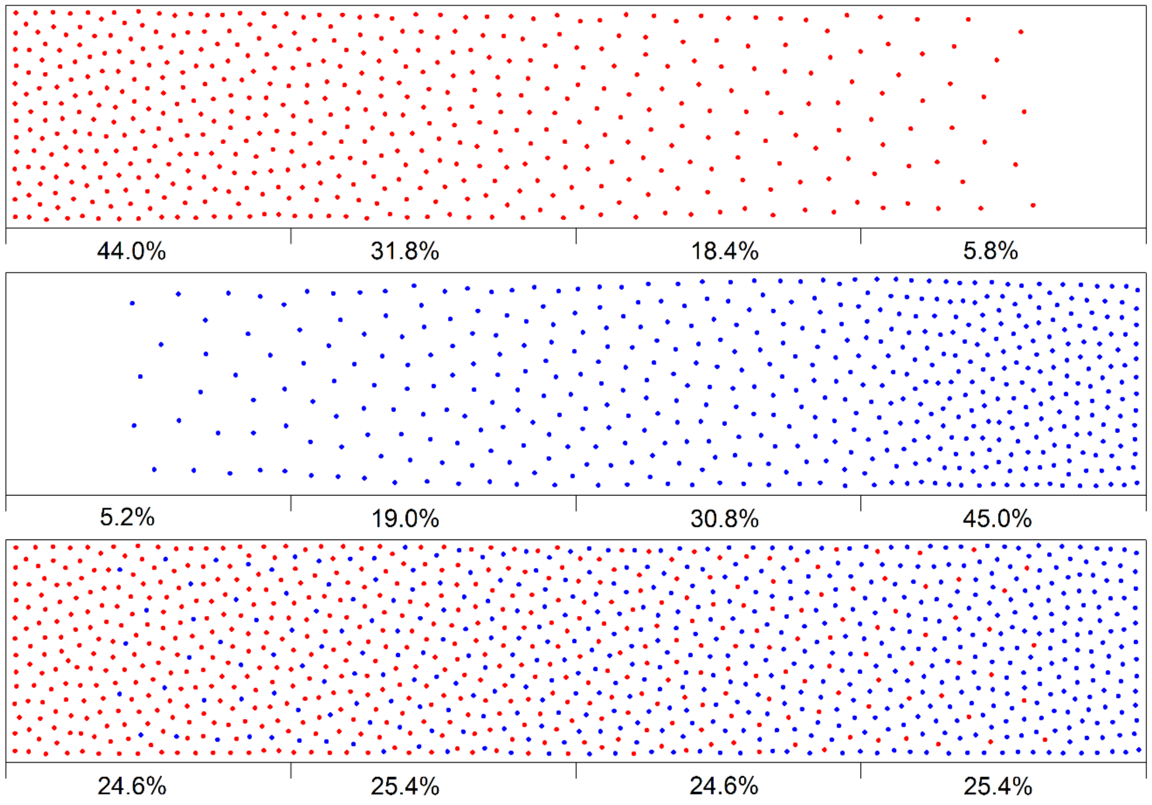


Adaptive two-class sampling



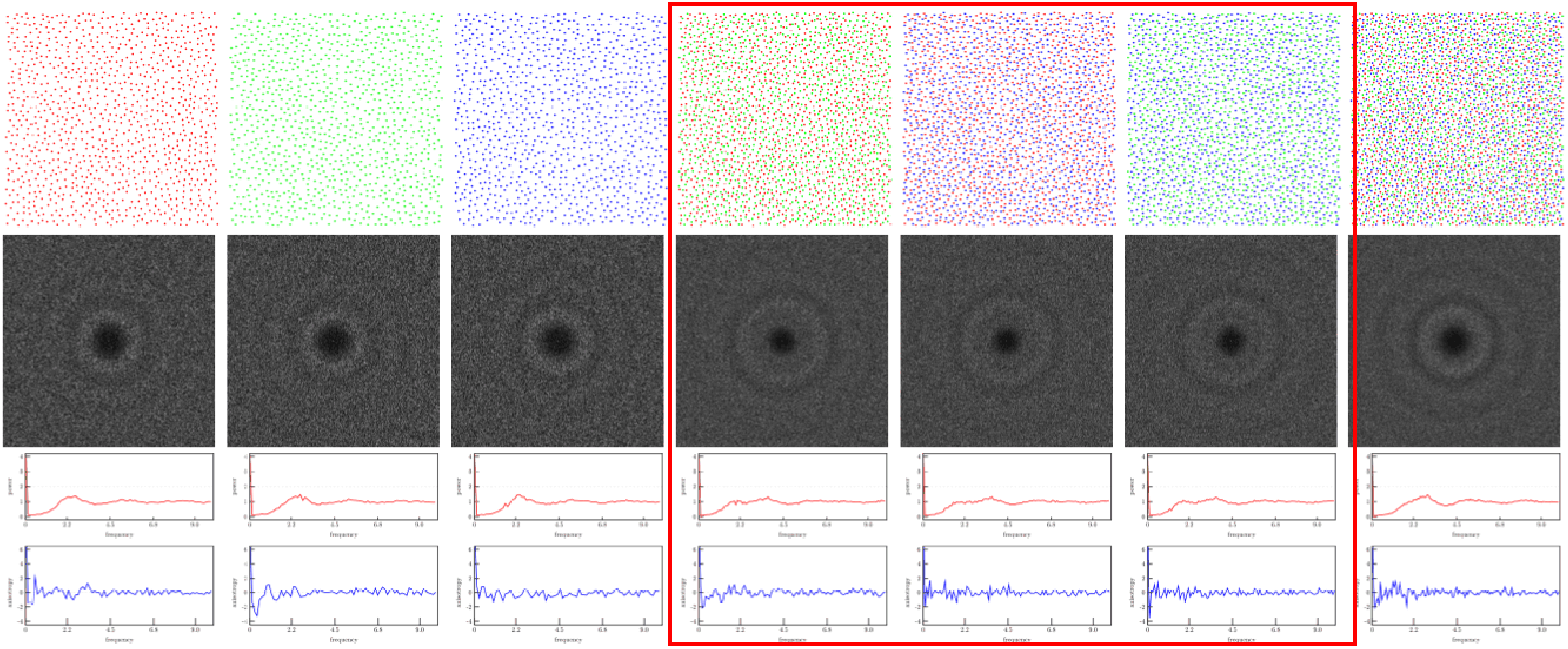


Dart throwing [Wei 2010]



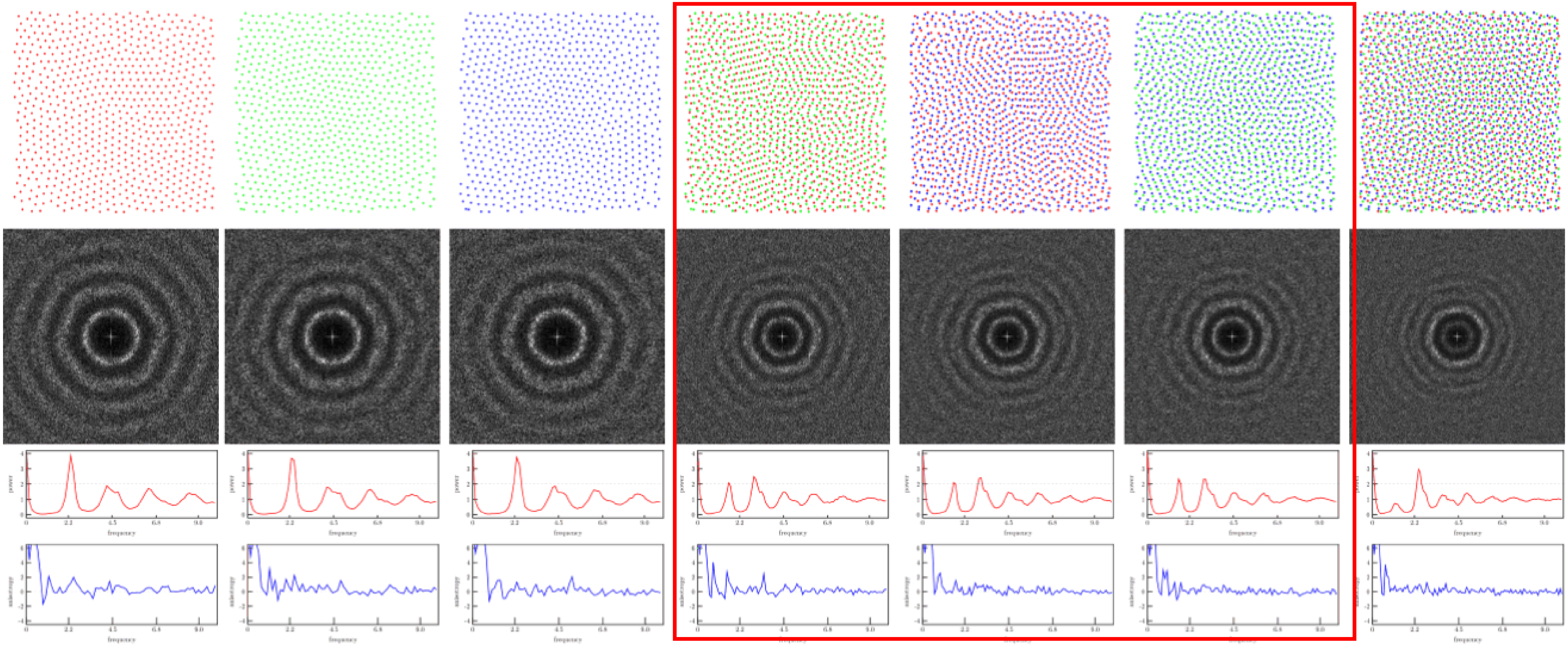
Ours

The comparison for three-class sampling



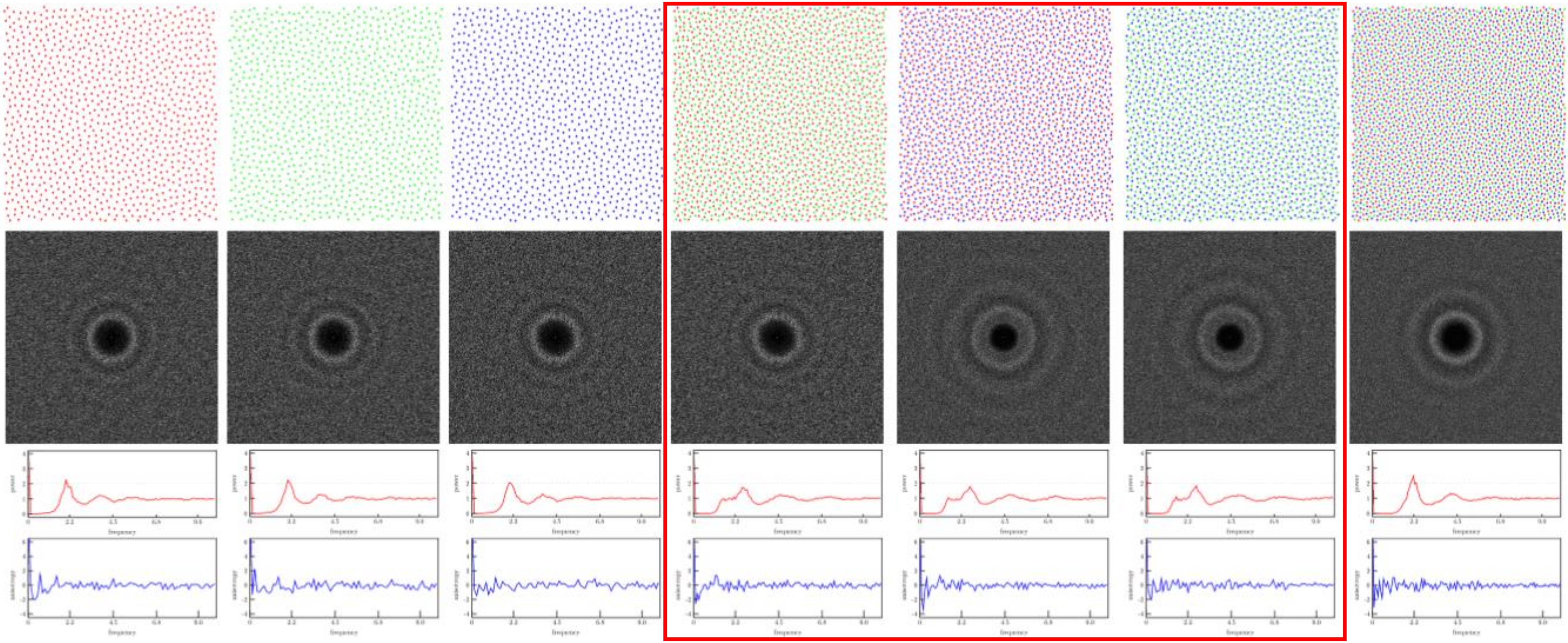
Dart throwing [Wei 2010]

The comparison for three-class sampling



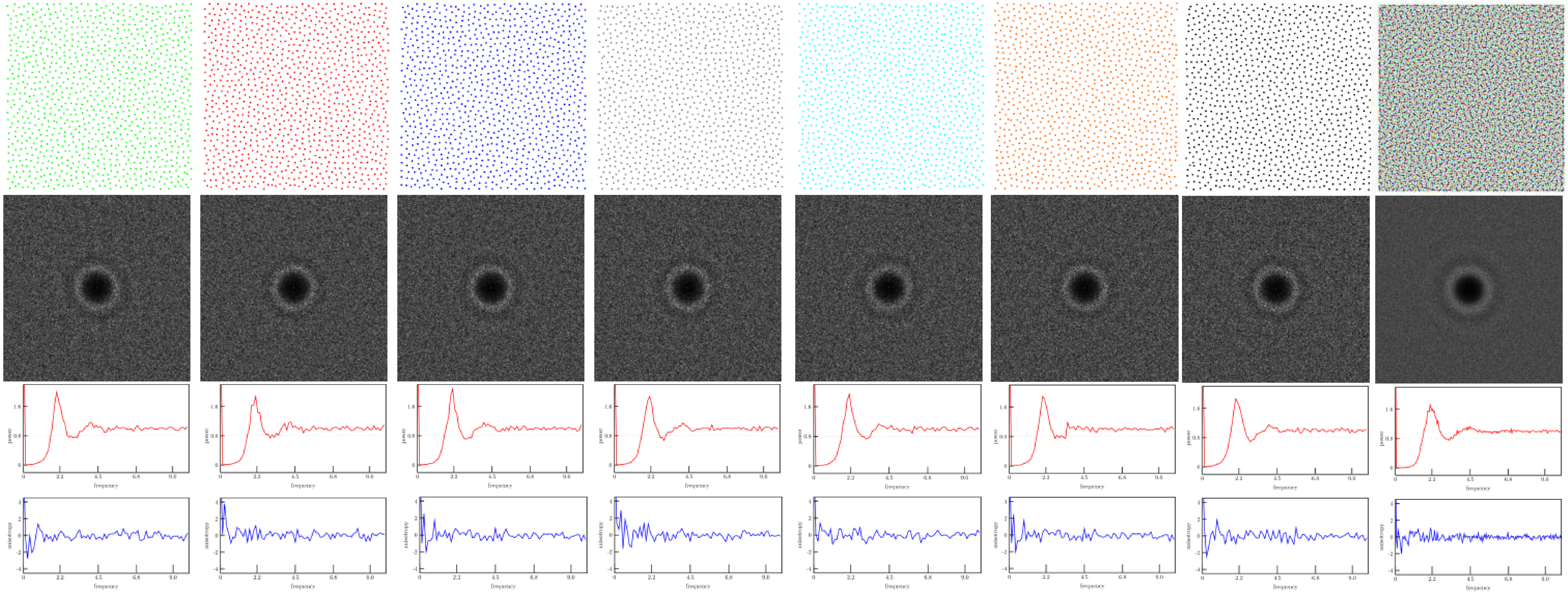
SPH [Jiang et al. 2015]

The comparison for three-class sampling

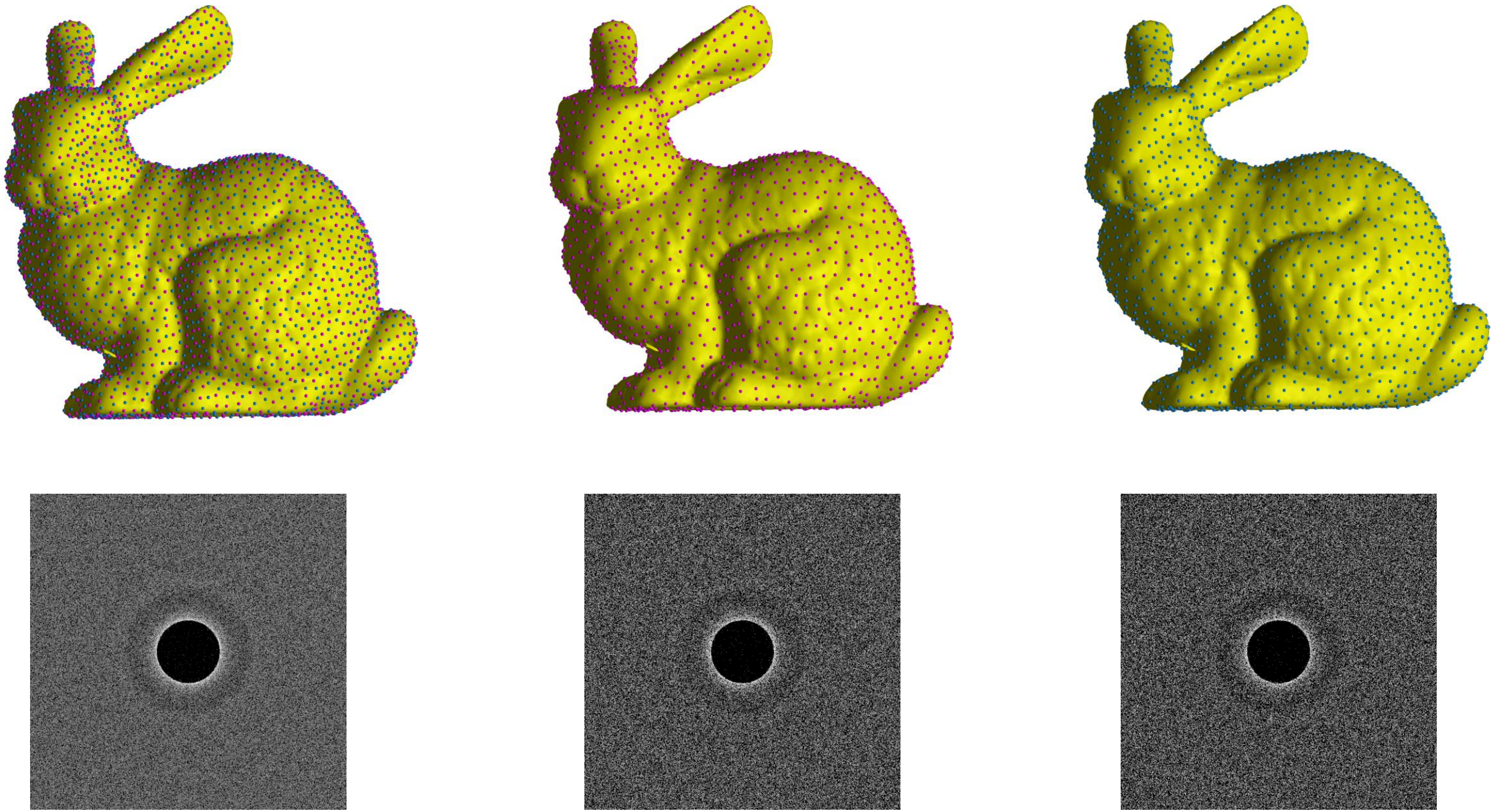


Ours

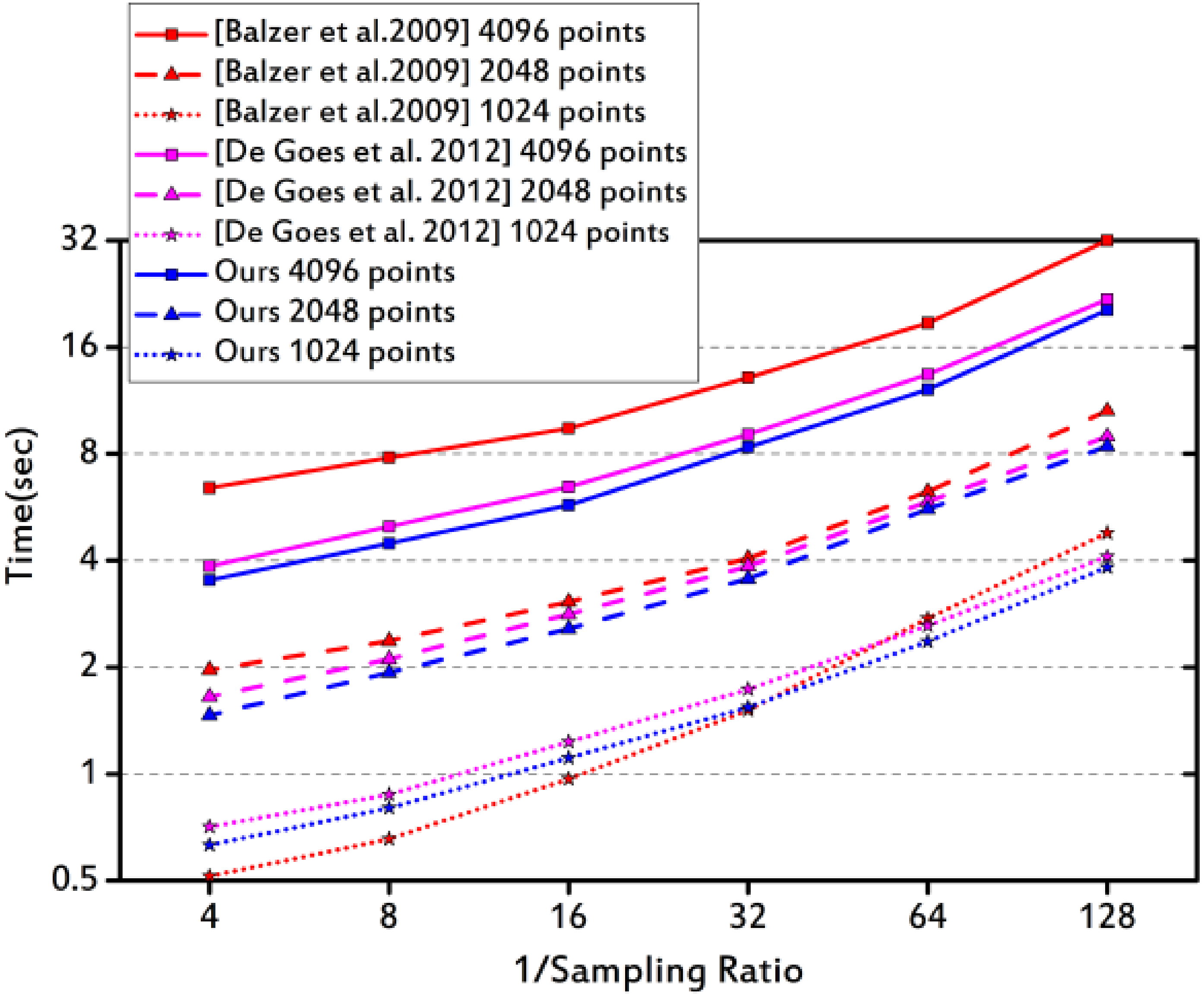
Seven-class sampling



Sampling on a point set surface



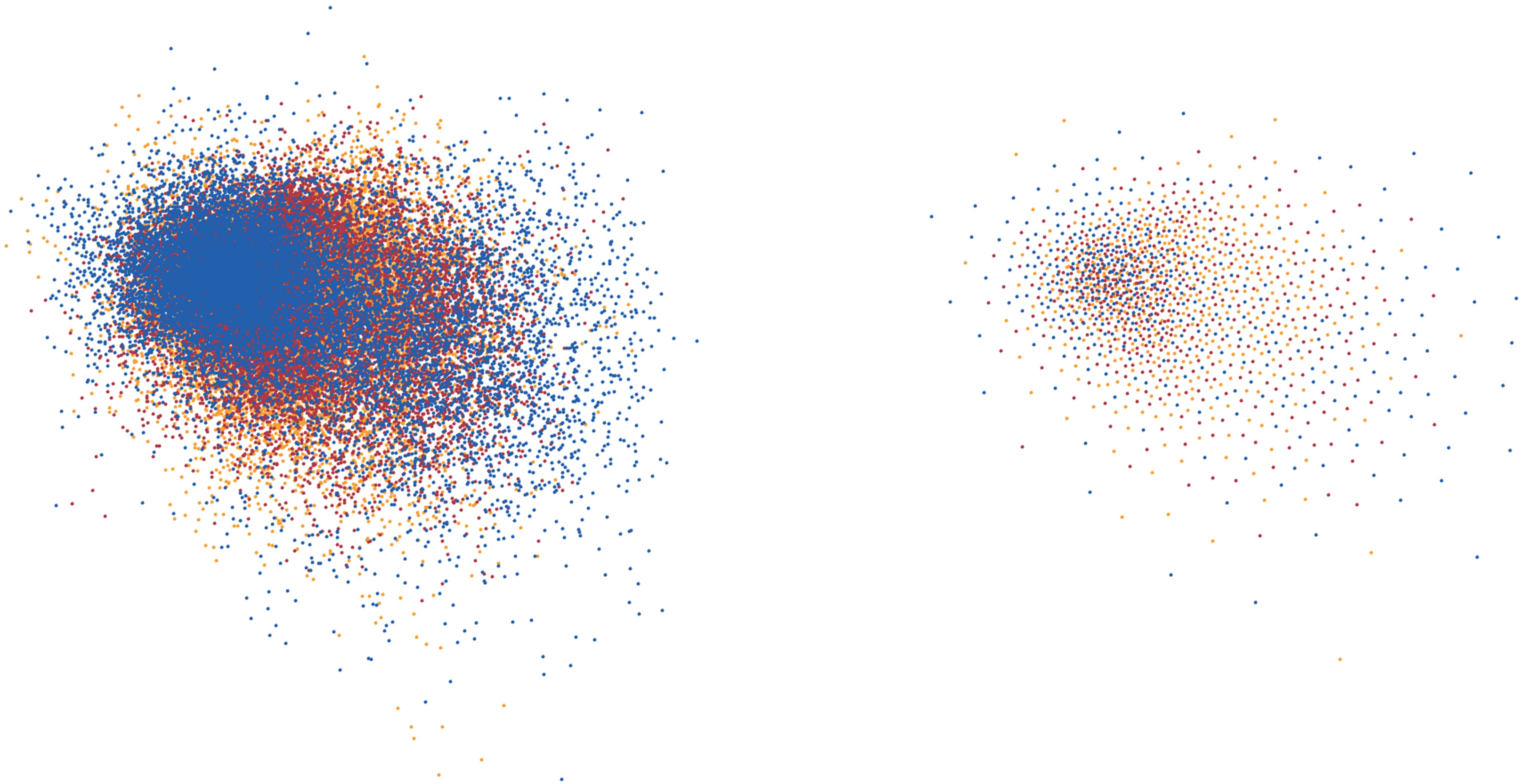
Running time in log scale for single sampling



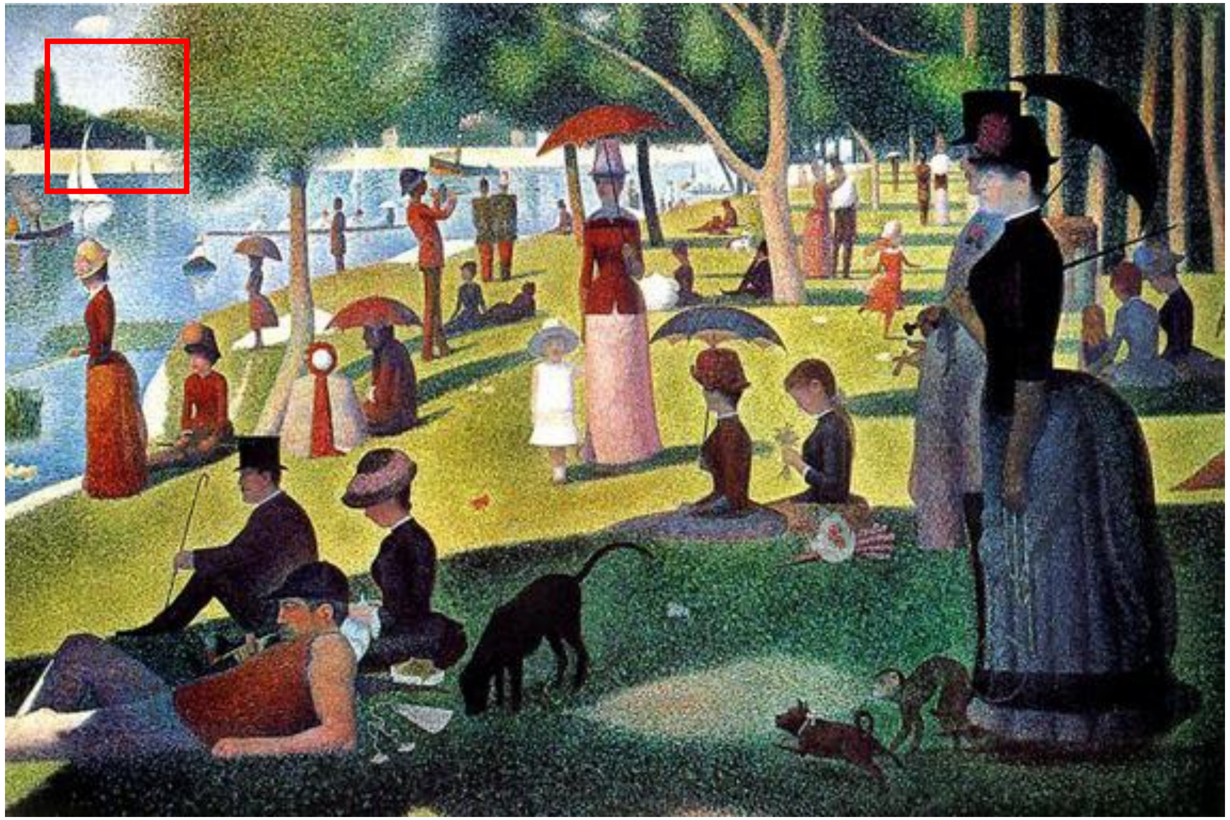
Running time and memory usage for multi-class sampling

Number of Class	Times(s)		Memory(M)
	[Wei 2010]	Ours	Ours
2	34.769	14.670	128.11
3	151.727	37.553	288.11
7	414.031	212.133	448.31

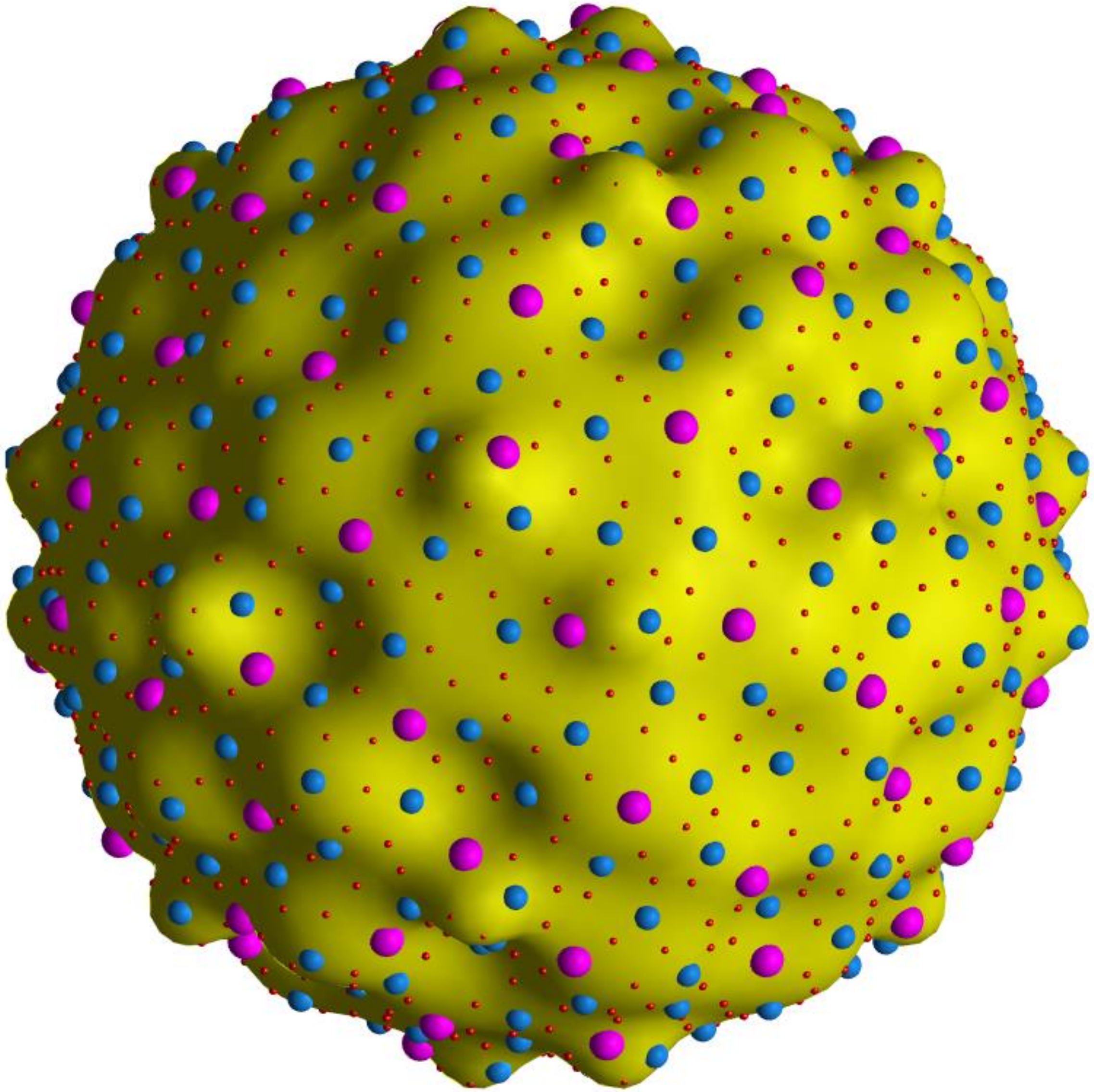
Applications: visual abstraction



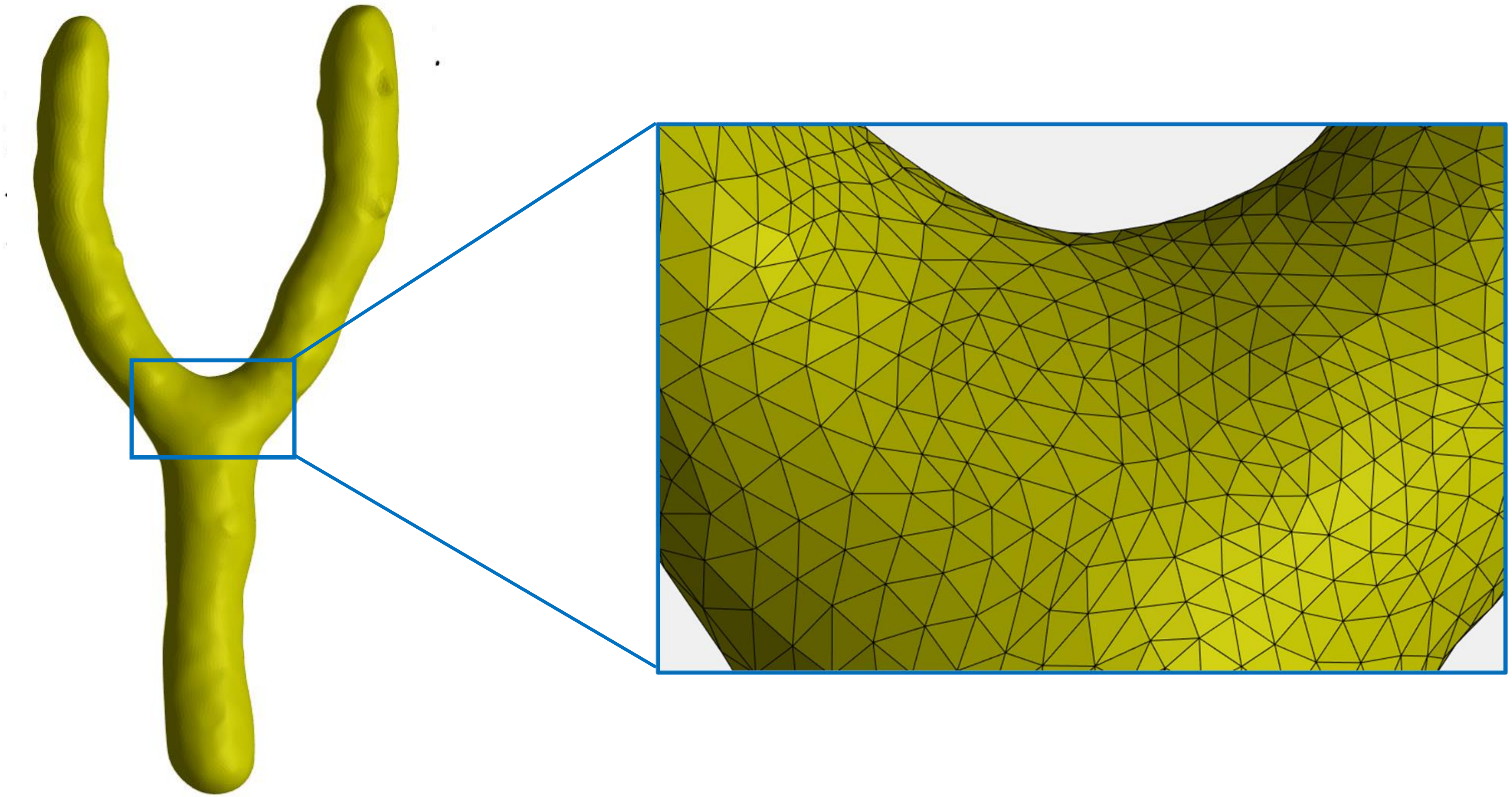
Applications: color stippling



Applications: object placement



Surface construction of noisy point clouds



Conclusions

- Multi-class blue noise sampling as the constrained Wasserstein barycenter

- Relaxing optimal transport problem via the entropic regularization term

Future work

- The synthesis of point distributions

- The efficiency for large datasets

Acknowledgments

- Reviewers

- Discussions and suggestions
  - Liyi Wei, The University of Hong Kong
  - Lin Lu, Shandong University
  - Yunhai Wang, Shandong University

- Data
  - Zhonggui Chen, Xiamen University

Thank you!

Code :
https://github.com/Hongxing-CQUPT/SamplingCUDA