适合等几何分析的样条工具

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Galerkin Projection

- To solve u, such that: $Lu = f \qquad in\,\Omega$

  with the boundary conditions

$$u = 0 \text{ on } \Gamma_{\text{D}} \text{ and } \langle \nabla u, n \rangle = h \text{ on } \Gamma_{\text{N}}$$

  Here L is a second order elliptic operator on the Lipschitz domain Ω

- Week form:

$$a(u,v) = l(v) \text{ for all } \mathrm{v} \in \mathrm{V} := \{\mathrm{u} \in \mathrm{H}^1(\Omega), u|_{\Gamma_{\mathrm{D}}} = 0\}$$

Galerkin Projection

- Replace the infinite space *V* with the finite dimensional space $S = span\{\phi_i, i = 1,2,\ldots,n\}$
- Numerical solution

$$u_h = \sum_{i=1}^{n} q_i \phi_i$$

- The coefficients are from a linear system *Aq=b*,

$$A_{i,j} = a(\phi_j, \phi_i), b_i = l(\phi_i)$$

Collocation

$$Lu = f \qquad in\,\Omega$$

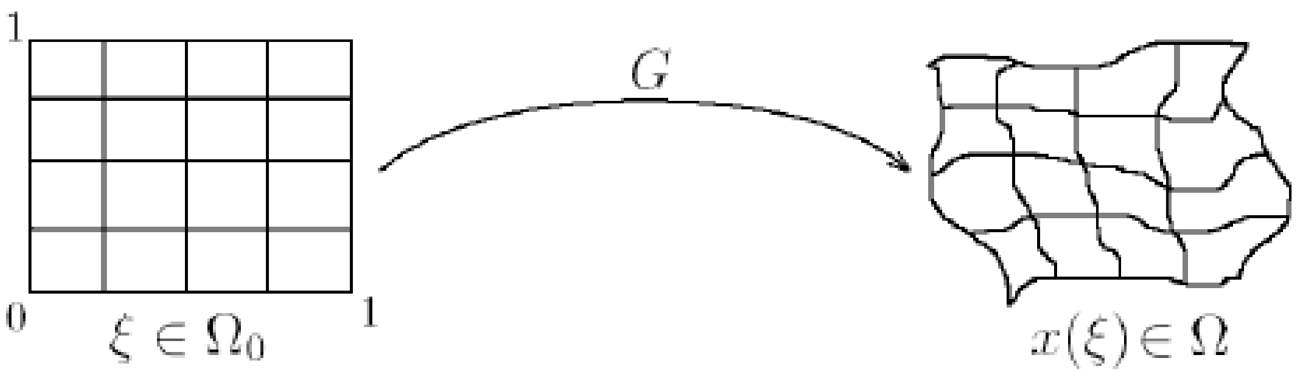
$$u_h = \sum_{i=1}^{n} q_i \phi_i$$

$$Lu_h(\alpha_j) = f(\alpha_j), j = 1,2,\ldots,n$$

Isogeometric analysis

- The geometry functions: $G(\xi) = x(\xi) = \sum_{j=1}^{n} N_j(\xi) x_j$
- The basis functions:

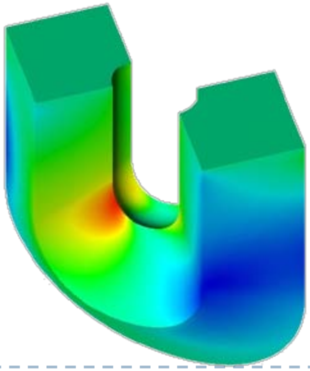
$$\phi_i = N_i \circ G^{-1}(x)$$



Isogeometric Analysis

- Based on technologies (e.g., NURBS, subdivision,T-splines, etc.) from *computational geometry* used in:
  - Design
  - Animation
  - Graphic art
  - Visualization

- Includes standard FEA as a special case, but offers other possibilities:
  - Precise and efficient geometric modeling
  - Simplified mesh refinement
  - Smooth basis functions with compact support
  - Superior approximation properties
  - *Integration* of design and analysis



Outline

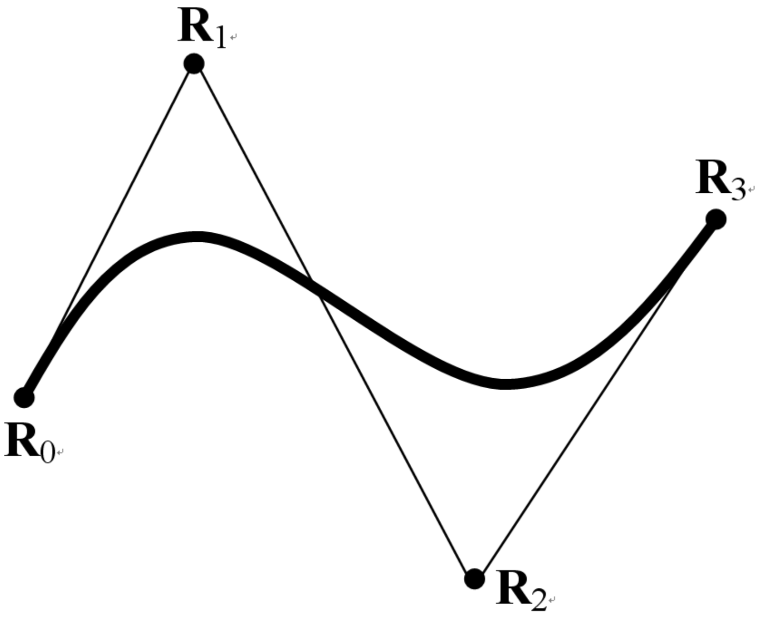
B-pline-based IGA: the advantages

T-Spline-based IGA: From design to analysis

B-spline based IGA

Bézier curves

Pierre Bézier (1910.9.1-1999.11.25)



Bézier curves

Bézier curve

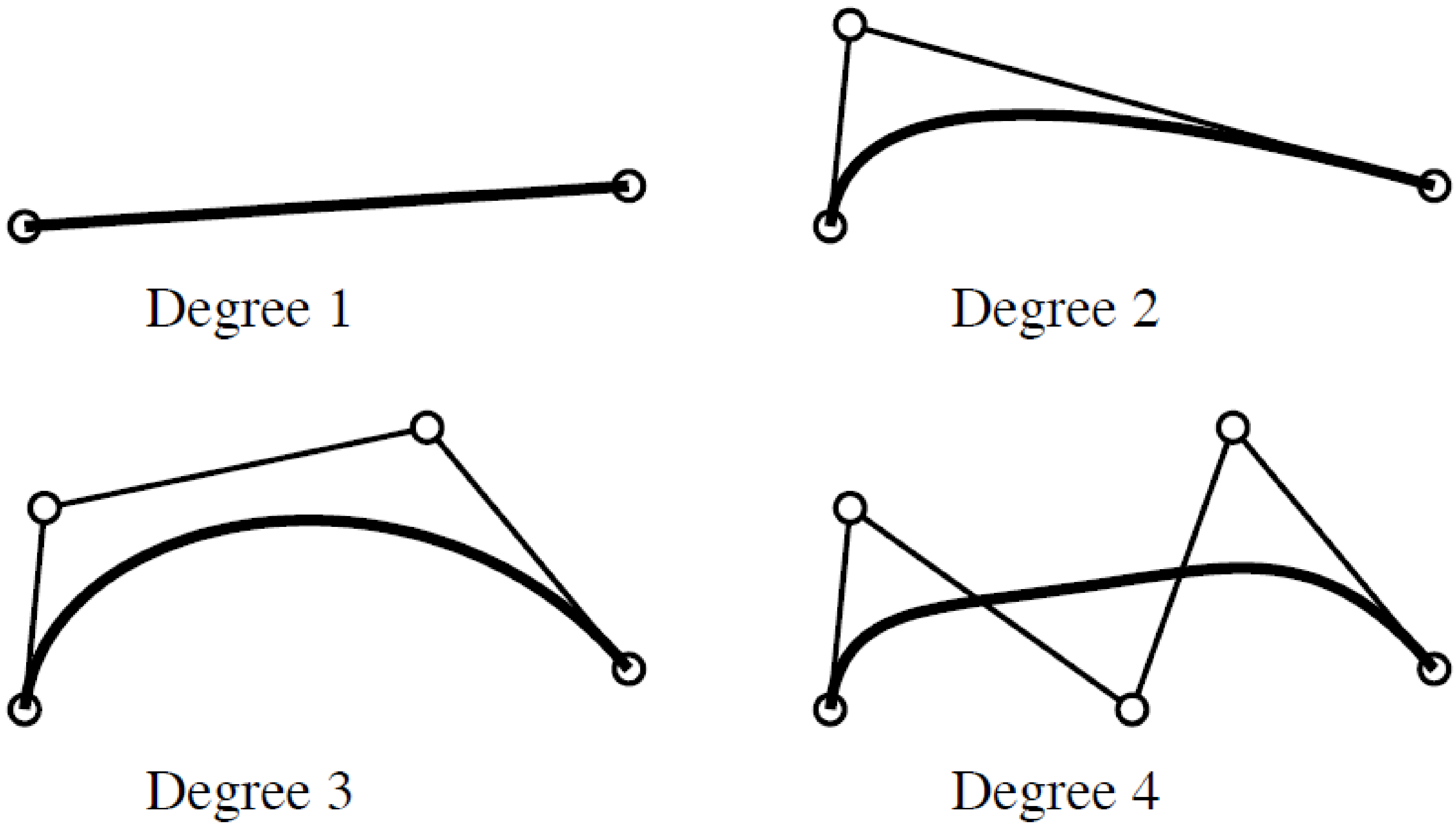
- A degree n Bézier curve:

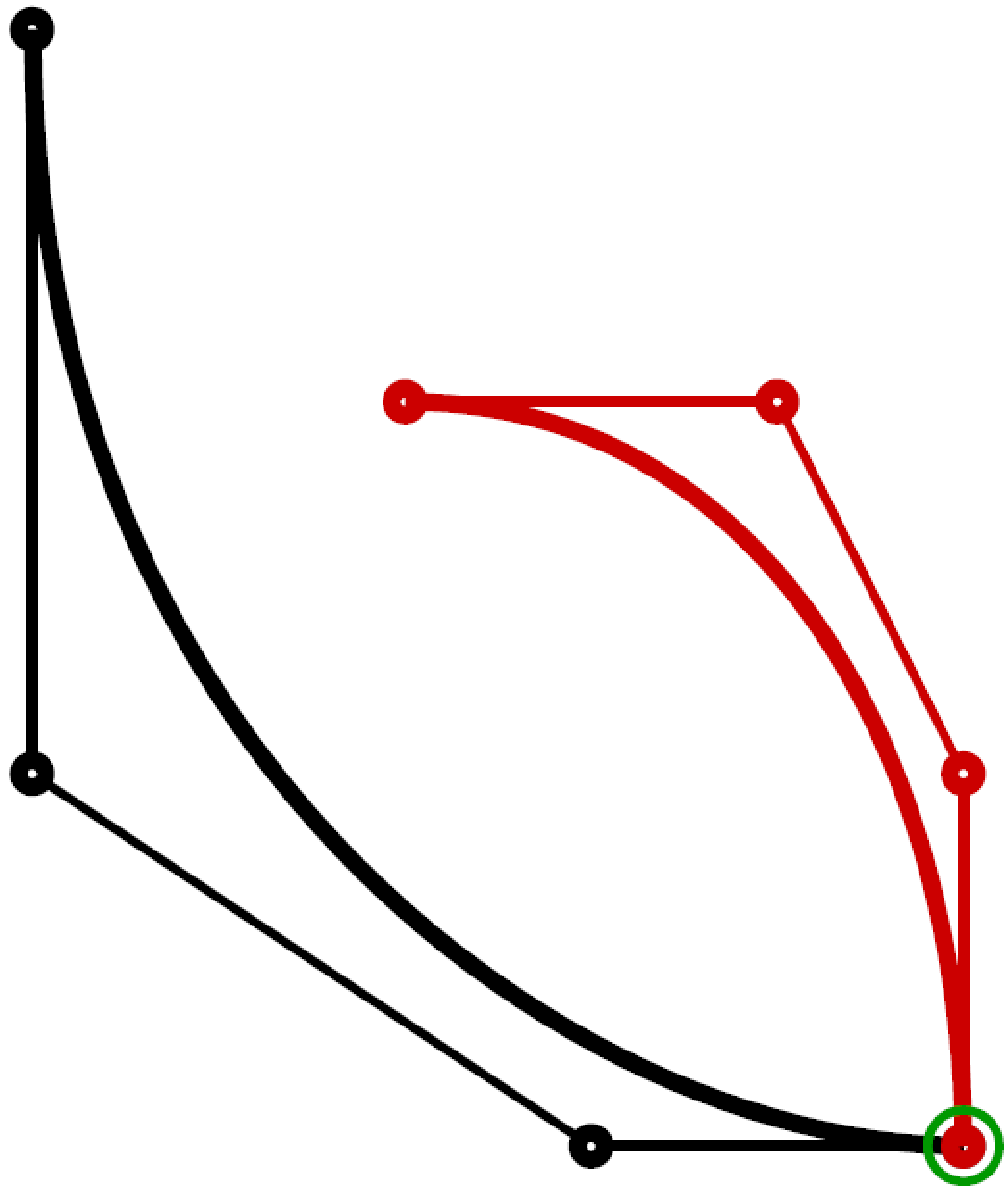
$$\mathbf{R}(t) = \sum_{i=0}^{n} \mathbf{R}_i B_{i,n}(t) \qquad 0 \le t \le 1$$

$\{B_{i,n}(t)\}$ are Bernstein basis functions:

$$B_{i,n}(t) = C_n^i (1-t)^{n-i} t^i$$

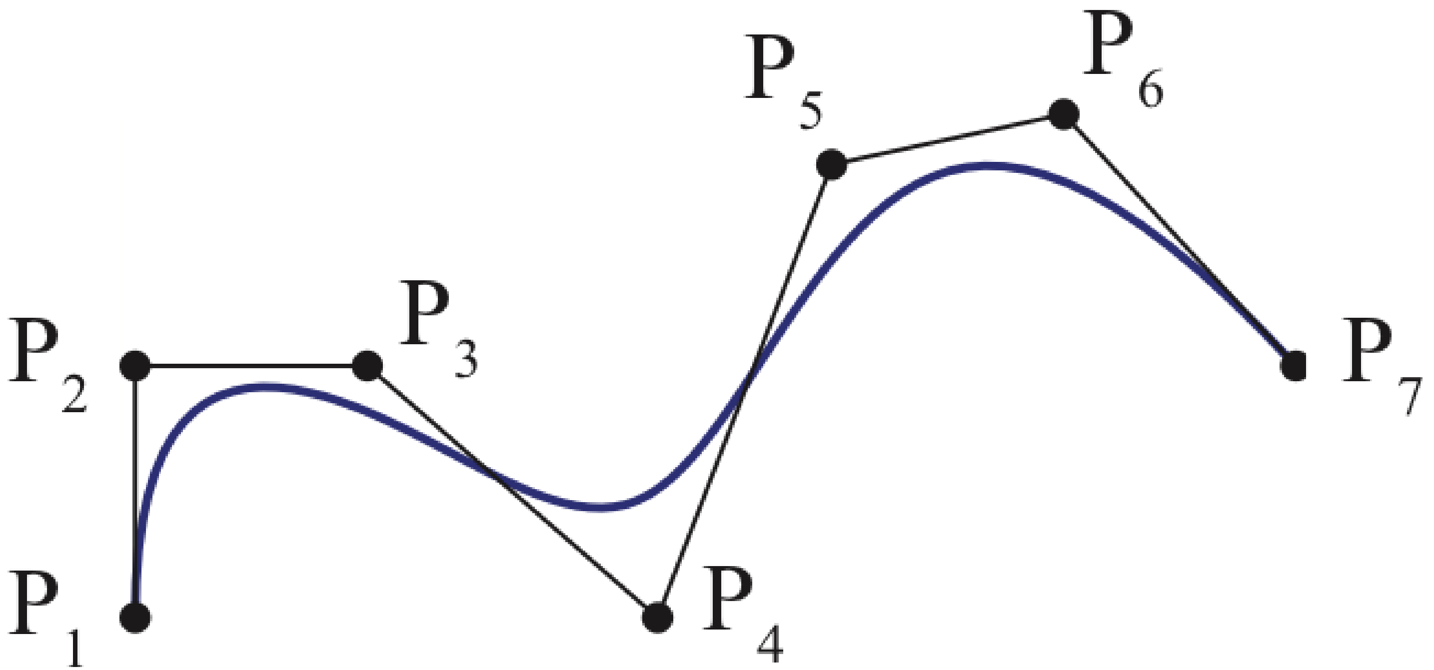
$$C_n^i = n! \big/ \left(i!(n-i)! \right)$$


Degree 1
Degree 2
Degree 3
Degree 4



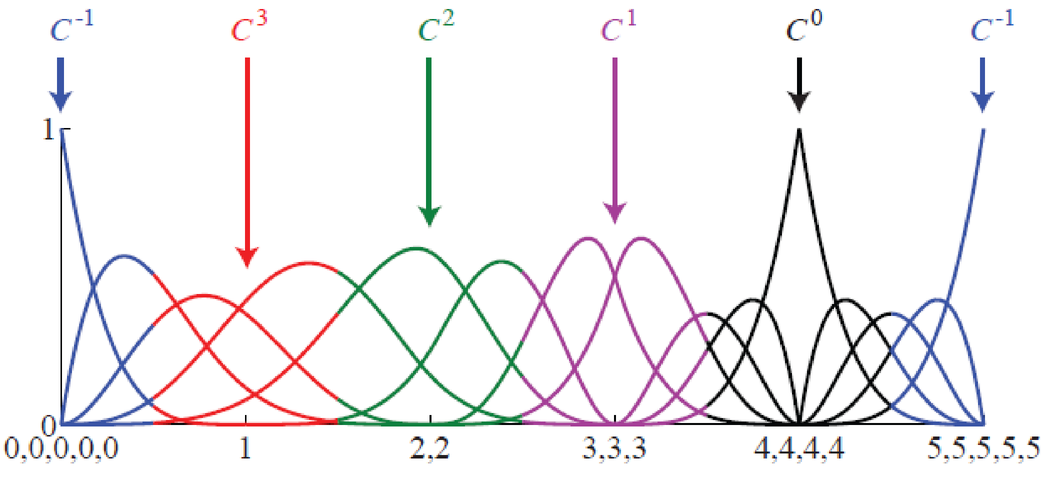
A NURBS Curve:

Automatically control the continuity



$$P(t) = \sum_{i=1}^{n} P_i \frac{\omega_i N_{i,k}(t)}{\sum_{i=1}^{n} \omega_i N_{i,k}(t)}$$

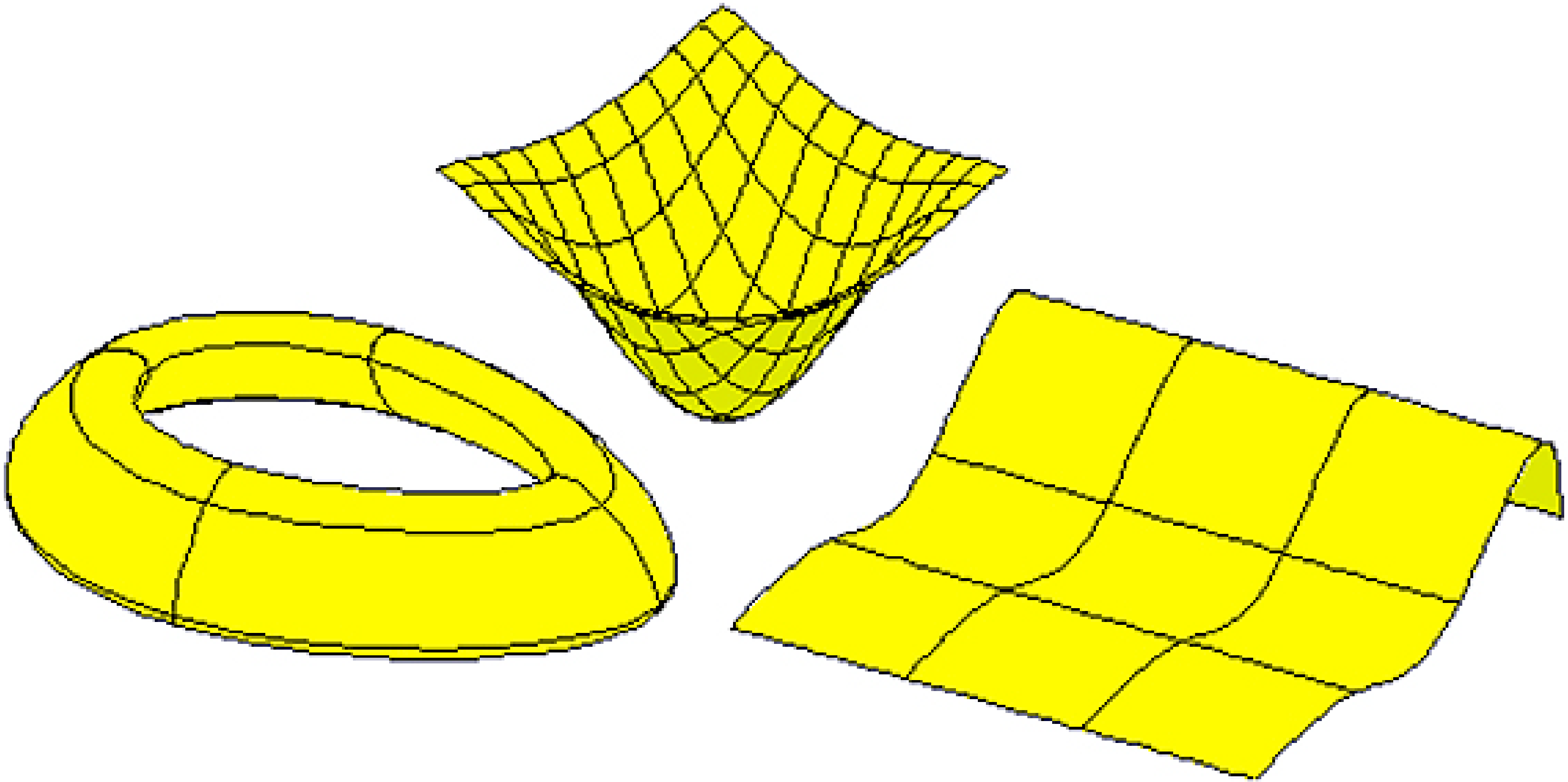
B-spline Basis Functions



Properties:

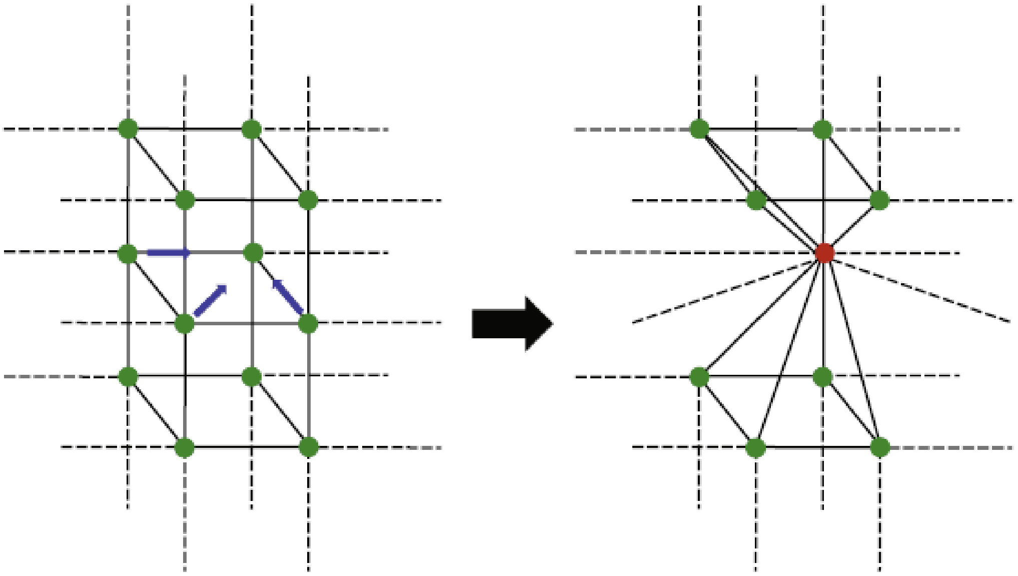
- ✓ Partition of unity
- ✓ Pointwise nonnegativity
- ✓ Linear independence
- ✓ Compact support
- ✓ Control of continuity
- ✓ Refineability

B-spline surfaces

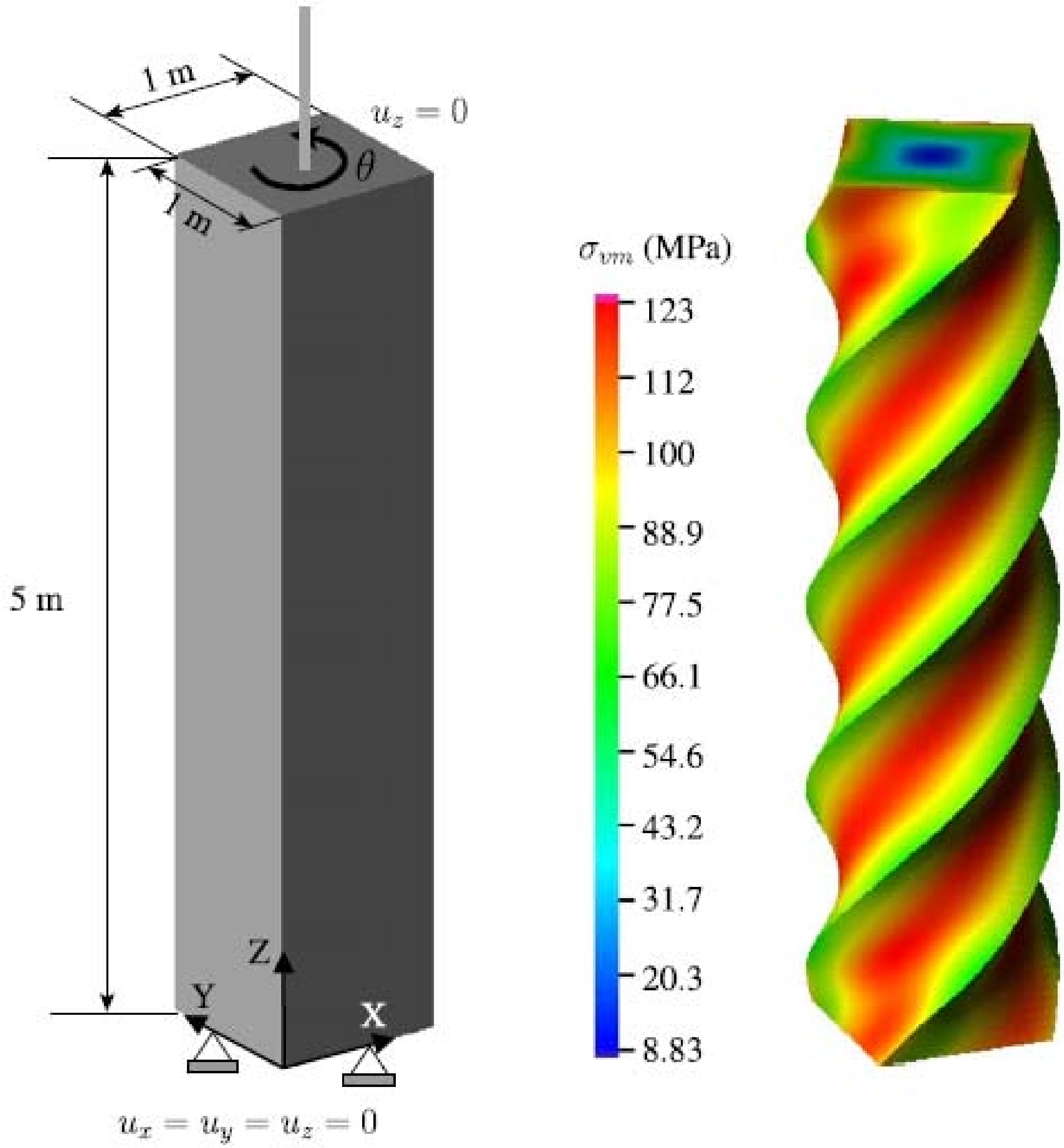


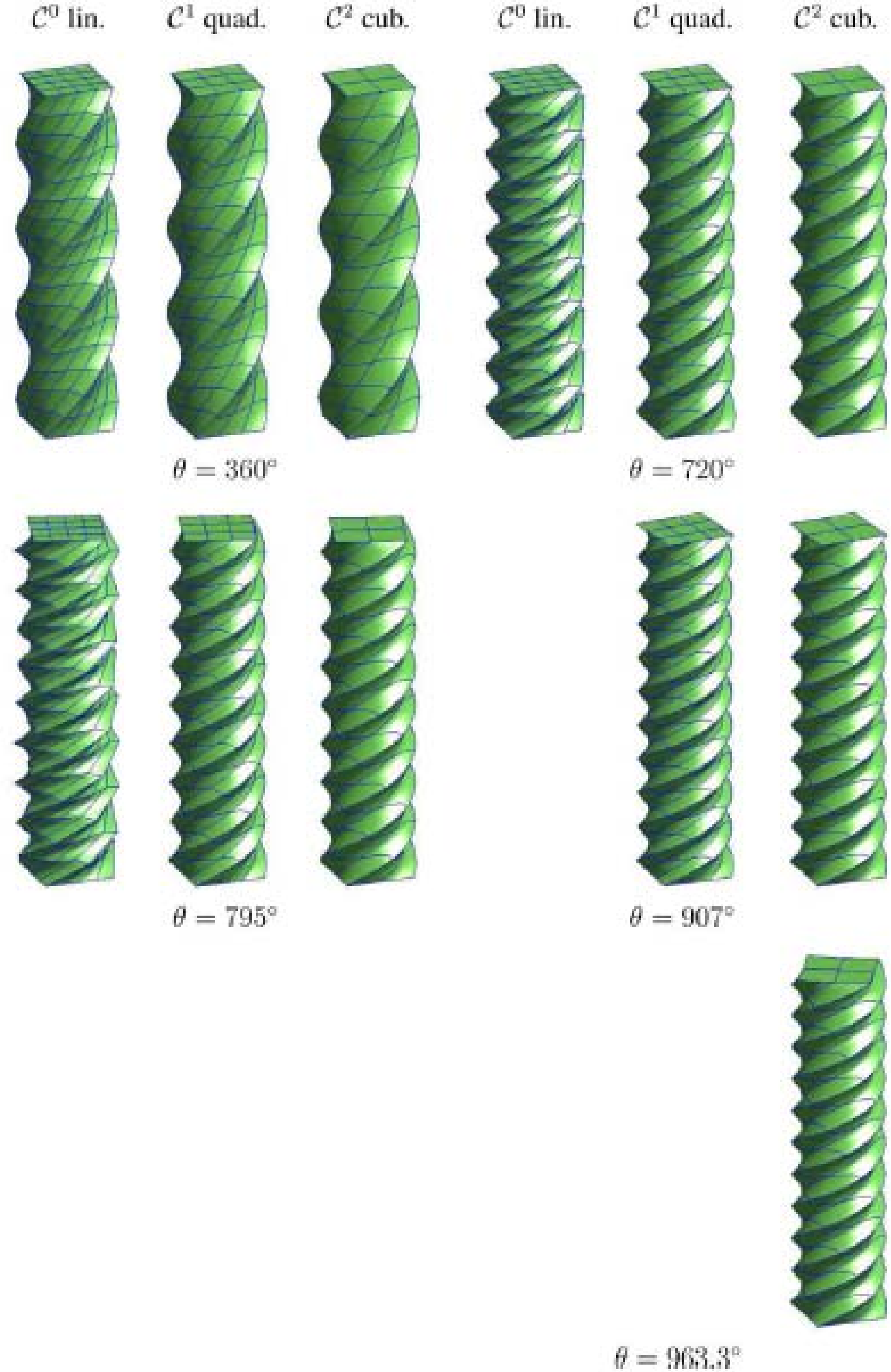
Robustness

- NURBS produce valid discretizations under increased control mesh distortion
- Using higher-order functions one may relax the requirements on the quality of the control mesh



S. Lipton a, J.A. Evans , Y. Bazilevs , T. Elguedj , T.J.R. Hughes, Robustness of isogeometric structural discretizations under severe mesh distortion, CMAME 2010.


1 m
$u_z = 0$
θ
1 m
5 m
Z
Y
X
$u_x = u_y = u_z = 0$
σ_{vm} (MPa)
123
112
100
88.9
77.5
66.1
54.6
43.2
31.7
20.3
8.83


$\mathcal{C}^0$ lin.
$\mathcal{C}^1$ quad.
$\mathcal{C}^2$ cub.
$\mathcal{C}^0$ lin.
$\mathcal{C}^1$ quad.
$\mathcal{C}^2$ cub.
$\theta = 360°$
$\theta = 720°$
$\theta = 795°$
$\theta = 907°$
$\theta = 963.3°$

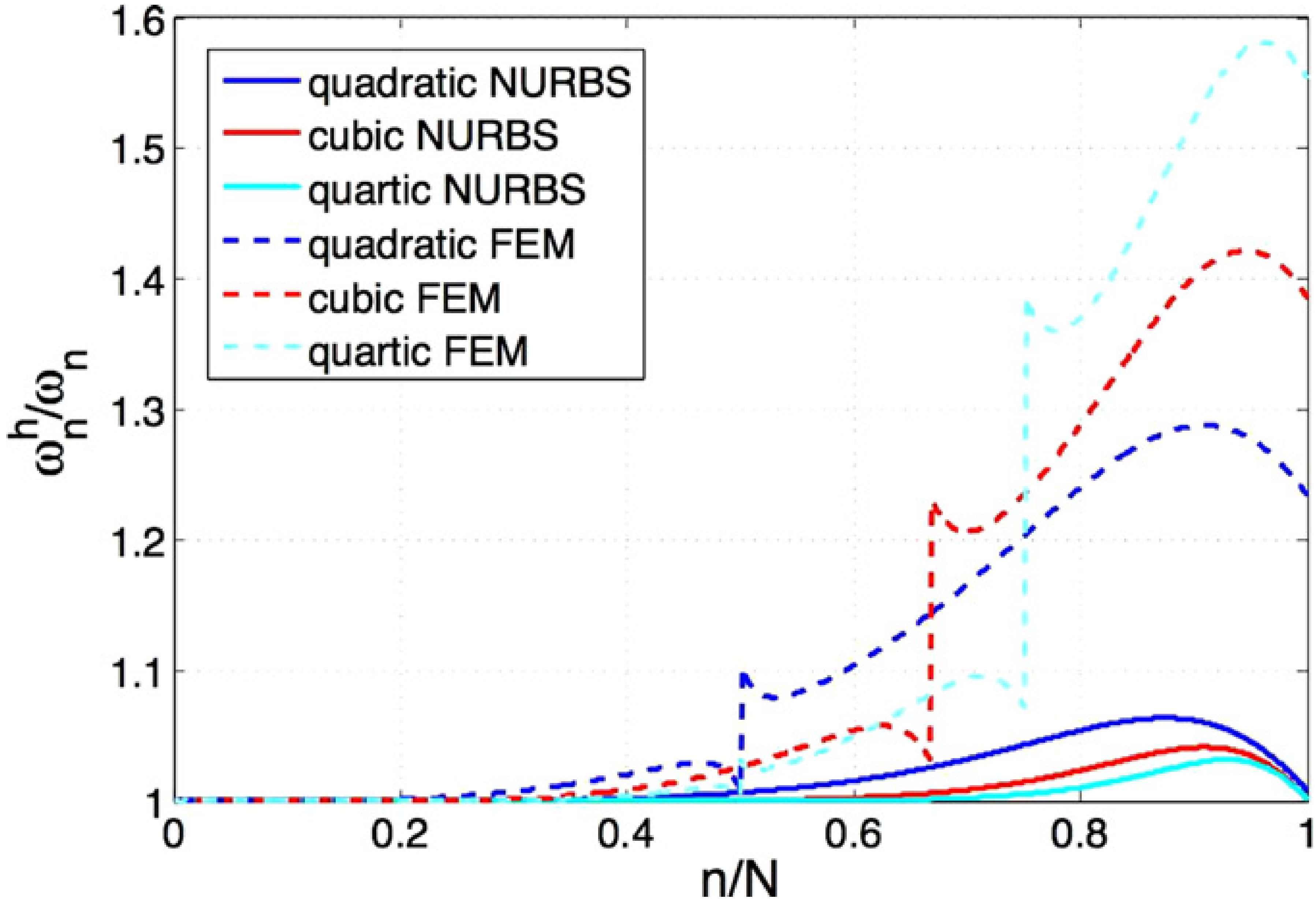
structural vibrations of an elastic fixed–fixed rod of unit length

$$u_{xx} + \varpi^2 u = 0, x \in]0,1[,$$

$$u(0) = u(1) = 0,$$

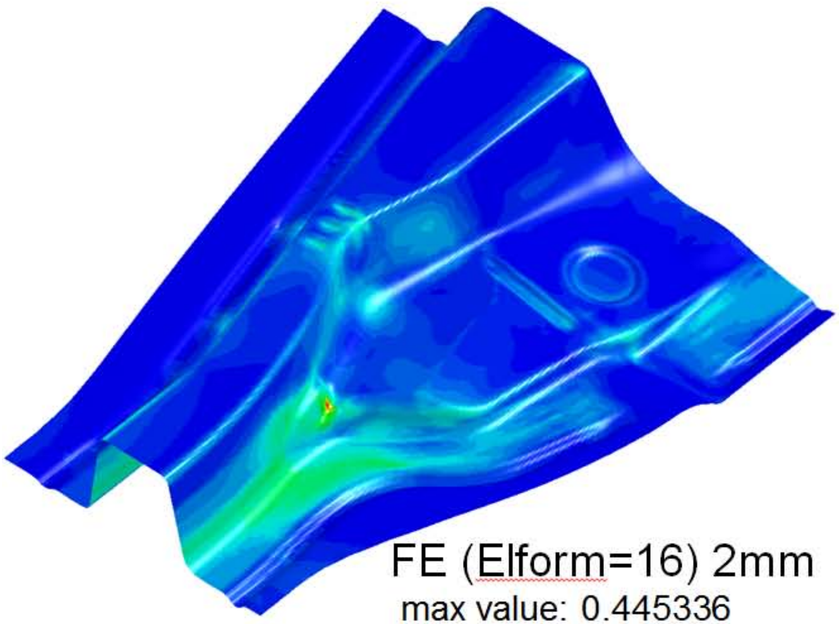
$$\textit{exact natural frequencies} \qquad \varpi_n = n\pi$$

J.A. Cottrell , T.J.R. Hughes, A. Reali, Studies of refinement and continuity in isogeometric structural analysis, CMAME 2007.

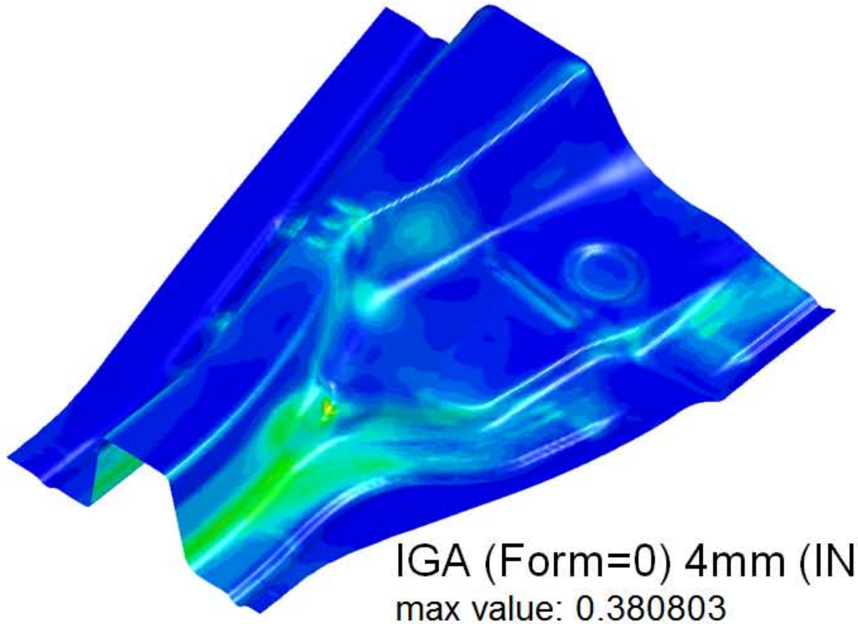


quadratic NURBS
cubic NURBS
quartic NURBS
quadratic FEM
cubic FEM
quartic FEM
ω^h_n/ω_n
n/N
1.6
1.5
1.4
1.3
1.2
1.1
1
0
0.2
0.4
0.6
0.8
1

Automobile crashworthiness



FE (Elform=16) 2mm
max value: 0.445336



IGA (Form=0) 4mm (INT=2)
max value: 0.380803

2011

smp (4 AMD 2.2 GHz)
no mass scaling
no thickness update

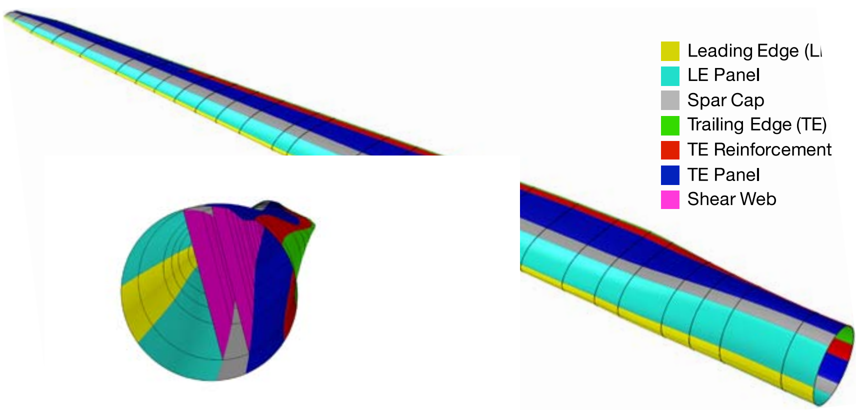
2016

mpp (4 Intel 2.67 GHz)
mass scaling (app. 10%)
with thickness update

CPU-time	(h:min)		CPU-time	(h:min)
- Std-2mm:	21:06	17.4 % →	- Std-2mm:	3:41
- P2-4mm:	14:31	15.1 % →	- P2-4mm_i0:	2:12 (59.7 %)
			- P2-4mm_i2:	1:34 (42.5 %)

NREL 5 MW Blade Material Design

Distinct material zones

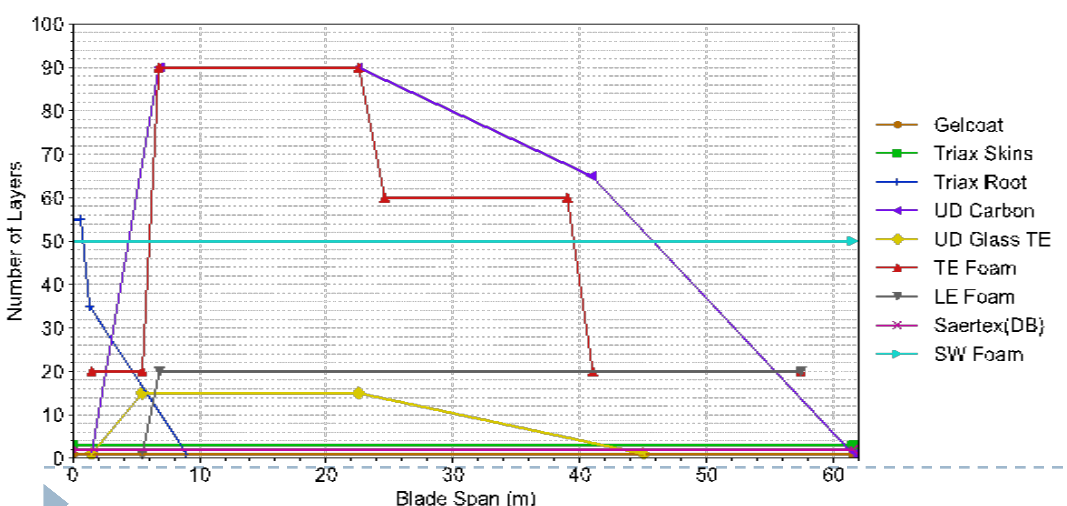


Layout of materials in each material zone

Material Zone	Stack Usage
Leading Edge (LE)	1, 2, 3, 2
LE Panel	1, 2, 3, 7, 2
Spar Cap	1, 2, 3, 4, 2
Trailing Edge (TE)	1, 2, 3, 2
TE Reinforcement	1, 2, 3, 5, 6, 2
TE Panel	1, 2, 3, 6, 2
Shear Web	8, 9, 8

Stack ID	Stack Name	Material
1	Gelcoat	Gelcoat
2	Triax Skins	SNL Triax
3	Triax Root	SNL Triax
4	UD Carbon	UD Carbon
5	UD Glass TE	E-LT-5500
6	TE Foam	Foam
7	LE Foam	Foam
8	Saertex	Saertex
9	SW Foam	Foam

Thickness distributions of materials



Classical laminated shell theory

$$\mathbf{K}_{\mathrm{exte}} = \int_{h_{\mathrm{th}}} \bar{\mathbf{C}}\, d\xi_3 = \sum_{k=1}^{n} \bar{\mathbf{C}}_k t_k$$

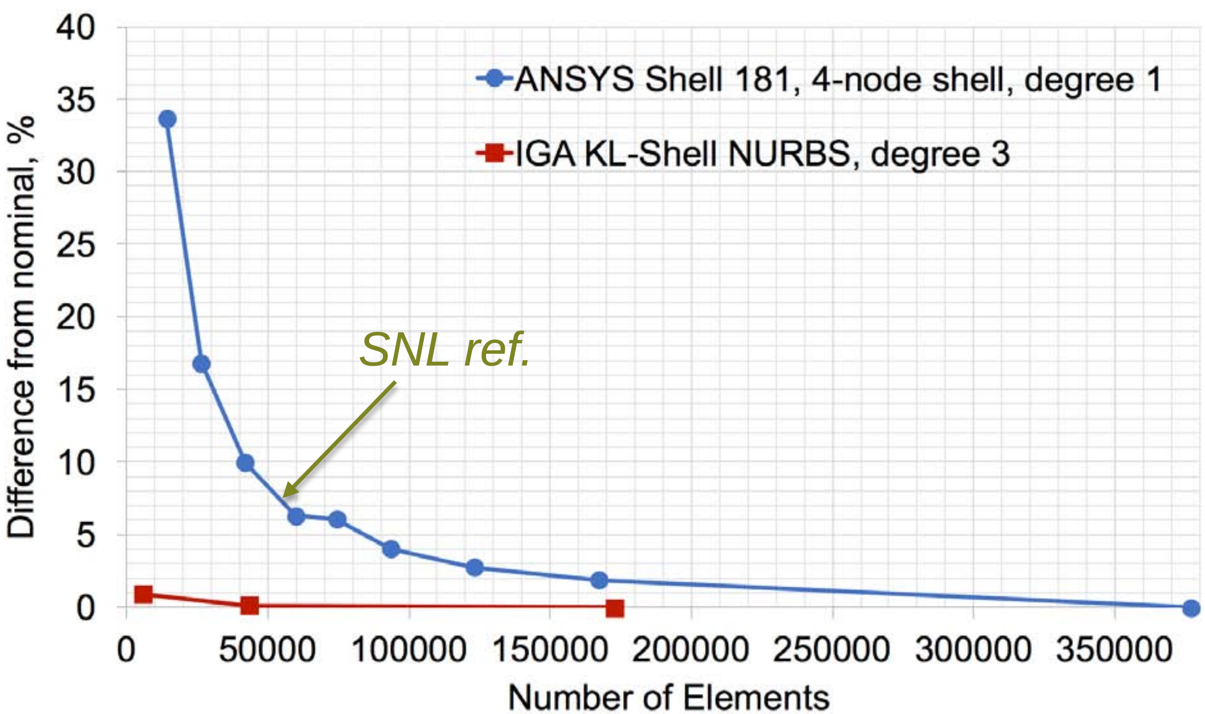
$$\mathbf{K}_{\mathrm{coup}} = \int_{h_{\mathrm{th}}} \xi_3 \bar{\mathbf{C}}\, d\xi_3 = \sum_{k=1}^{n} \bar{\mathbf{C}}_k t_k \bar{z}_k$$

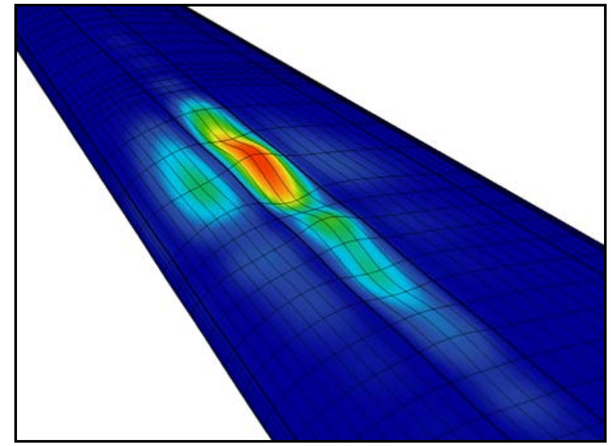
$$\mathbf{K}_{\mathrm{bend}} = \int_{h_{\mathrm{th}}} \xi_3^2 \bar{\mathbf{C}}\, d\xi_3 = \sum_{k=1}^{n} \bar{\mathbf{C}}_k \left(t_k \bar{z}_k^2 + \frac{t_k^3}{12} \right)$$

Credits: Austin Herrema, Ming-Chen Hsu

Mesh Convergence: IGA vs. FEM

- Model verification – mesh convergence
  - Linear buckling analysis
  - Investigation of the convergence of the 1st eigenvalue (buckling-inducing load multiplier)



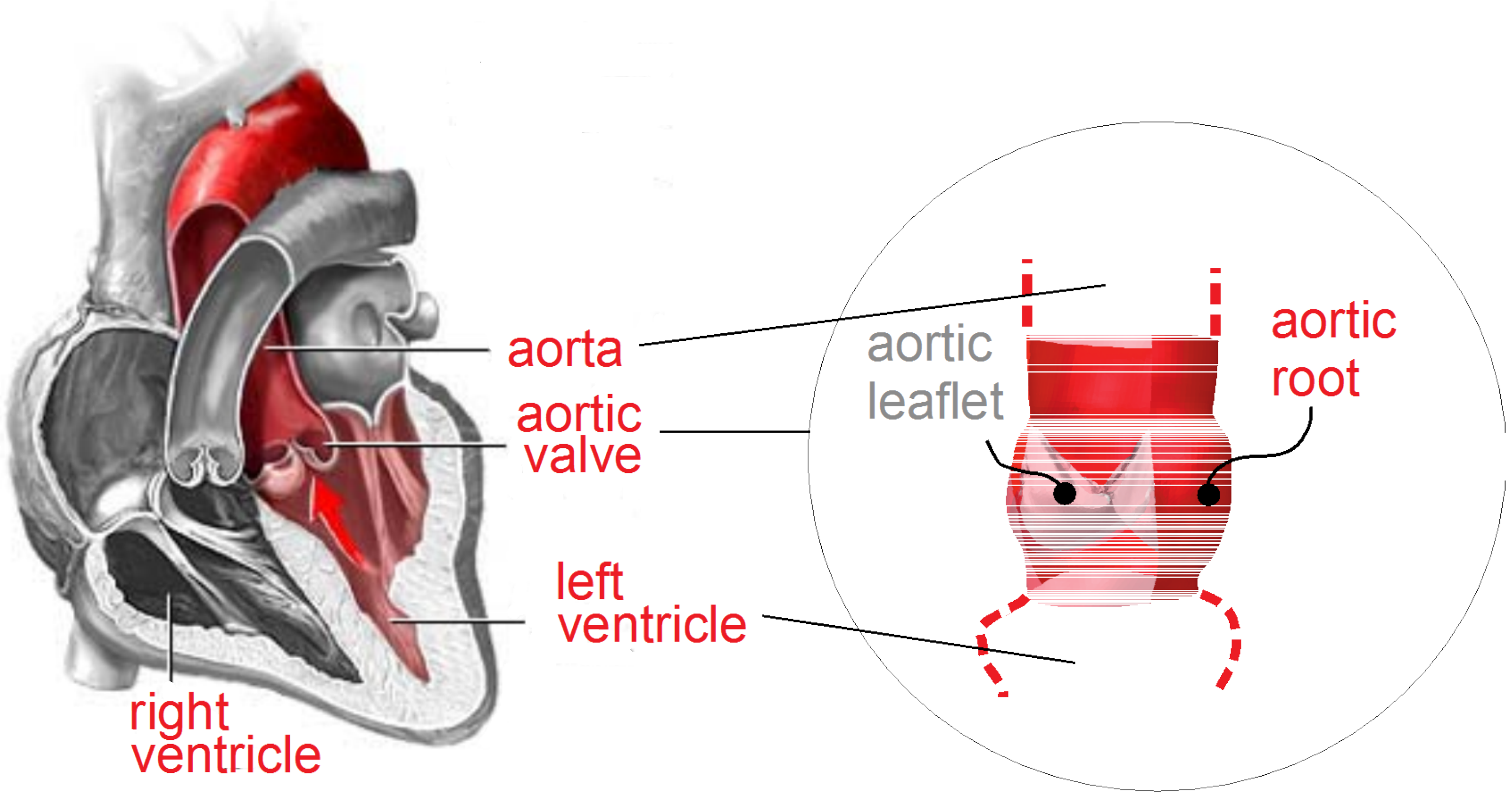


	Ref.	*IGA*
Buckling Load Factor	1.63	1.56
Root Bending Moment (kN-m)	22,740	23,723

Ref: Resor BR. Definition of a 5MW/61.5m wind turbine blade reference model. *Technical Report SAND2013-2569*, Sandia National Laboratories, Albuquerque, NM, 2013.

Credits: Austin Herrema, Ming-Chen Hsu

Aortic Valve closure



"Patient-specific isogeometric structural analysis of aortic valve closure," S. Morganti, F. Auricchio, D. Benson, F.I. Gambarin, S. Hartmann, T.J.R.H., A. Reali, *CMAME*, 2015.

Solution times for comparable accuracy

Analysis	# Nodes	# CPUs	Time step	# Increments	Total analysis time
IGA	762	12	2.30e-07	4347390	1h 15m
FEA	153646	12	2.65e-08	37787314	550h 23m

Why is IGA so much faster than traditional FEA?

1. Much more accurate per degree of freedom.

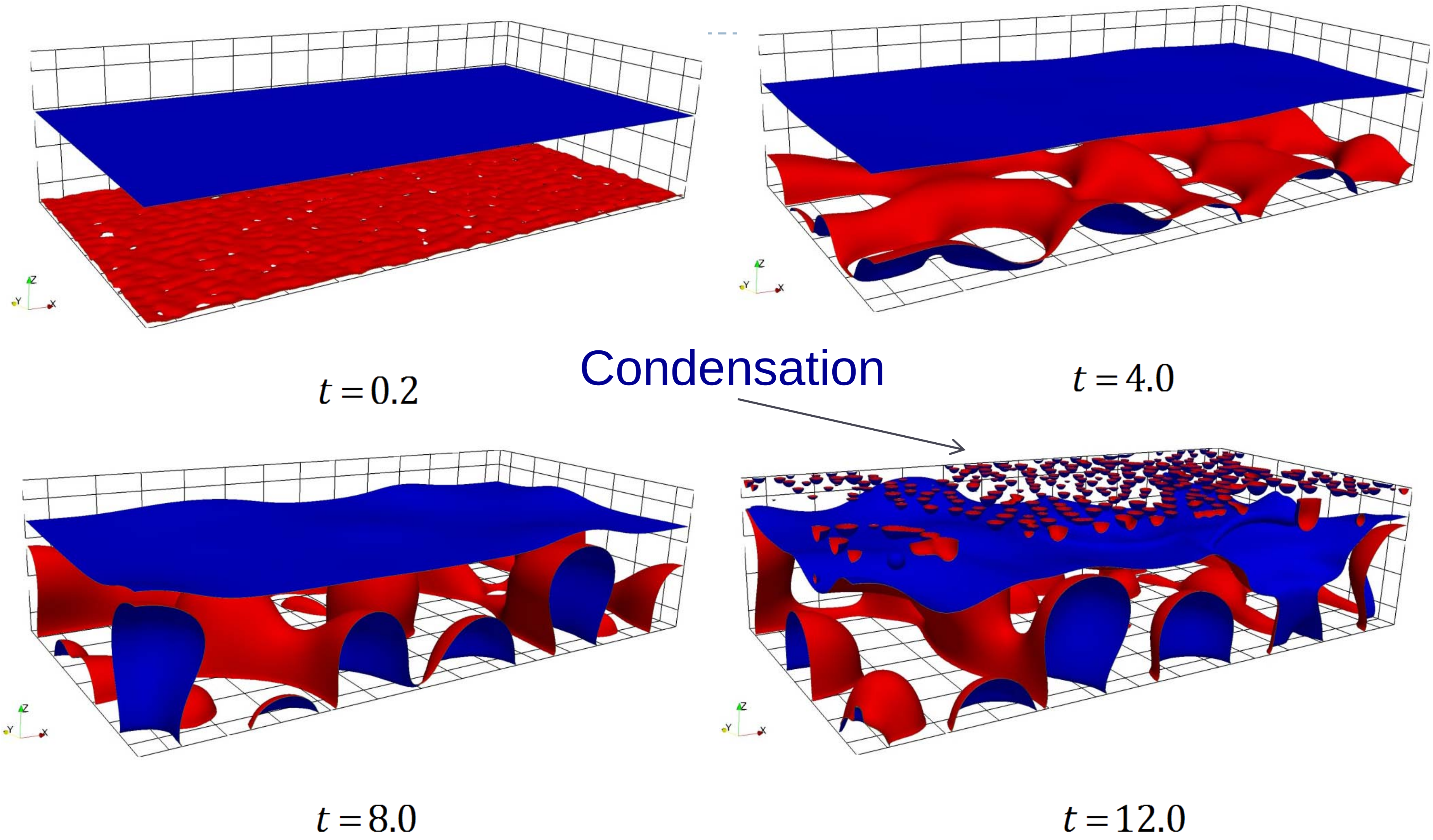
2. Efficient dynamics, e.g., large time steps.

3. Quality of contact surface provided by smooth geometry and smooth basis functions.

Navier-Stokes-Korteweg Equations

- Liquid-vapor two-phase flows
- Third-order spatial derivatives require smooth basis function

Three-dimensional Boiling



J. Liu, C.M. Landis, H. Gomez, and T.J.R. Hughes, "Liquid-Vapor Phase Transition: Thermomechanical Theory, Entropy Stable Numerical Formulation, and Boiling Simulations." *Computer Methods in Applied Mechanics and Engineering,* 297:476-553, 2015.

Matrix forming

Consider the mass matrix

$$\mathbb{M} = \{m_{i,j}\} = \left\{ \int_{[0,1]^d} c(\zeta)\, \hat{B}_i(\zeta)\, \hat{B}_j(\zeta)\, d\zeta \right\}$$

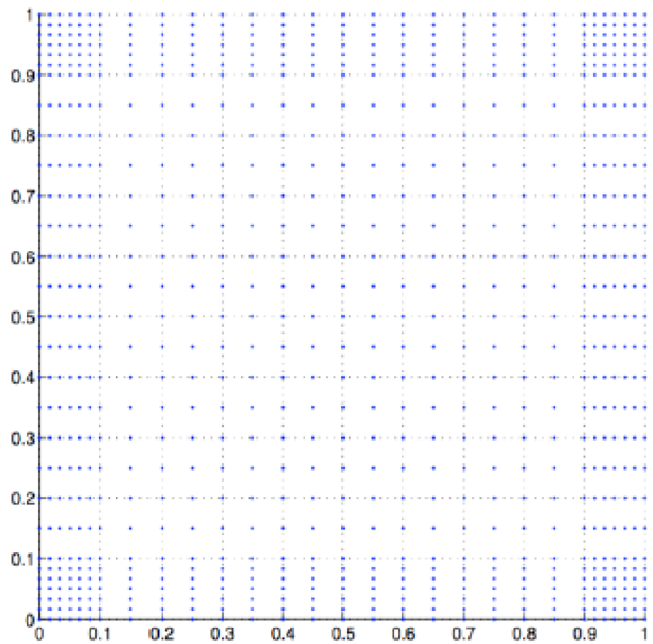
Gauss Quadrature (GQ) (or Generalized GQ)

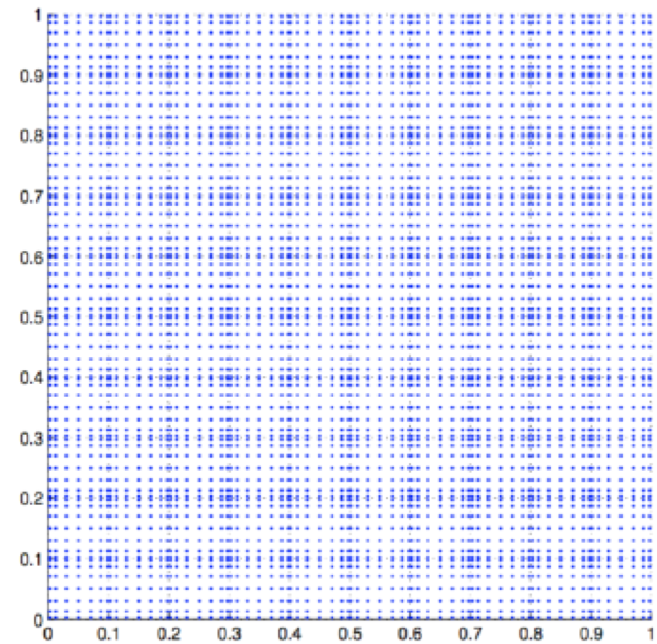
$$m_{i,j} \approx \mathfrak{Q}^{\mathrm{GQ}}(c\hat{B}_i\hat{B}_j) = \sum_{\boldsymbol{q}} w_{\boldsymbol{q}}^{\mathrm{GQ}} c(\boldsymbol{x}_{\boldsymbol{q}}^{\mathrm{GQ}}) \hat{B}_i(\boldsymbol{x}_{\boldsymbol{q}}^{\mathrm{GQ}}) \hat{B}_j(\boldsymbol{x}_{\boldsymbol{q}}^{\mathrm{GQ}})$$

weighted quadrature (WQ)

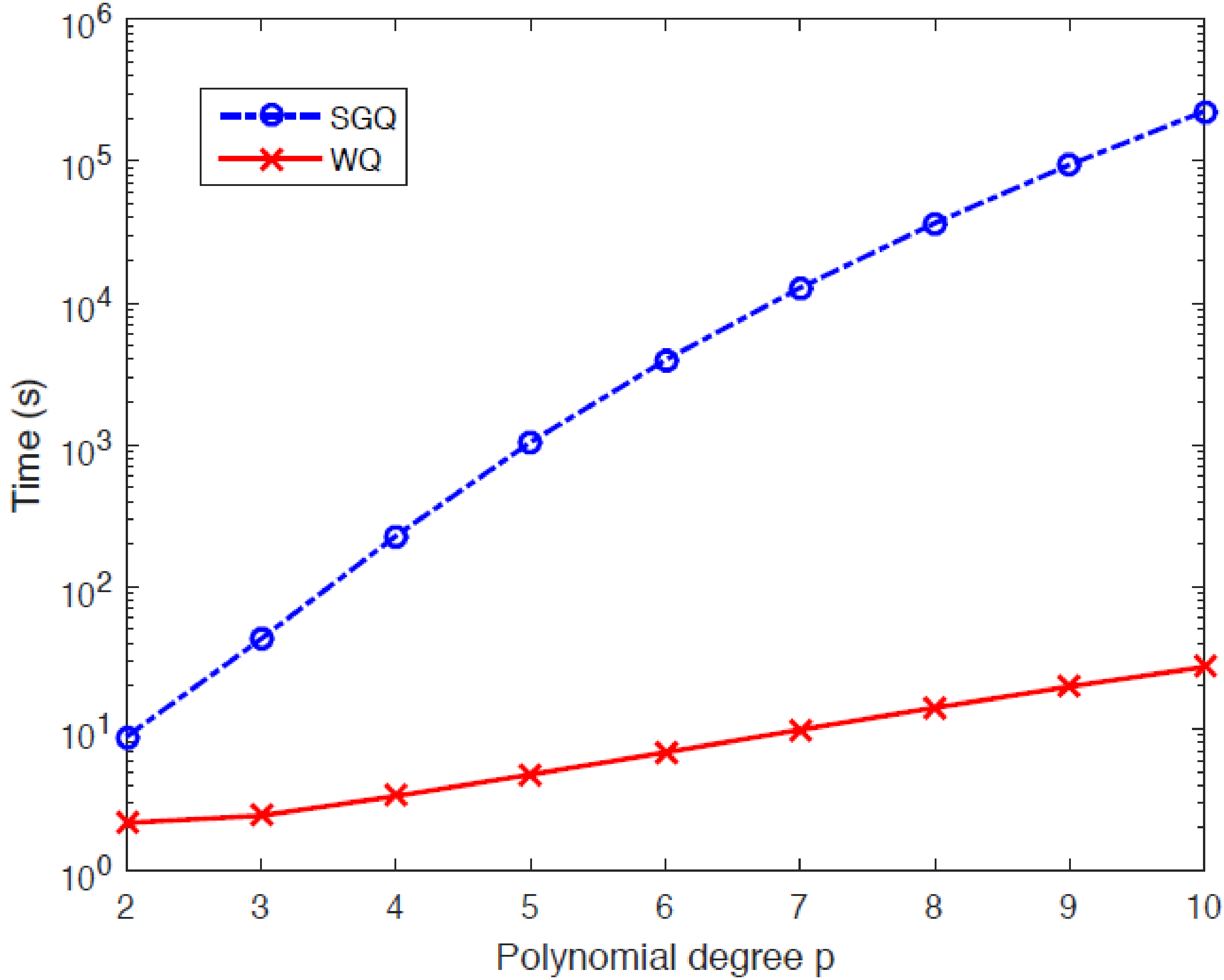
$$m_{i,j} = \int_{\hat{\Omega}} c(\zeta)\, \hat{B}_j(\zeta)\, (\hat{B}_i(\zeta) d\zeta) \approx \mathfrak{Q}_i^{\mathrm{WQ}}(c\hat{B}_j) = \sum_{\boldsymbol{q}} w_{\boldsymbol{q},i}^{\mathrm{WQ}} c(\boldsymbol{x}_{\boldsymbol{q}}^{\mathrm{WQ}}) \hat{B}_j(\boldsymbol{x}_{\boldsymbol{q}}^{\mathrm{WQ}})$$

F. Calabro, G. Sangalli, M. Tani, Fast formation of isogeometric Galerkin matrices by weighted quadrature, CMAME 2017.





$$\text{cost for WQ} = O(N_{\mathsf{DOF}}p^{d+1}) \qquad \text{cost for SGQ} = O(N_{\mathsf{DOF}}p^{3d})$$



SGQ
WQ
Time (s)
Polynomial degree p
10^0
10^1
10^2
10^3
10^4
10^5
10^6
2
3
4
5
6
7
8
9
10

Variational Collocation

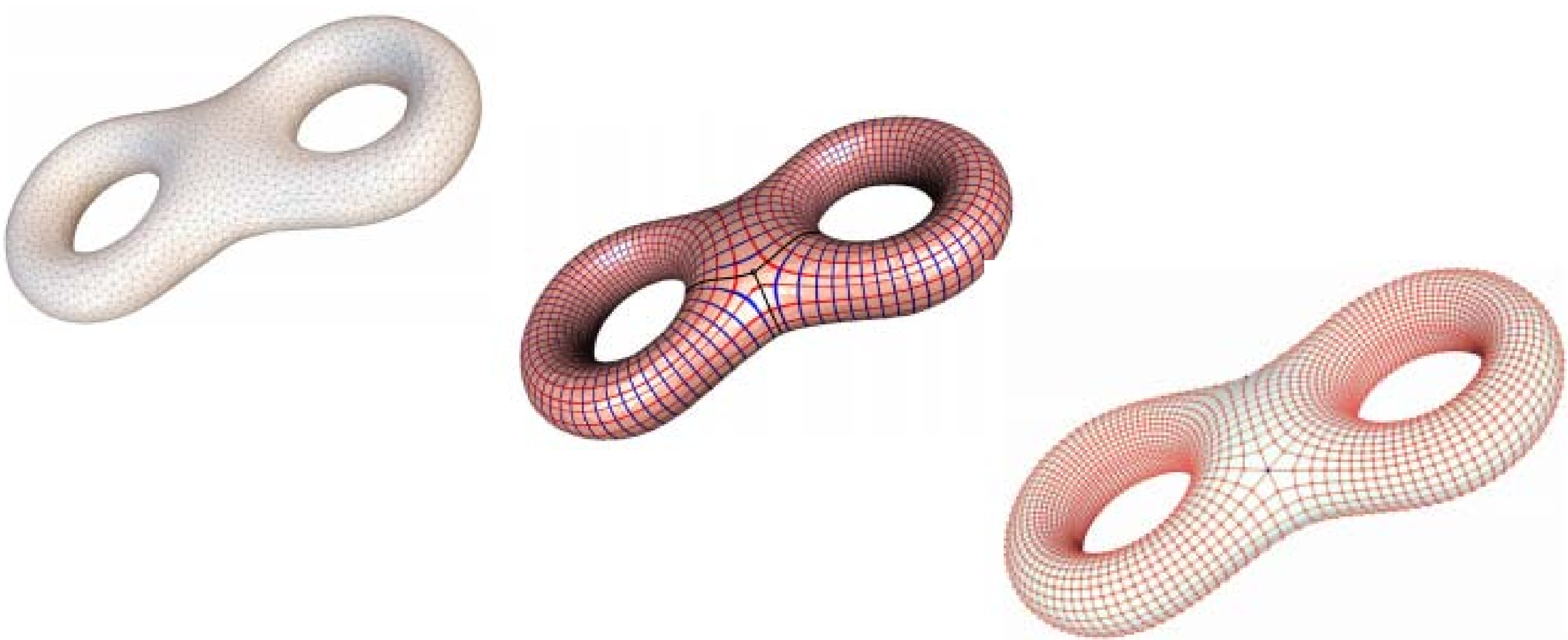
$$u_h = \sum_{i=1}^{n} q_i \phi_i$$

$$Lu_h(\alpha_j) = f(\alpha_j),\ j = 1,\dots,n,$$

- IGA collocation, the rate of convergence is different for odd and even degrees;
- For NURBS, there exist collocation sites that produce the Galerkin solution exactly;

Hector Gomez, Laura De Lorenzis, The variational collocation method, CMAME 2017.

3D parameterization

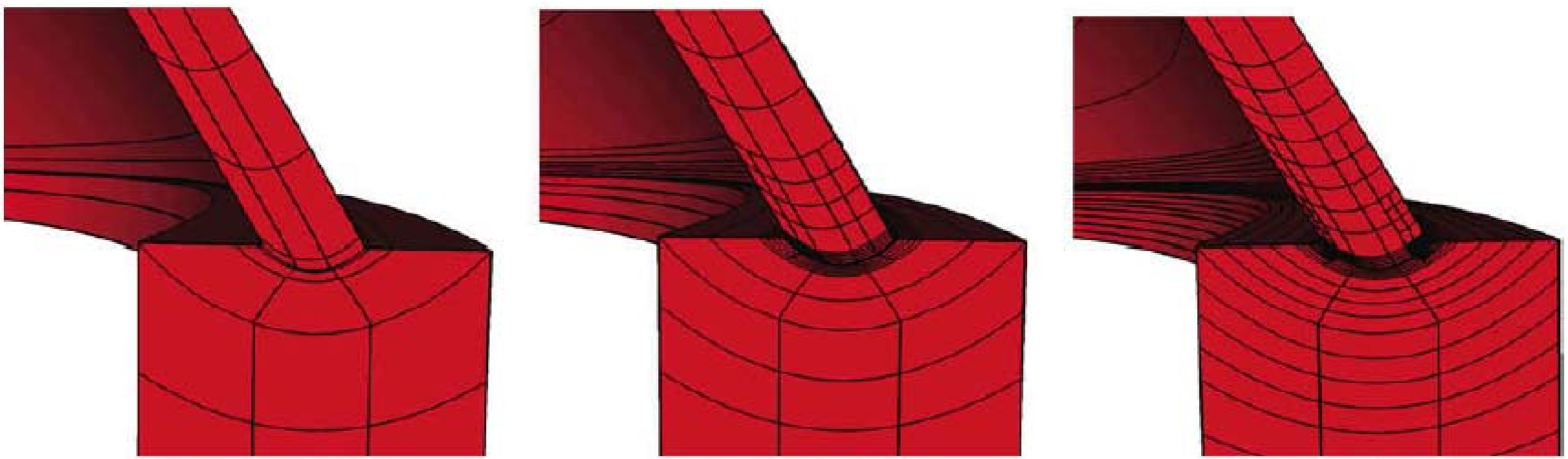


Na Lei, Xiaopeng Zheng, Jian Jiang, Yu-Yao Lin, David Xianfeng Gu, Quadrilateral and Hexahedral Mesh Generation Based on Surface Foliation Theory, CMAME 2017.

T-splines-based IGA

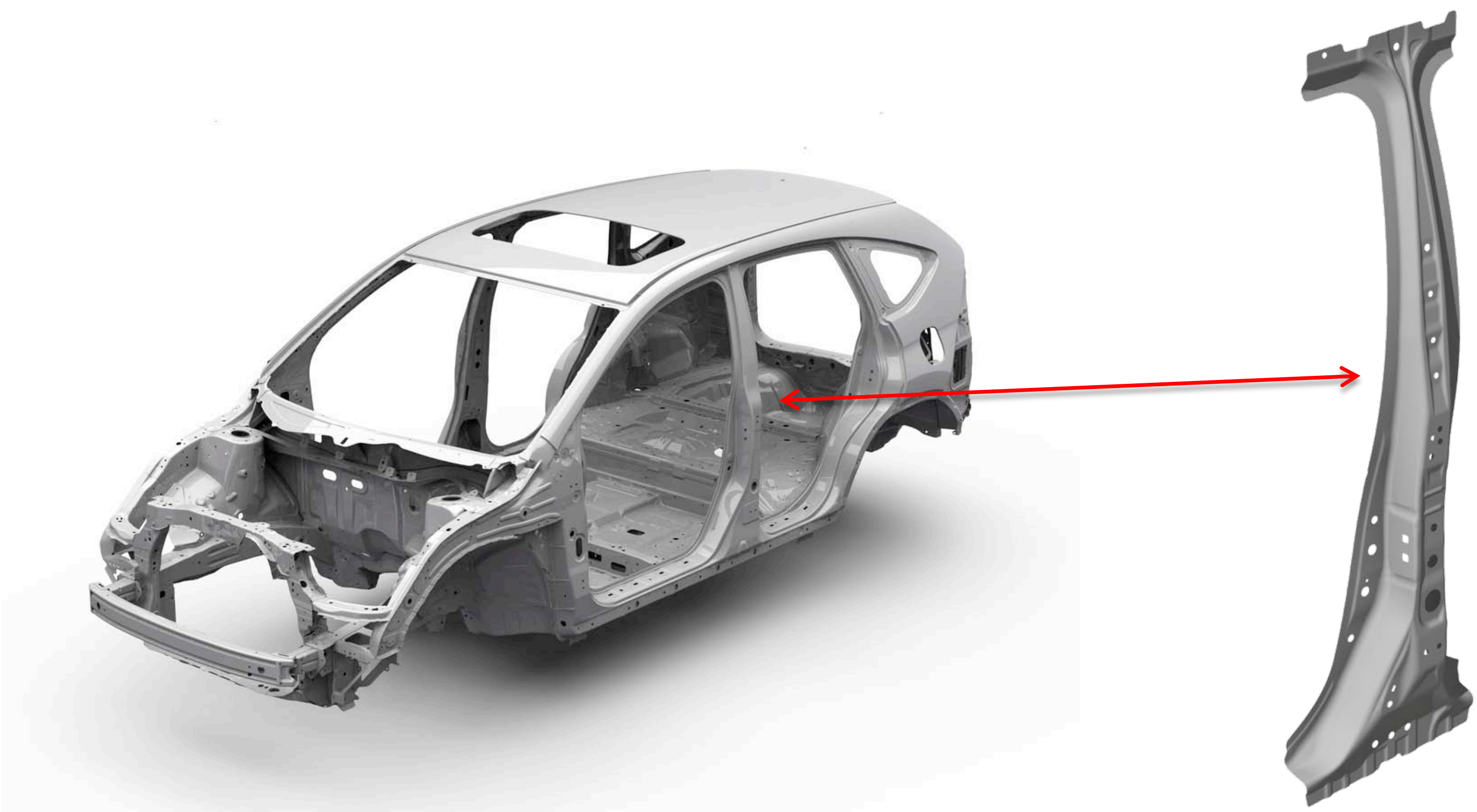
From design to analysis

Why T-splines



NDOF = 972 NDOF = 2340 NDOF = 4248

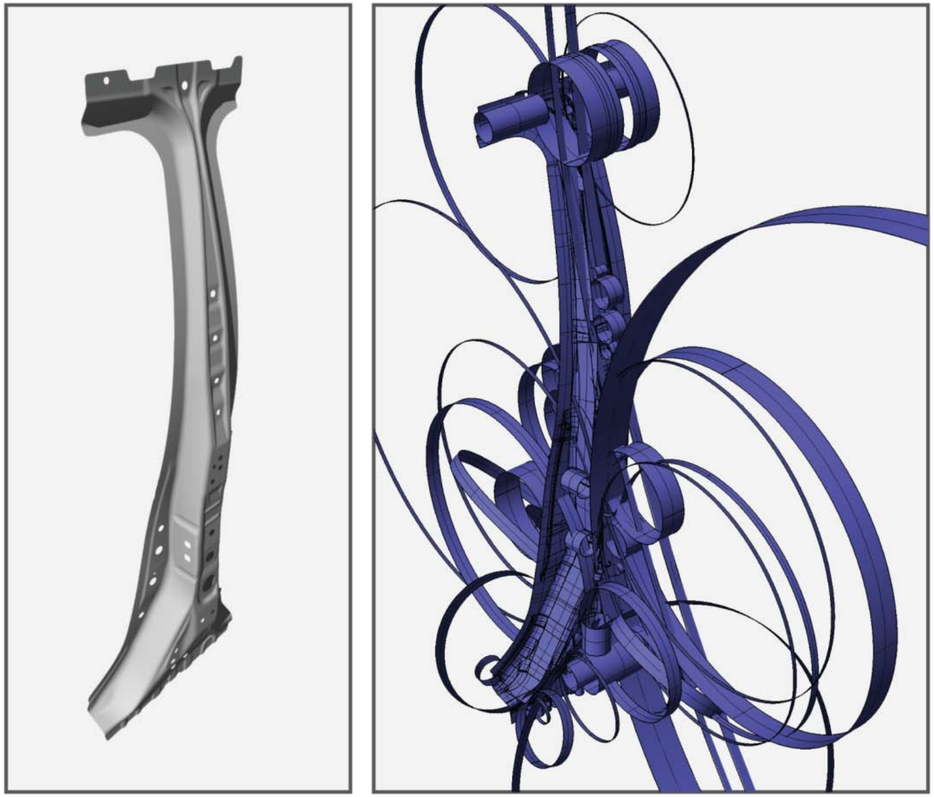
Why is meshing so time consuming?



Honda B-pillar

Why is it so time consuming?

Graphics

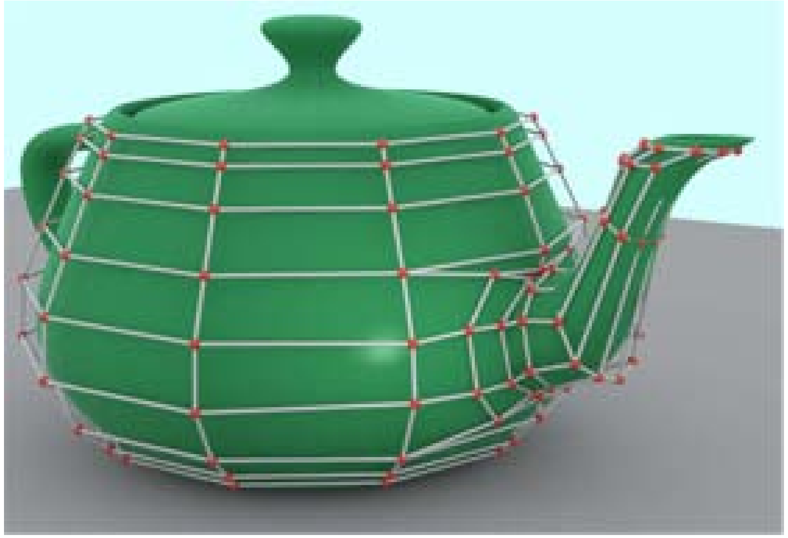


Data structure

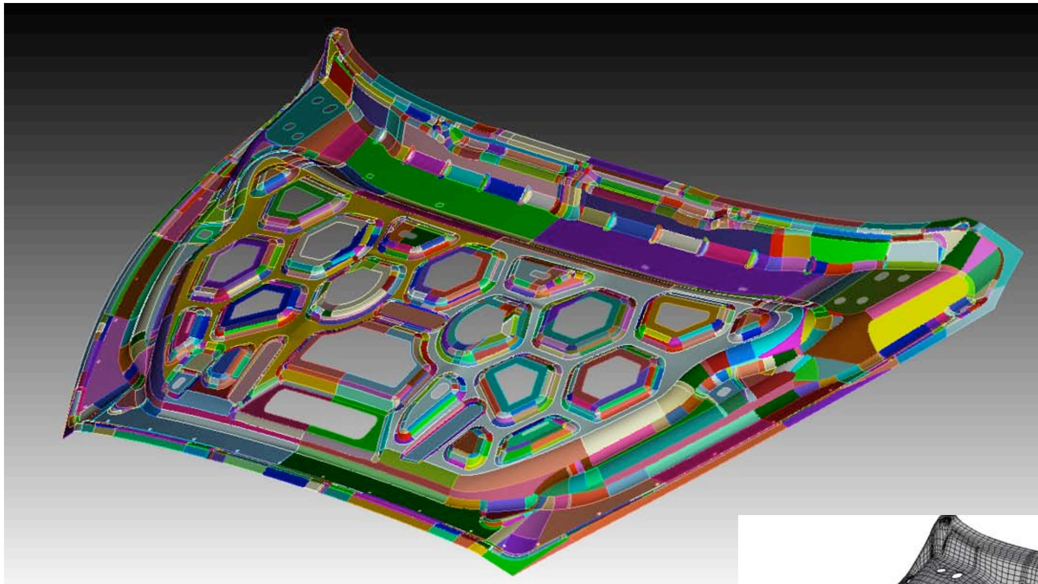
1280 trimmed surfaces, that are **not** watertight

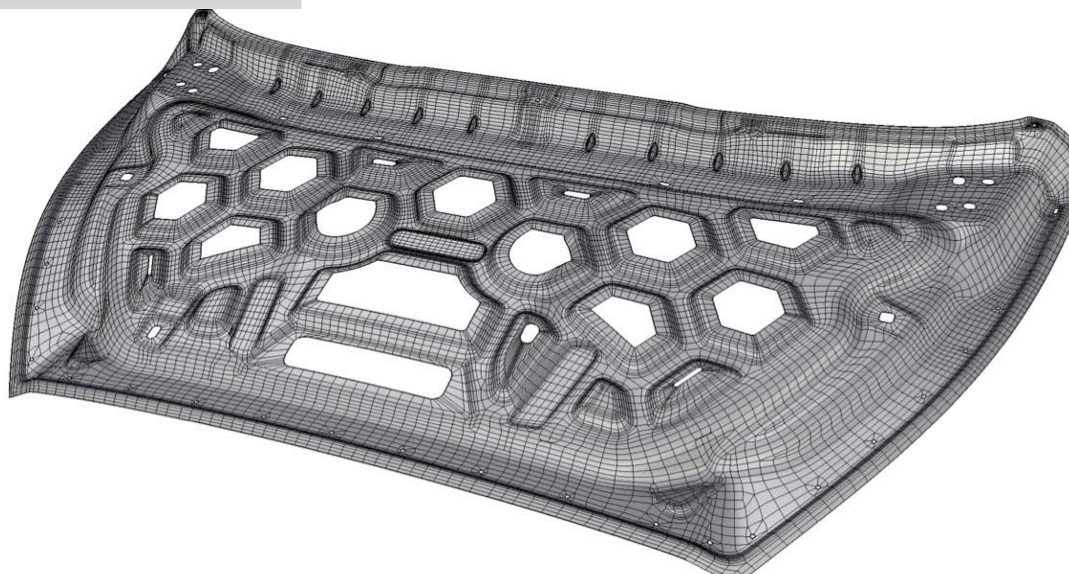
T-spline-based technology

- Watertight;
- Editable;
- Isogeometric;
- NURBS compatible;



T.W. Sederberg, Finnigan, Xin Li, etal , Watertight trimmed NURBS, Siggraph 2008.





T-splines

The T-Spline technology addresses some important limitations that are inherent in conventional NURBS surfaces. T-Splines are based on solid mathematical principles. An important practical consideration is that T-Splines are forward and backward compatible with NURBS.

--Dr. Rich Riesenfeld, Founder of B-splines in CAD

The technology acquisition will strengthen our Digital Prototyping portfolio with more flexible free-form modeling and will help achieve even closer integration between industrial design and engineering workflows.

-- Buzz Kross, senior vice president of Autodesk

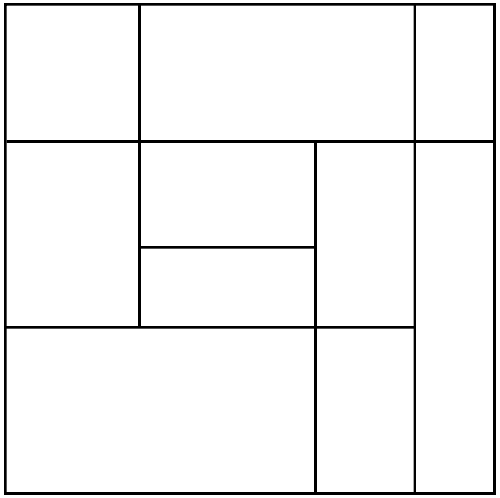
Trimming technology for IGA

- Other T-spline technology can also handle the trimming problem;
  - PHT, HB, LR splines;
- Immersed boundary method for IGA
  - Cannot directly contact with CAD
- B++ spline method

Locally refinable splines

- Polynomial spline spaces over T-meshes;
- Hierarchical B-splines;
- T-splines;
- LR B-splines;

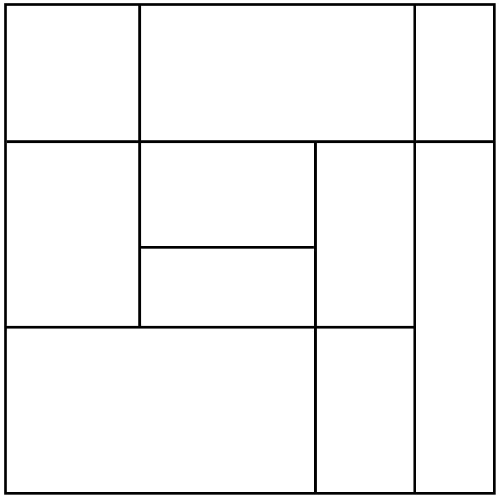
Polynomial spline spaces over T-meshes



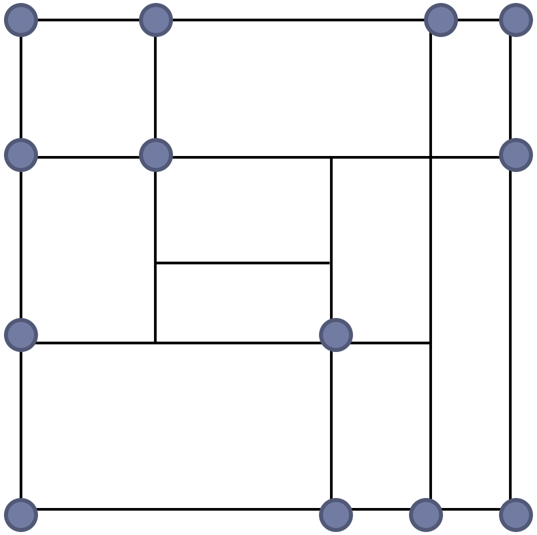
$$S(m,n,\alpha,\beta,T)=\{f(s,t)\in C^{\alpha,\beta}(\Omega): s|_{\phi}\in P^{m,n}\}$$

$\mathrm{T}:\mathrm{t-mesh}, \phi:\text{a face}, C^{\alpha,\beta}(\Omega):\text{continuous with}$

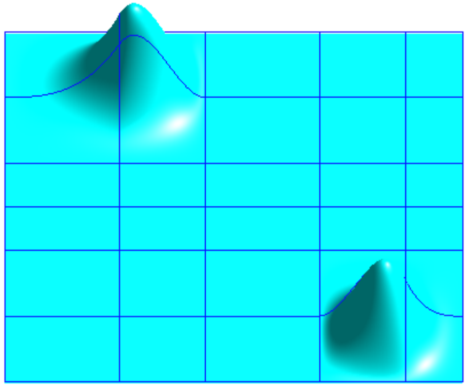
order α and β in s and t directions



Given a T-mesh in the parametric domain



Compute the dimension of the space



Construct the basis functions

图 3.11: 拟合igea的开模型

Application

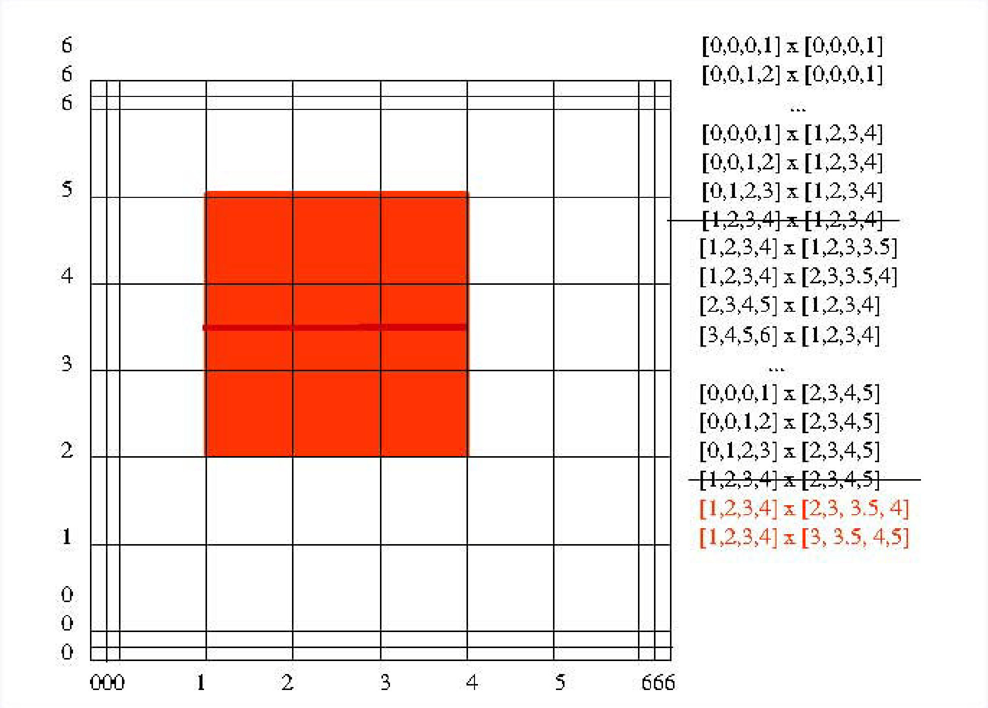
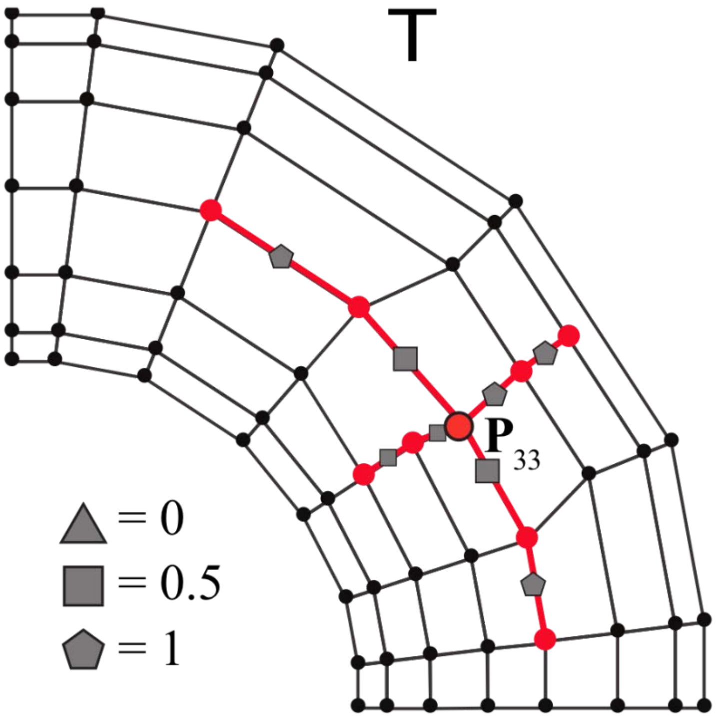
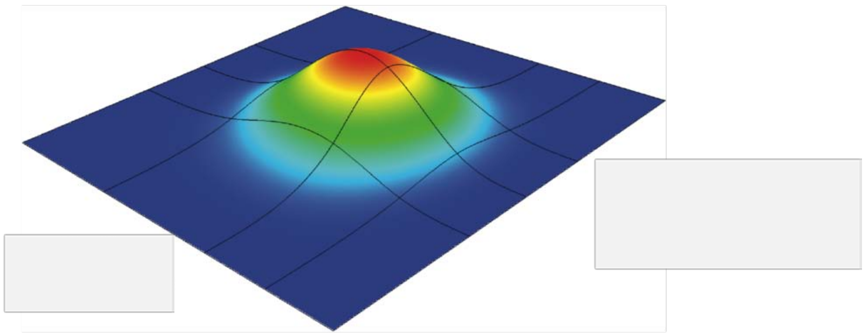


Figure: Locally Refined splines

T-splines





T-splines： $T(\xi) = \sum_{i=1}^{n} C_i N_i(\xi)$

Two approaches for local refinement splines

- space-based:
  - PHT
  - Difficult to compute the dimension and to construct the basis
- Blending functions-based
  - T-splines;
  - HB-splines;
  - LR B-splines;
  - Difficult to prove any mathematical properties;

Requirement for T-splines in IGA

Find a class of T-meshes such that

- Linear Independent:

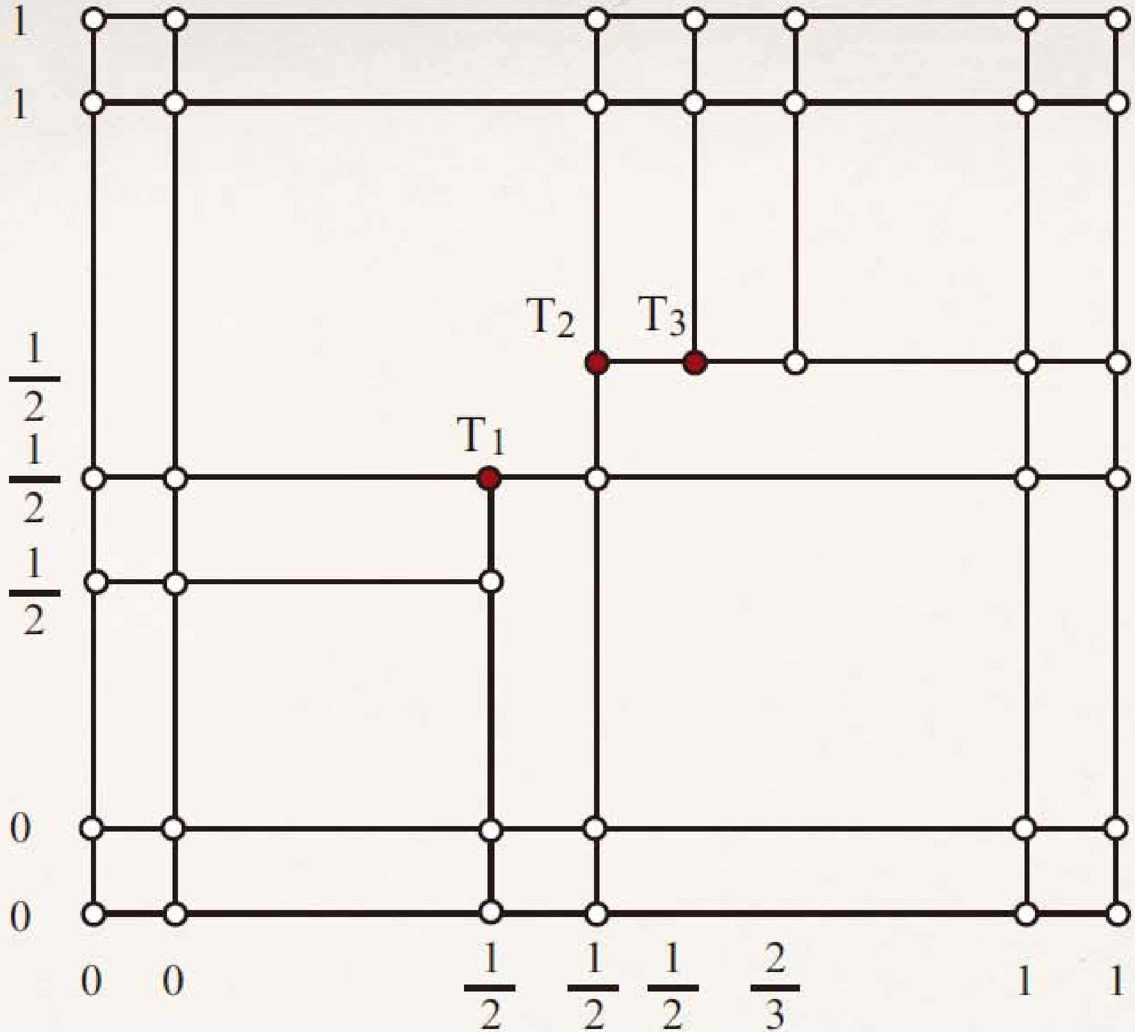
$$\sum_{i=1}^{n} C_i T_i(\xi) = 0 \longrightarrow C_i = 0$$

- Nesting: Find the condition for T^1 and T^2 such that

$$span\{T_i^1(\xi)\} \subset span\{T_i^2(\xi)\}$$

- Completeness: Given a piecewise polynomial function $f(\xi)$, check

$$f(\xi) \in span\{T_i(\xi)\}$$

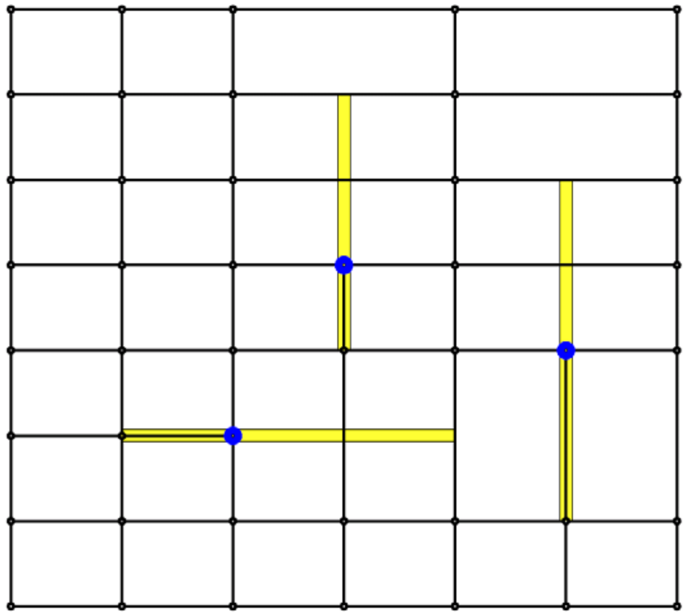


Linear independence of the T-spline blending functions associated with some particular T-meshes

A. Buffa, D. Cho, G. Sangalli

Computer Methods in Applied Mechanics and Engineering Volume 199, Issues 23-24, 15 April 2010, Pages 1437-1445

Definition



- Xin Li, T.W.Sederberg, Jianmin Zheng, T. Hughes, M. A. Scott. On linear independency of Tsplines blending functions, CAGD, 29(1): 63-76, 2012.
- M. A. Scott, Xin Li, T.W.Sederberg, T, Hughes. Local refinement of Analysis-suitable T-splines, CMAME, Volumes 213-216, 1-3,206-222, 2012.
- Xin Li, M.A. Scott. Analysis-suitable T-splines: characterization, refineability, and approximation, M3AS,Vol. 24, No. 06, 1141-1164 , 2014.

T-splines-based IGA

- Backwards compatible with NURBS
- Used in design
- Higher-order smoothness
- Partition of unity
- Locally refinement
- Watertight

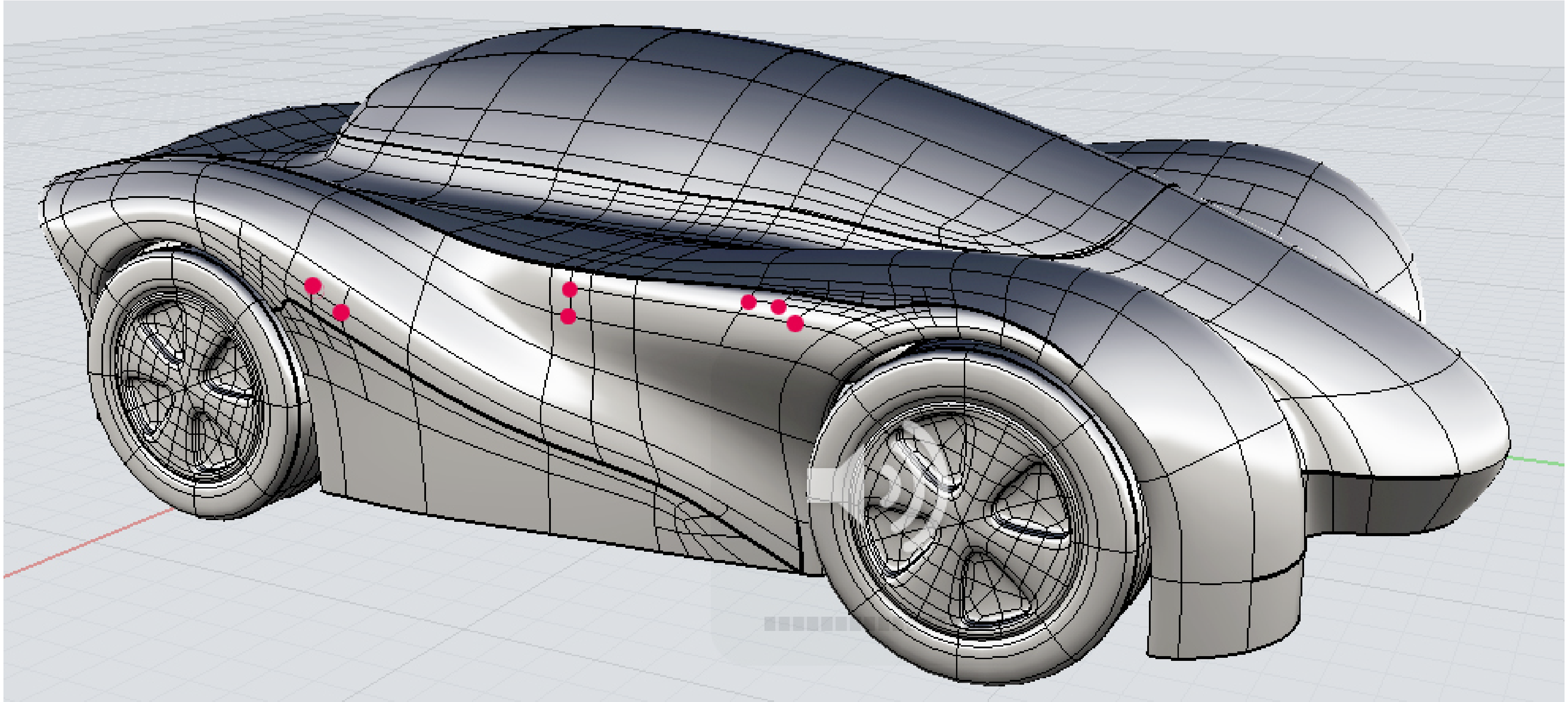
- Simply implemented in FEA codes
- Linearly independent
- Affine covariance
- Watertight
- Locally refinement
- Optimal convergence

Analysis-suitable T-splines

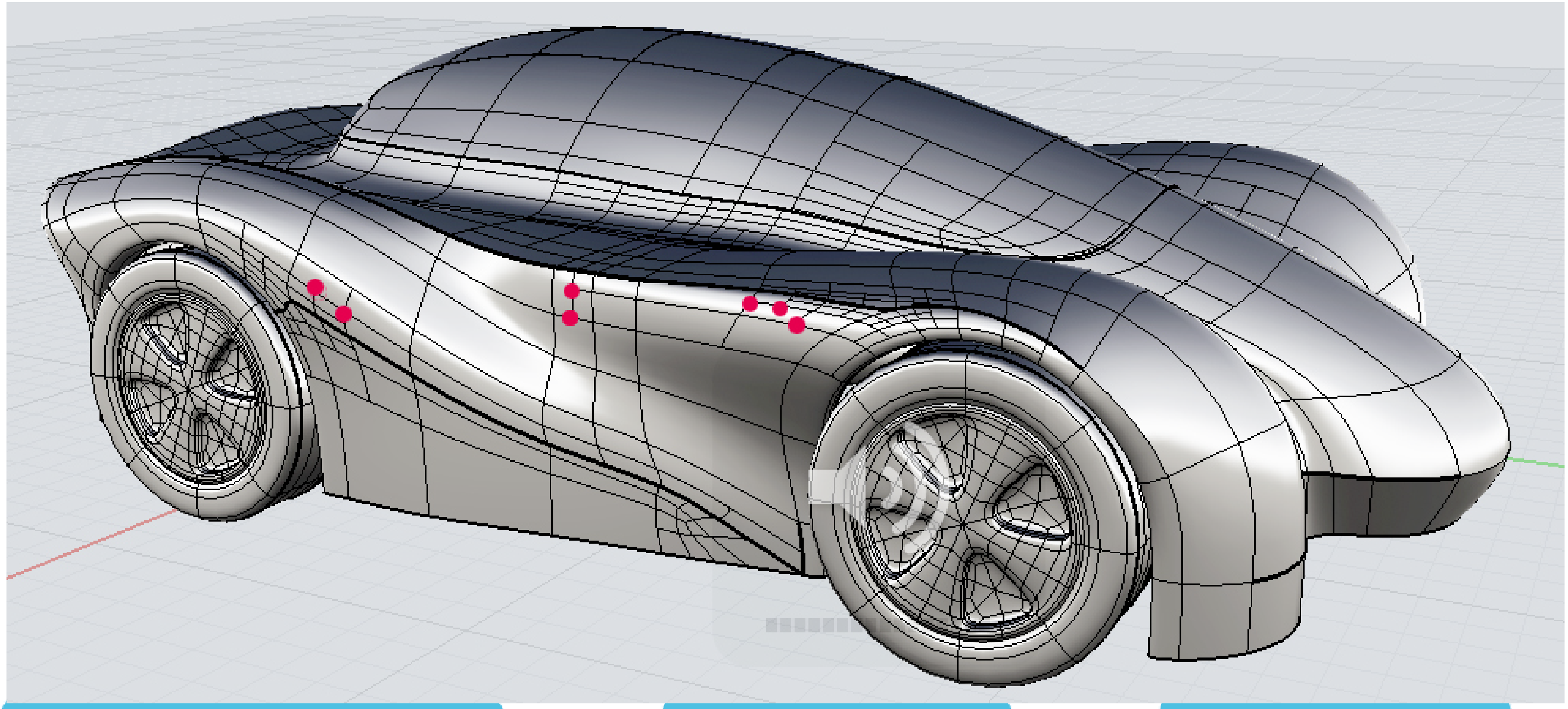
AS T-splines

- Academic Development
  - Prof. A. Buffa：Dual Basis
  - Prof. D. Zorin：T-spline subdivision
  - Prof. G. Zavarise：Local refinement
  - Prof. D. Cho：Non-polynomial T-splines
- Applications: crack, fracture, wave, electromagnetics
  - Hughes，
  - T. Rabczuk，
  - A.Buffa，
  - Y. Bazilevs，

Not AS T-splines



Not AS T-splines



Watertight T-spline models from CAD software → Convert to AS T-splines → Perform IGA analysis

AS++ T-splines

Definition

A T-mesh is called an analysis-suitable++ T-mesh (for short, AS++ T-mesh) if and only if:

1. For any two T-junctions $\mathbf{T}_i$, $\mathbf{T}_j$ which are not parallel, denote $V = ext_2^f(\mathbf{T}_i) \cap ext_2^f(\mathbf{T}_j)$, then either $ext_2^f(\mathbf{T}_i) \cap ext_2^f(\mathbf{T}_j) = \varnothing$ (no V exists) or for any $\mathbf{V}_i$, $V \notin VK(\mathbf{V}_i)$;
2. $\mathrm{T}_{ext} = \mathrm{T}_{elem}$.

An AS++ T-spline is a T-spline defined on an AS++ T-mesh.

- Xin Li, Jingjing Zhang. AS++ T-splines: linear independence and approximation, CMAME, 2018.
- Jingjing Zhang, Xin Li. Local refinement of Analysis-suitable++ T-splines, CMAME.
- Xin Li, Characterization of Analysis-suitable++ T-splines, Prepared to submite to M3AS.

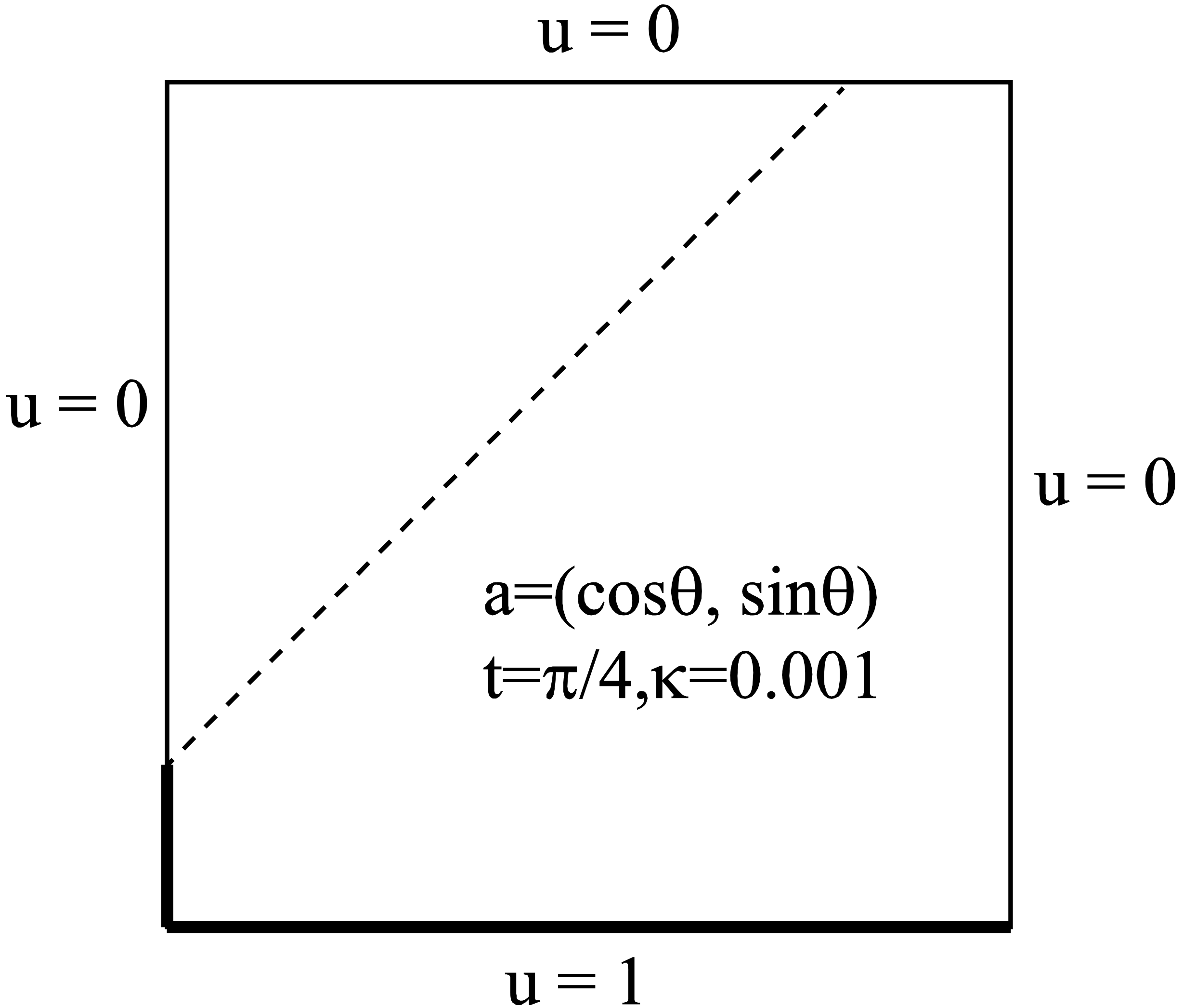
Property

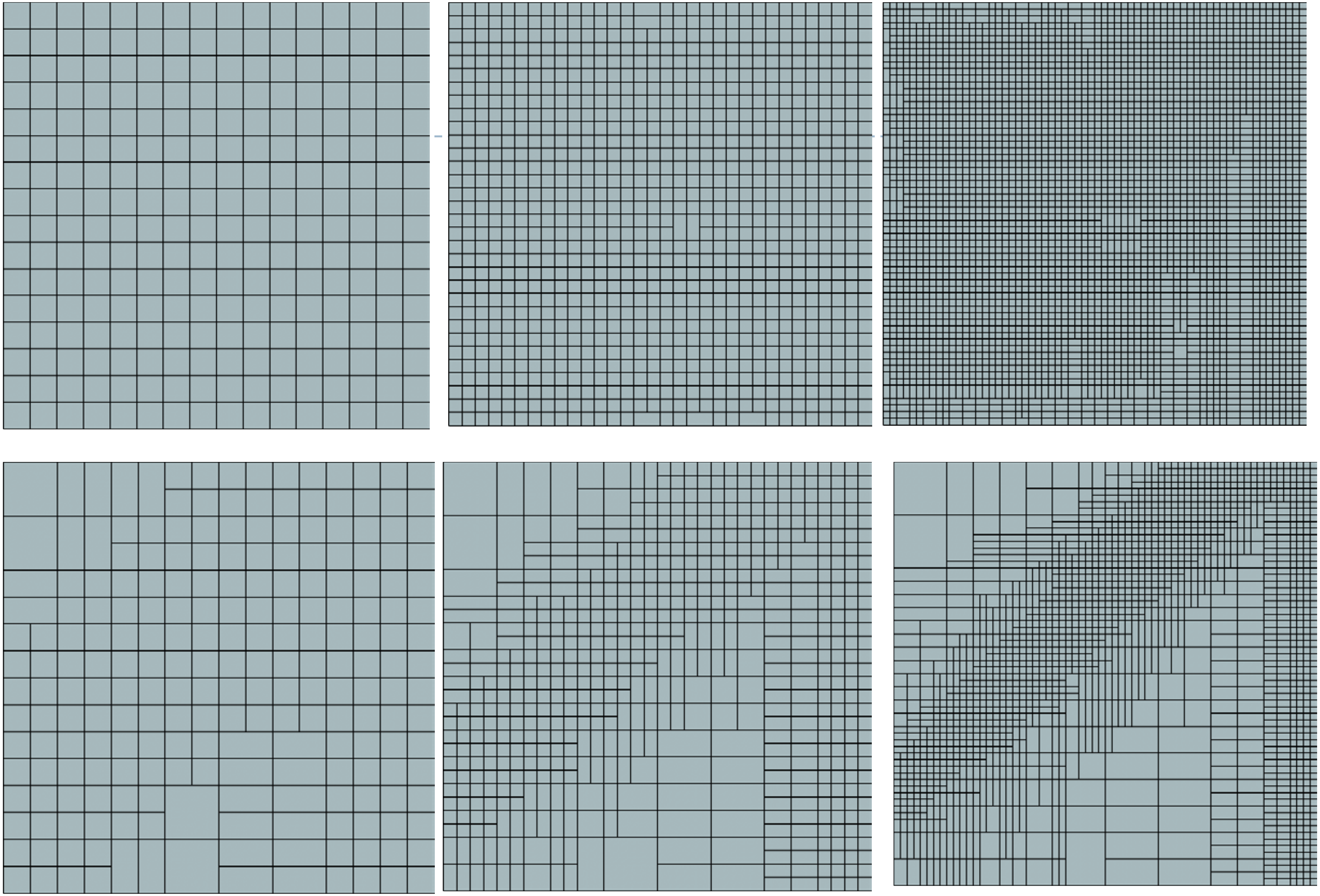
Theorem

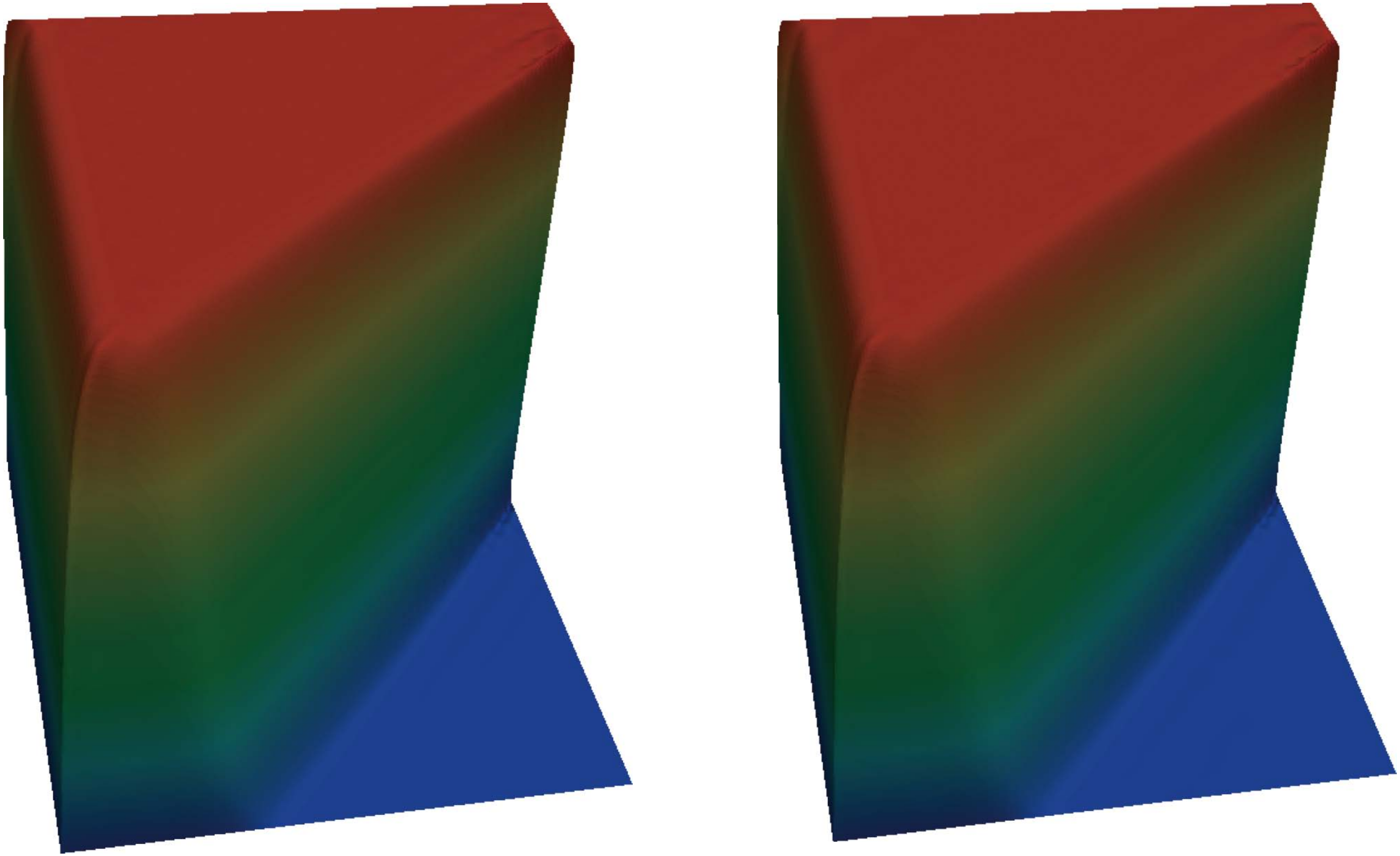
The T-splines producing by insertion control points from any AS++ T-splines using any existing local refinement algorithm are still AS++ T-splines.

- Create the basic shape:
  - From primitives or lines:
  - From a set of curves: lofting or skinning;
  - From surfaces: NURBS or meshes;
- Editing tools:
  - Local refinement;
  - Extrusion (Adding star points and local refinement);
  - Merging (local refinement and adding star points);
  - etal.
- In the editing part, the T-junctions are introduced by local refinement algorithm

advection-diffusion problem





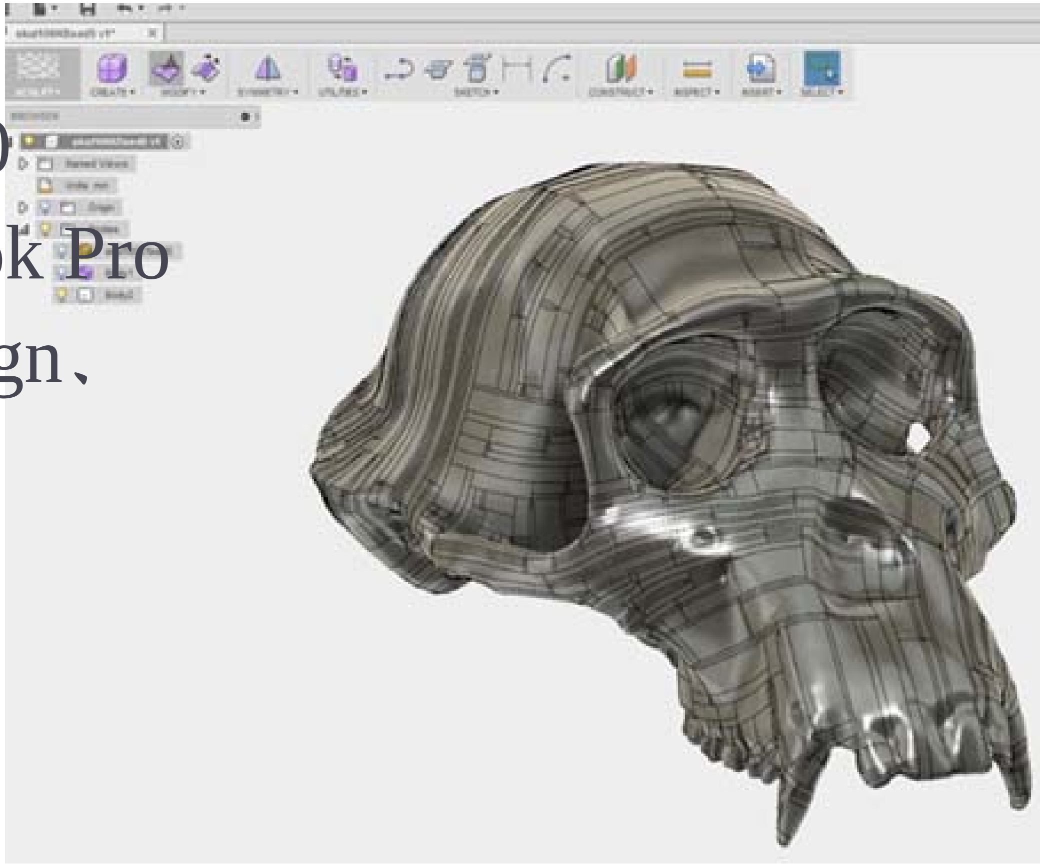


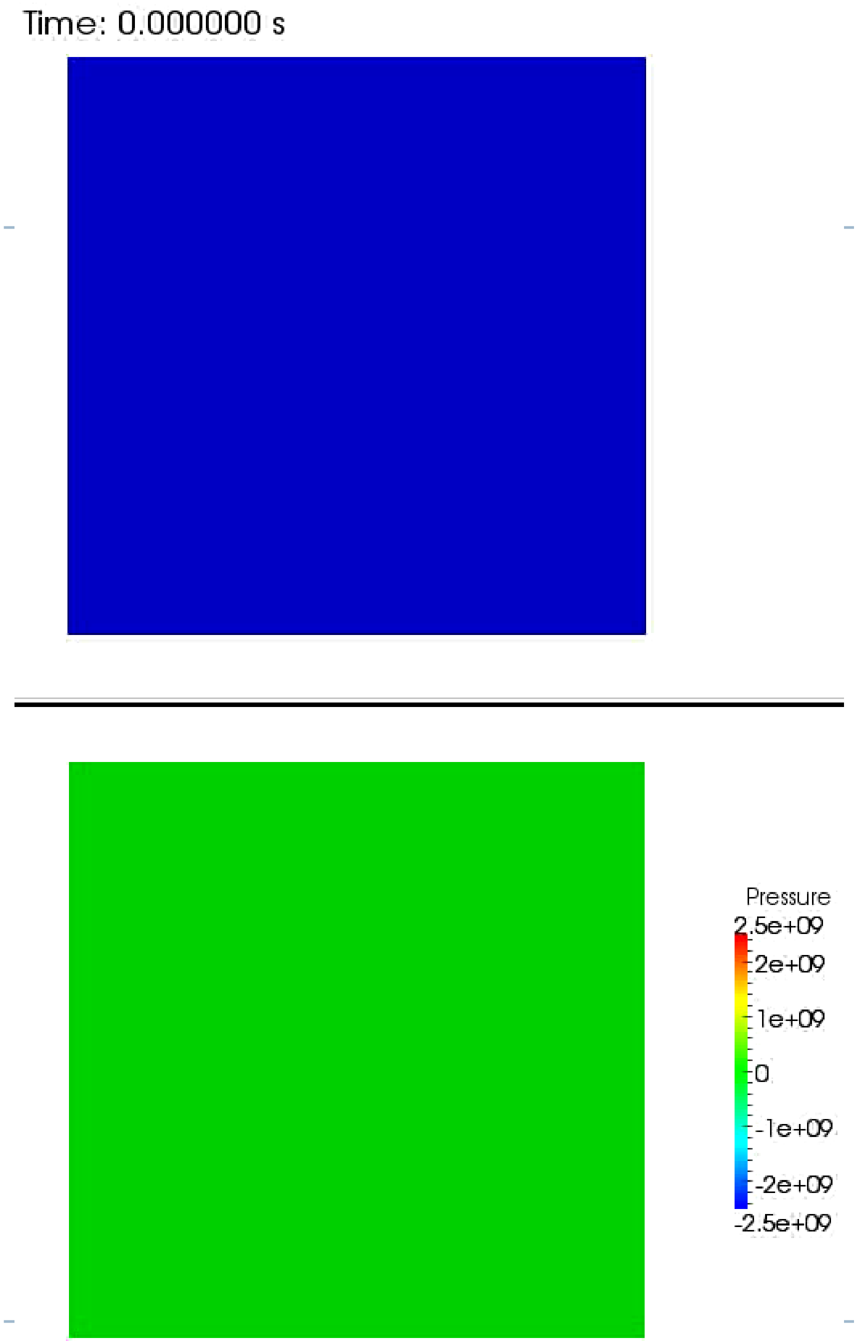
AS++ T-splines

- Include AS T-splines as a special case;
- A class of T-spline poss all the mathematics properties of AS T-splines except the locally linear independence;
- Poss the geometric modeling ability (no need conversion to analysis) from T-splines with a less propagation local refinement algorithm;

T-splines in industry

- Autodesk T-Splines Plug-in for Rhino
- Autodesk Products
  - Autodesk Fusion 360
  - Autodesk SketchBook Pro
  - Autodesk Alias Design、
  - Autodesk Inventor
  - Autodesk Showcase

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Pressure
2.5e+09
2e+09
1e+09
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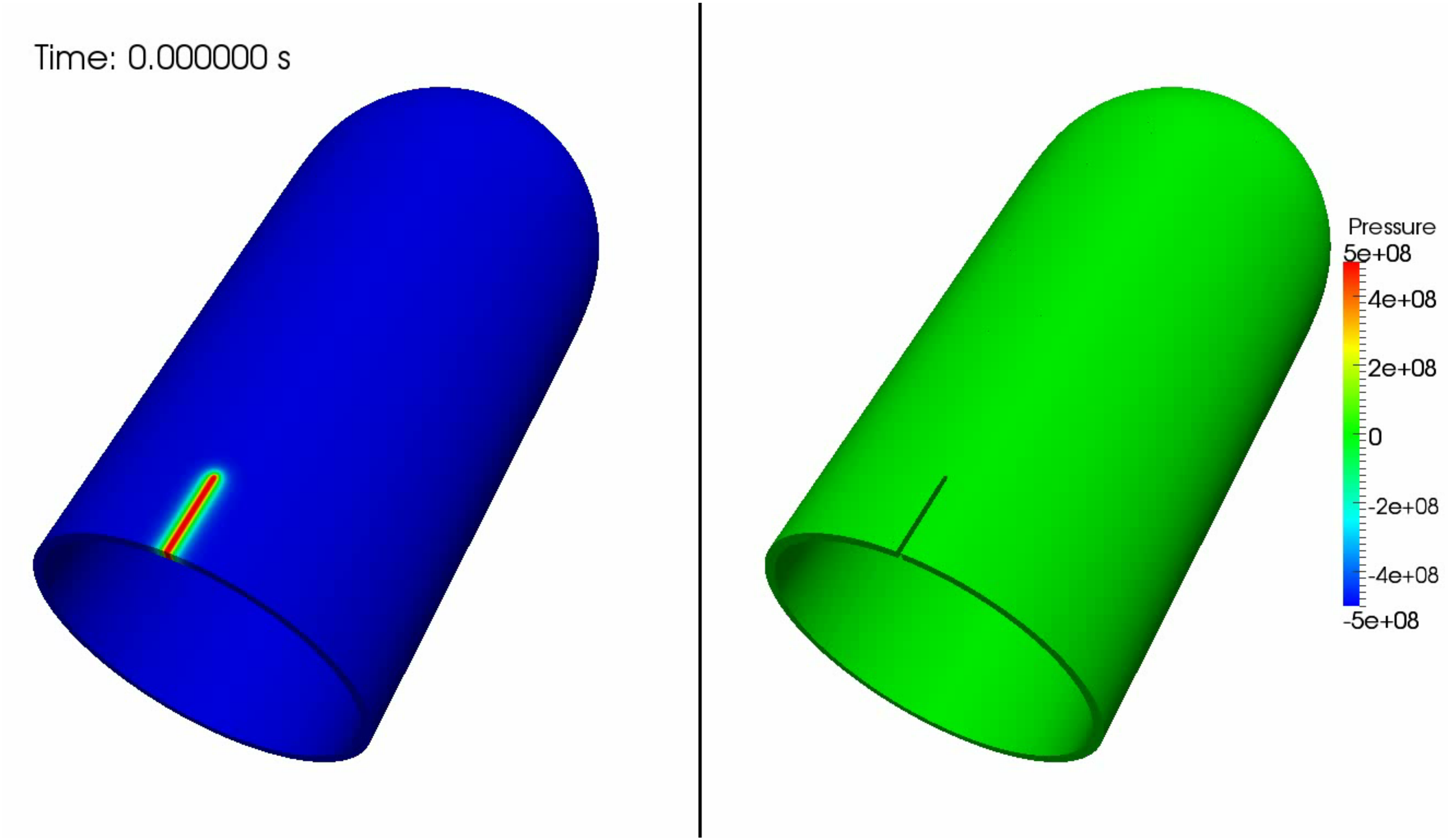
-1e+09
-2e+09
-2.5e+09



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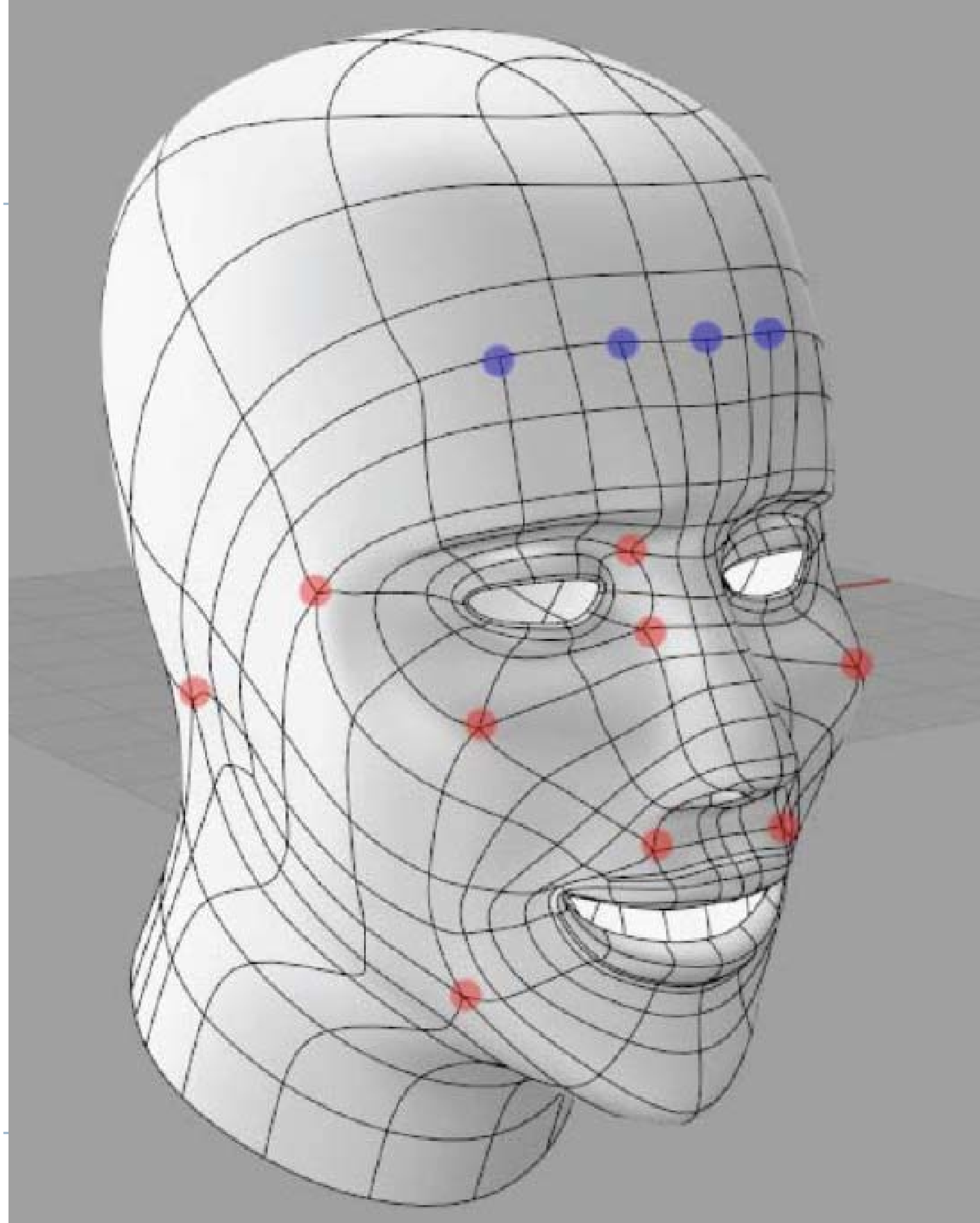
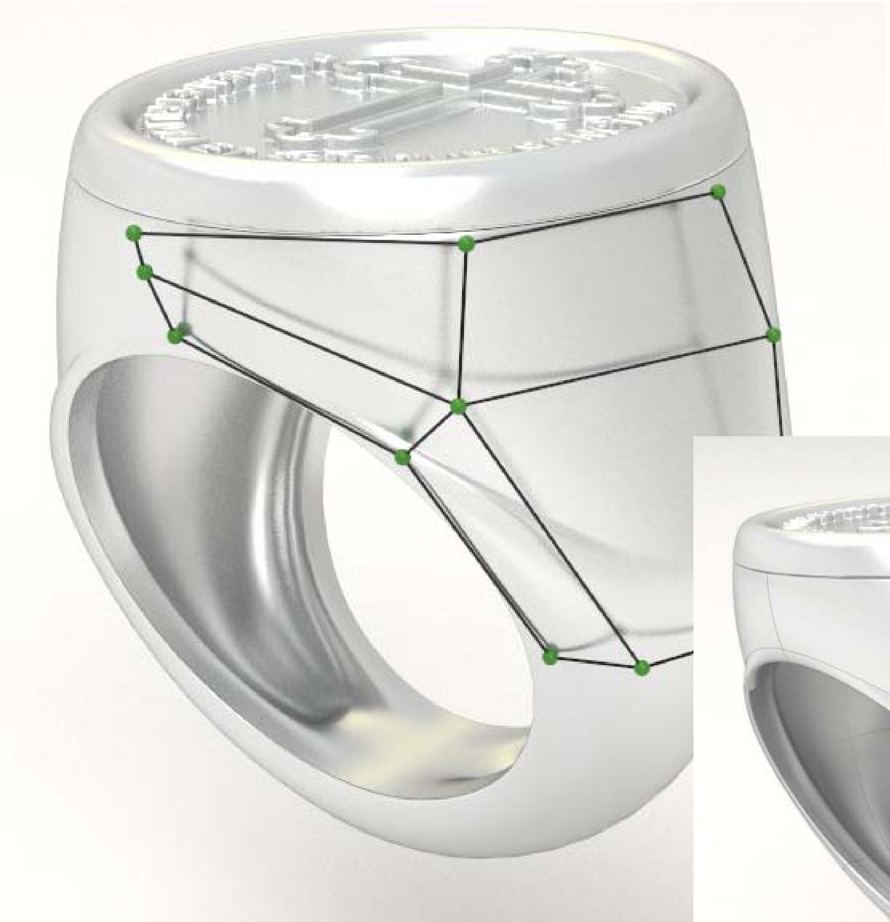
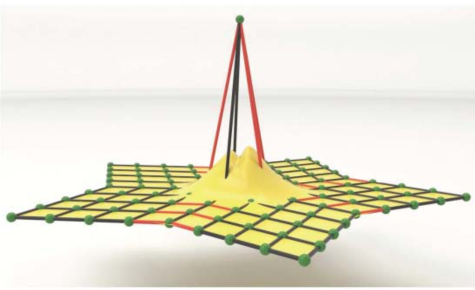


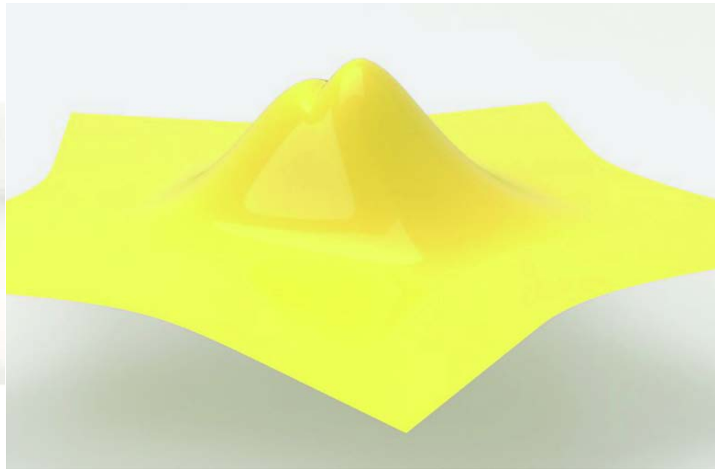
Extraordinary points



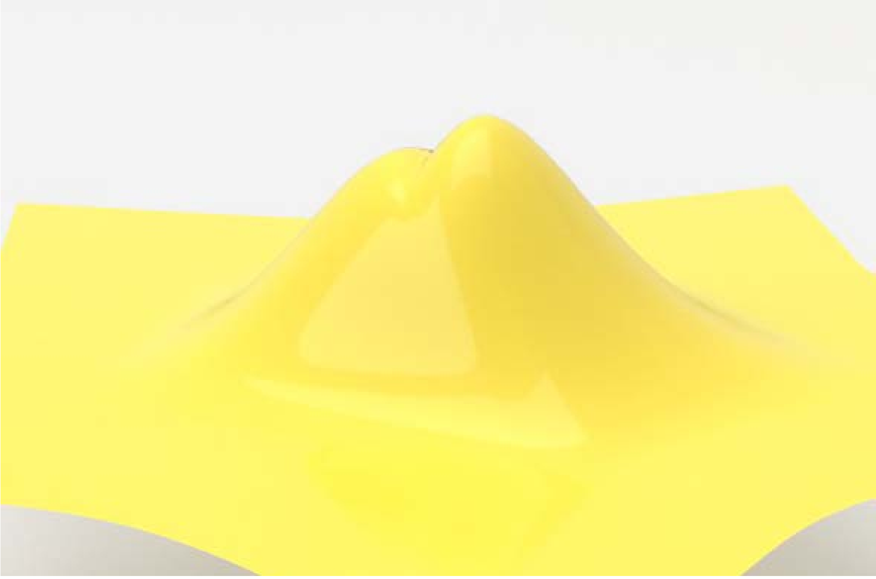
Extraordinary points



Red = 10
Black = 1



Eigen-polyhedron-based technology

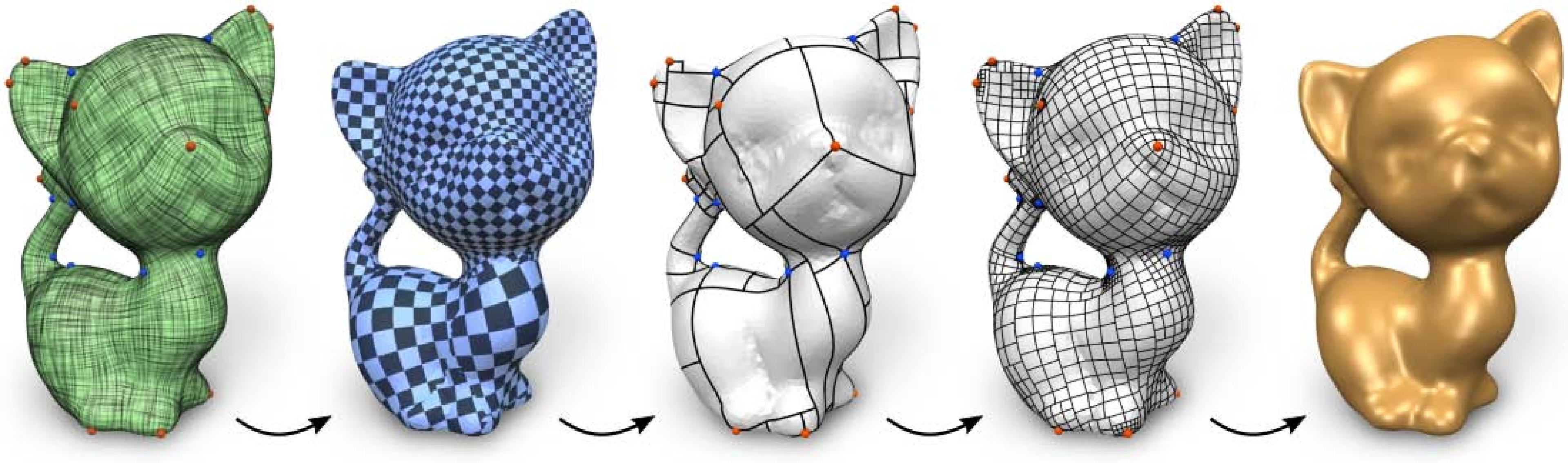


- Xin Li, Finnigan, T. W. Sederberg, G^1 Non-Uniform Catmull-Clark Surfaces, Siggraph 2016.

Conclusion

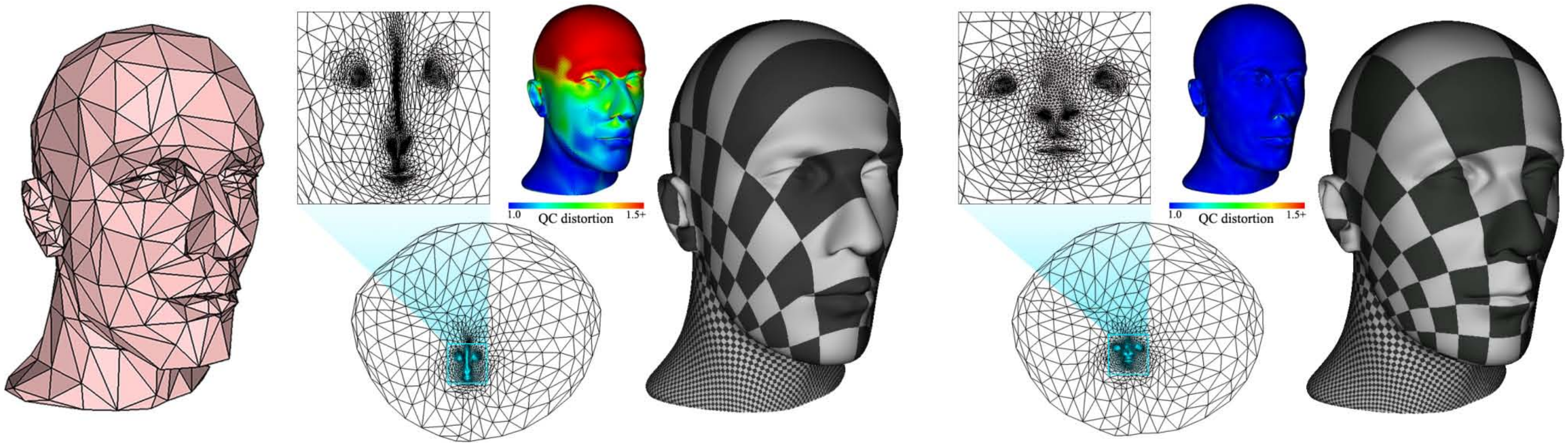
- Isogeometric analysis is very promising approach for analysis both on the efficiency and stability;
- T-splines based IGA makes several very important contributions for design-through-analysis process;
  - Spline technology is well done for currently;
  - Spline-based IGA algorithm still needs lots of work;
  - Trimming NURBS conversion still needs more consideration for industry application;
  - Volumetric parameterization;
  - Can IGA suitable for computer graphics?

From mesh to T-splines



MARCEL CAMPEN and DENIS ZORIN, Similarity Maps and Field-Guided T-Splines: a Perfect Couple, Siggraph 2017.

Geometric processing



Fernando de Goes, Mathieu Desbrun, Mark Meyer, Tony DeRose. Subdivision Exterior Calculus for Geometry Processing, Siggraph 2016.

Thanks！