IGA Solver

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内容简介

1. IGA for Linear Elastic Problems
2. Isogeometric Shell Analysis
3. Trimmed Isogeometric Analysis
4. B++ Splines with Applications to IGA

有限元和等几何分析求解器发展历程

1. 1943 年，Courant 发表了第一篇使用三角形区域的多项式函数来求解扭转问题的论文。
2. 1956 年波音公司的Turner，Clough，Martin 和Topp 在分析飞机结构时系统研究了离散杆、梁、三角形的单元刚度表达式
3. 1960 年Clough 在处理平面弹性问题，第一次提出并使用“有限元方法”(finite element method)的名称。
4. 1967 年Zienkiewicz 和Cheung 出版了第一本有关有限元分析的专著。
5. 1970 年以后，有限元方法开始应用于处理非线性和大变形等问题。著名学者如Ted Belytschko 和Thomas Hughes.
6. 2005年，Thomas Hughes教授提出等几何分析.

弹性力学方程

$$\begin{cases} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + P_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + P_y = 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + P_z = 0 \end{cases} \qquad \boxed{\sigma_{ij,j} + f_i = 0}$$

$$\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E(1-\mu)}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1 & \frac{\mu}{1-\mu} & \frac{\mu}{1-\mu} & 0 & 0 & 0 \\ & 1 & \frac{\mu}{1-\mu} & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & \frac{1-2\mu}{2(1-\mu)} & 0 & 0 \\ & & & & \frac{1-2\mu}{2(1-\mu)} & 0 \\ & & & & & \frac{1-2\mu}{2(1-\mu)} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

$$\boxed{\sigma_{ij} = 2Ge_{ij} + \lambda e_{kk}\delta_{ij}}$$

$$\begin{cases} \epsilon_x = \frac{1}{E}[\sigma_x - \mu(\sigma_y + \sigma_z)] \\ \epsilon_y = \frac{1}{E}[\sigma_y - \mu(\sigma_z + \sigma_x)] \\ \epsilon_z = \frac{1}{E}[\sigma_z - \mu(\sigma_x + \sigma_y)] \end{cases} \begin{cases} \gamma_{xy} = \frac{1}{G}\tau_{xy} \\ \gamma_{yz} = \frac{1}{G}\tau_{yz} \\ \gamma_{zx} = \frac{1}{G}\tau_{zx} \end{cases}$$

$$\boxed{e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)}$$

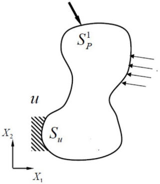
平衡方程

$$\sigma_{ij,j} + f_i = 0$$

$$u_i = g_i$$

$$\sigma_{ij} n_j = h_i$$

物理方程



几何方程

虚功原理

$$\int (\sigma_{ij,j} + f_i)\delta u_i dV - \int (\sigma_{ij} n_j - \overline{T}_i)\delta u_i dS = 0$$

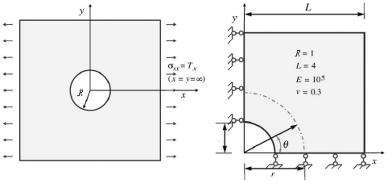
$\delta \prod = 0$ 最小势能原理（势能泛函）

$$\prod = \int_B \left(\frac{1}{2}\varepsilon_{ij}\sigma_{ij} - u_i f_i\right) d\Omega - \int_{\Gamma_N} u_i \overline{T}_i d\Gamma$$

$$\prod = \int_\Omega \frac{1}{2}\varepsilon\sigma d\Omega - \mathbf{u}^T\mathbf{f}\mathbf{d} - \int_{\partial\Omega} \mathbf{u}^T\mathbf{T}\mathbf{d}$$

中间带孔无限长板问题

下面以中间带孔无限长板平面应变问题为例，来介绍线弹性问题等几何分析



其解析解为：

$$\sigma_{r,r}(r,\theta) = \frac{T_x}{2}(1-\frac{R^2}{r^2}) + \frac{T_x}{2}(1-4\frac{R^2}{r^2}+3\frac{R^4}{r^4})cos2\theta \quad (1)$$

$$\sigma_{\theta,\theta}(r,\theta) = \frac{T_x}{2}(1+\frac{R^2}{r^2}) - \frac{T_x}{2}(1-3\frac{R^4}{r^4})cos2\theta \quad (2)$$

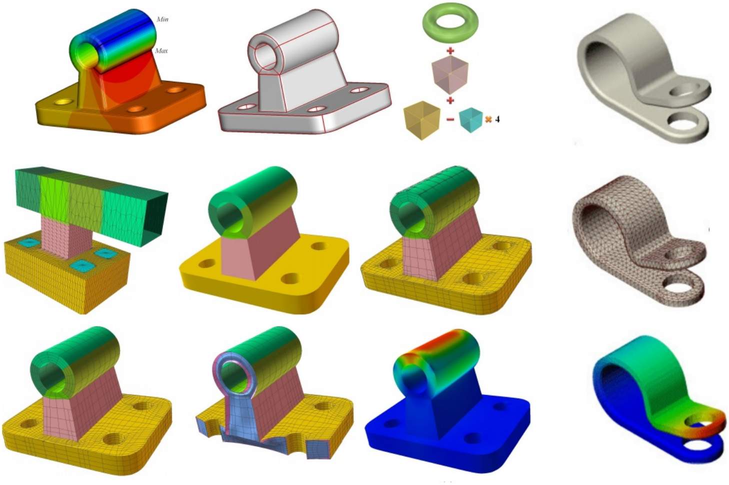
$$\sigma_{r,\theta}(r,\theta) = \frac{T_x}{2}(1+2\frac{R^2}{r^2}-3\frac{R^4}{r^4})sin2\theta \quad (3)$$

线弹性问题等几何分析的基本步骤

线弹性问题等几何分析的基本步骤可归纳为以下步骤:

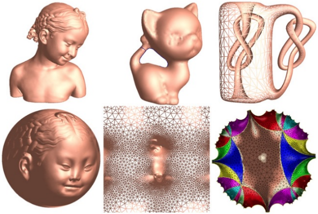
- 结构几何参数化
- 单元分析-单刚
- 整体分析-总刚
- 施加Dirichlet边界条件和载荷边界条件
- 求解位移场
- 后处理

Jessica Zhang, Gang Xu, Xin Li, Na Lei et al.

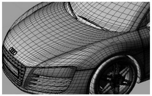


by Jessica Zhang et al.

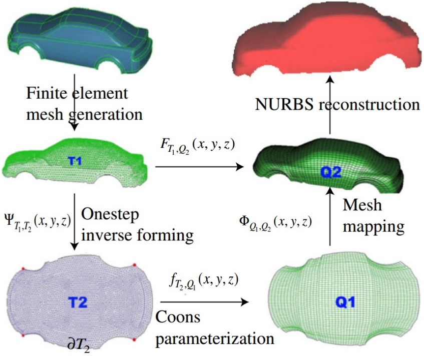
Xianfeng Gu, Xin Li, Jessica Zhang, Gang Xu et al.



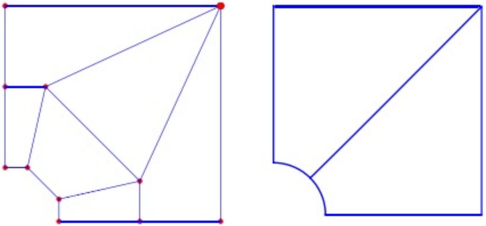
By Xianfeng Gu et al.



AST Splines, by Xin Li et al.

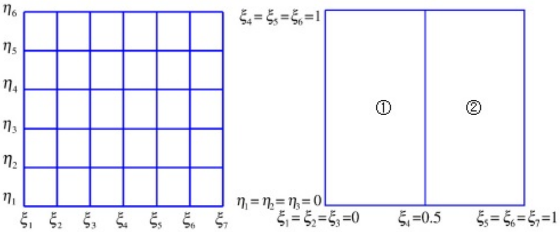


结构几何参数化-平面问题



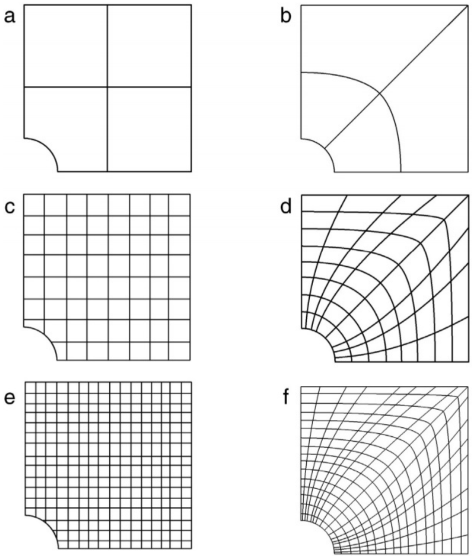
(a)控制点 （b） 网格

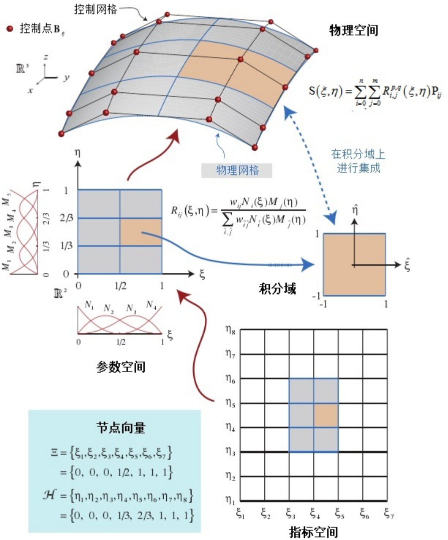
控制点及网格



（a） （b）

结构几何参数化-平面问题

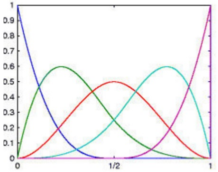


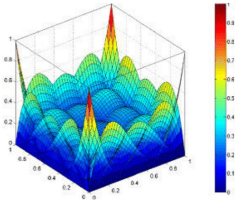

控制网格
控制点B_{ij}
物理空间
$S(\xi,\eta)=\sum_{i=0}^{n}\sum_{j=0}^{m}R_{i,j}^{p,q}(\xi,\eta)P_{ij}$
在积分域上
进行集成
物理网格
$R_{ij}(\xi,\eta)=\frac{w_{ij}N_i(\xi)M_j(\eta)}{\sum_{i,j}w_{ij}N_i(\xi)M_j(\eta)}$
积分域
参数空间
节点向量
$\Xi=\{\xi_1,\xi_2,\xi_3,\xi_4,\xi_5,\xi_6,\xi_7\}$
$=\{0,\ 0,\ 0,\ 1/2,\ 1,\ 1,\ 1\}$
$\mathcal{H}=\{\eta_1,\eta_2,\eta_3,\eta_4,\eta_5,\eta_6,\eta_7,\eta_8\}$
$=\{0,\ 0,\ 0,\ 1/3,\ 2/3,\ 1,\ 1,\ 1\}$
指标空间

选择位移函数(等参)

$$\mathbf{S} = \sum_{i}^{n} R_i(\xi, \eta) P_i$$

$$\mathbf{u} = \sum_{i}^{n} R_i(\xi, \eta) d_i$$





$$\prod = \int_\Omega \frac{1}{2}\varepsilon\boldsymbol{\sigma} d\Omega - \int_\Omega \mathbf{u}^T \mathbf{f} d\Omega - \int_{\partial\Omega} \mathbf{u}^T \overline{\mathbf{T}} d\Gamma$$

上述泛函列式可转化为每个单元上的泛函列式和:

$$\prod = \sum_e \prod^e = \sum_e \left(\int_{\Omega_e} \frac{1}{2}\varepsilon_{\mathbf{e}}^T \boldsymbol{\sigma}_{\mathbf{e}} d\Omega - \int_{\Omega_e} \mathbf{u_e}^T \mathbf{f}_e d\Omega - \int_{\partial\Omega_e} \mathbf{u}_e^T \overline{\mathbf{T}} d\Gamma\right)$$

其中

$$\boldsymbol{\sigma}_e = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \mathbf{D}\varepsilon_e \quad \varepsilon_{\mathbf{e}} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = L\mathbf{u}_e$$

$$\mathbf{u}_e = \begin{bmatrix} u \\ v \end{bmatrix} \mathbf{L} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

假设n_{eq}是某个单元相关控制点数目

$$\mathbf{u}_e = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \mathrm{R}_1 & 0 & \cdots & \mathrm{R}_i & 0 & \cdots & \mathrm{R}_{n_{eq}} & 0 \\ 0 & \mathrm{R}_1 & \cdots & 0 & \mathrm{R}_i & \cdots & 0 & \mathrm{R}_{n_{eq}} \end{bmatrix} \begin{bmatrix} d_1^x \\ d_1^y \\ \vdots \\ d_i^x \\ d_i^y \\ \vdots \\ d_{n_{eq}}^x \\ d_{n_{eq}}^y \end{bmatrix}$$

$$\prod^e = \sum_e \mathbf{d}_e^T \int_{\Omega_e} \frac{1}{2}\mathbf{B}^T\mathbf{D}\mathbf{B}d\Omega\mathbf{d}_e^T - \mathbf{d}_e{}^T \int_\Omega \mathbf{R}^T\mathbf{f}_e d\Omega - \mathbf{d}_e^T \int_{\partial\Omega_e} \mathbf{R}^T\bar{\mathbf{T}}d\partial\Omega$$

势能泛函对自由度求导

$$\frac{\partial \prod}{\partial \mathbf{d}} = 0$$

对于每一个单元泛函类似求导得到如下列式

$$\frac{\partial \prod}{\partial \mathbf{d}} = \sum_e \left(\int_{\Omega_e} \frac{1}{2}\mathbf{B}^T\mathbf{D}\mathbf{B}d\Omega\mathbf{d}_e^T - \int_\Omega \mathbf{R}^T\mathbf{f}_e d\Omega - \int_{\partial\Omega_e} \mathbf{R}^T\bar{\mathbf{T}}dS \right) = 0$$

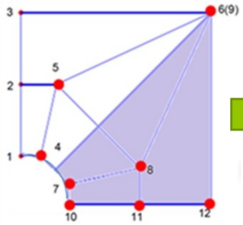
最后得到单元刚度方程列式

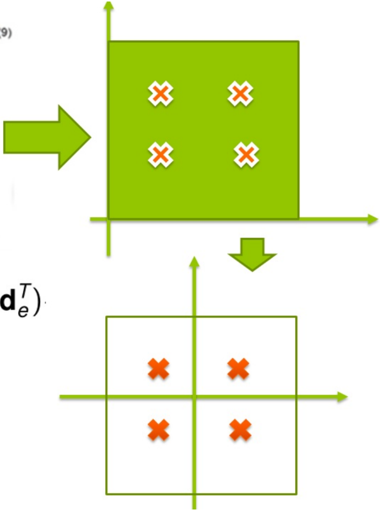
$$\mathbf{K}_e\mathbf{d}_e = \mathbf{f}_e$$

其中k_e是单元刚度矩阵，d_e是单元节点自由度，f_e是单元载荷向量。

$$K_{ij}^e = \int_{\Omega^e} \mathrm{B}_i^{\mathrm{T}} \mathrm{D} \mathrm{B}_j d\Omega^e = \int_{\Omega^e} D_0 \begin{bmatrix} \frac{\partial R_i}{\partial x} & 0 & \frac{\partial R_i}{\partial y} \\ 0 & \frac{\partial R_i}{\partial y} & \frac{\partial R_i}{\partial x} \end{bmatrix} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \frac{\partial R_j}{\partial x} & 0 \\ 0 & \frac{\partial R_j}{\partial y} \\ \frac{\partial R_j}{\partial y} & \frac{\partial R_j}{\partial x} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial R_i}{\partial \xi} \\ \frac{\partial R_i}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial R_i}{\partial x} \\ \frac{\partial R_i}{\partial y} \end{bmatrix} = \mathbf{J} \begin{bmatrix} \frac{\partial R_i}{\partial x} \\ \frac{\partial R_i}{\partial y} \end{bmatrix}$$



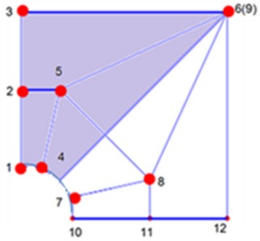
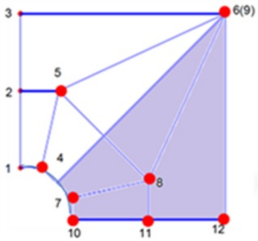


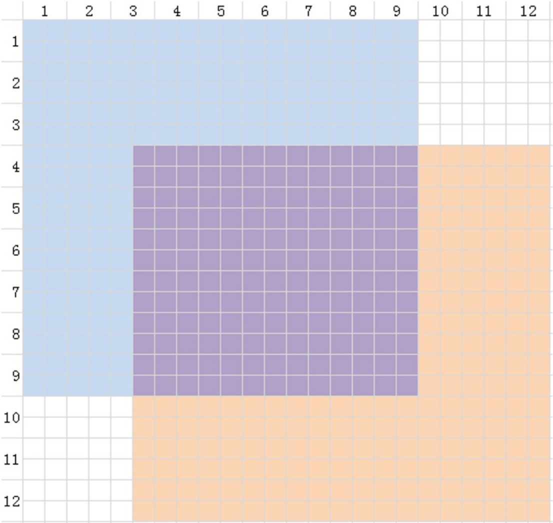
$$\sum_e \left(\int_{\Omega_e} \frac{1}{2} \mathbf{B}^T \mathbf{D} \mathbf{B} \, d\Omega \, \mathbf{d}_e^T \right)$$

$$\sum_e \left(\int_{\Omega} \mathbf{R}^T \mathbf{f}_e \, d\Omega \right)$$

$$\sum_e \left(\int_{\partial\Omega_e} \mathbf{R}^T \bar{\mathbf{T}} \, dS \right)$$

III 整体分析

单元索引	NURBS 单元	相关联控制点索引映射
①		1→ (1,1) →1 2→ (1,2) →2 3→ (1,3) →3 4→ (2,1) →4 5→ (2,2) →5 6→ (2,3) →6 7→ (3,1) →7 8→ (3,2) →8 9→ (3,3) →9
②		1→ (2,1) →4 2→ (2,2) →5 3→ (2,3) →6 4→ (3,1) →7 5→ (3,2) →8 6→ (3,3) →9 7→ (4,1) →10 8→ (4,2) →11 9→ (4,3) →12



单元刚度矩阵组装到整体刚度矩阵对号入座示意图

整体平衡方程:

$$\mathbf{Kd} = \mathbf{f}$$

式中，**k**为整体刚度矩阵；**d**为整体节点的位移向量；**f**为整体载荷向量。

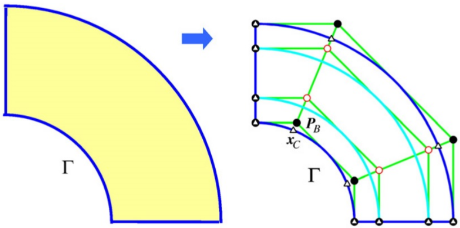
- 方程求解在引入边界条件之前，整体平衡方程是奇异的，即整体方程不可解。但在引入边界条件后，方程可解。

NURBS不满足Kronecker delta 性质，因此无法直接施加位移边界条件，王东东教授解决这一问题。他将位移边界的相关控制点转化为边界配点，则整体控制点自由度和边界配点自由度可以建立联系：

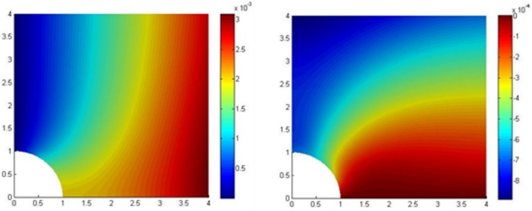
$$\mathbf{d} = \mathbf{T}\mathbf{d}^*$$

则$\mathbf{K}\mathbf{d} = \mathbf{f}$变为：

$$\mathbf{T}^T\mathbf{K}\mathbf{T}\mathbf{d}^* = \mathbf{T}^T\mathbf{f}$$

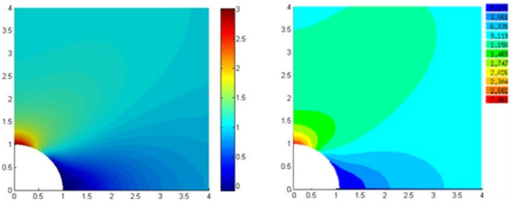


IV 后处理结果



(a) (b)

x 和 y 方向位移云图

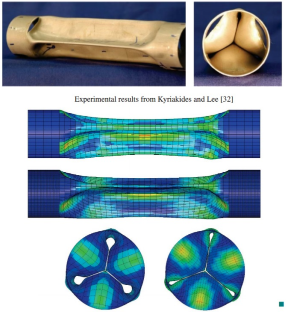


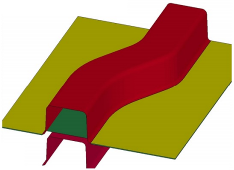
(a) (b)

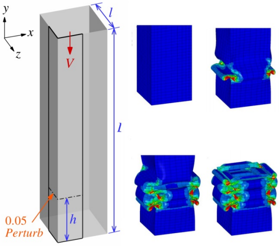
x 方向应力云图

内容简介

Isogeometric Shell Analysis







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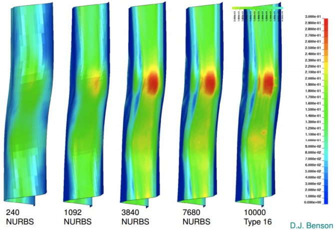
- Plastic Strain: Rotation-Free Shell vs. Reference Solution



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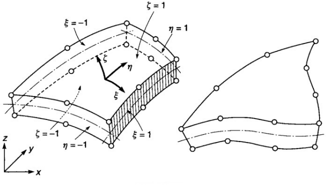
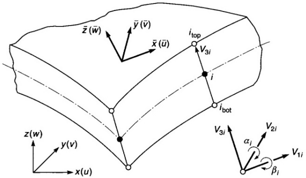
Isogeometric Shell Analysis

常用壳体理论一般包括Reissner-Mindlin理论和kirchhoff-Love理论。后者要求逼近几何至少C^1连续，而前者则不需要。





Isogeometric Shell Analysis

壳体几何函数为:

$$\mathbf{x}(\xi,\eta,\varsigma)=\mathbf{x}^N(\xi,\eta)+\varsigma\frac{h(\xi,\eta)}{2}\mathbf{f}(\xi,\eta)$$

根据等参思想,可列出壳体位移函数

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}=\sum_i N_i(\xi,\eta)\left(\begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix}+\frac{1}{2}\varsigma h_i\begin{bmatrix} V_{1i} & -V_{2i} \end{bmatrix}\begin{Bmatrix} \alpha_i \\ \beta_i \end{Bmatrix}\right)$$

向量形式为:

Isogeometric Shell Analysis

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{N}\mathbf{a}^{\mathbf{e}}; \quad \mathbf{a}^{\mathbf{e}} = \begin{Bmatrix} a_i^e \\ \vdots \\ a_j^e \end{Bmatrix} \quad with \quad a_i^e = \begin{pmatrix} u_i \\ v_i \\ w_i \\ \alpha_i \\ \beta_i \end{pmatrix} \tag{4}$$

$$\begin{aligned} \mathbf{N} &= \left\{ \begin{matrix} \mathbf{N_1} & \mathbf{N_2} & \cdots & \mathbf{N_9} \end{matrix} \right\} \\ \mathbf{a}^{\mathbf{e}} &= \left\{ \begin{matrix} u_1 & v_1 & w_1 & \varphi_1 & \psi_1 & \cdots & u_9 & v_9 & w_9 & \varphi_9 & \psi_9 \end{matrix} \right\} \end{aligned} \tag{5}$$

其中

$$\mathbf{N_i} = \begin{bmatrix} N_i(\xi,\eta) & 0 & 0 & \frac{\zeta}{2} h_i l_{1i} N_i(\xi,\eta) & -\frac{\zeta}{2} h_i l_{2i} N_i(\xi,\eta) \\ 0 & N_i(\xi,\eta) & 0 & \frac{\zeta}{2} h_i m_{1i} N_i(\xi,\eta) & -\frac{\zeta}{2} h_i m_{2i} N_i(\xi,\eta) \\ 0 & 0 & N_i(\xi,\eta) & \frac{\zeta}{2} h_i n_{1i} N_i(\xi,\eta) & -\frac{\zeta}{2} h_i n_{2i} N_i(\xi,\eta) \end{bmatrix} \tag{6}$$

Isogeometric Shell Analysis

全局坐标和局部坐标之间的关系：

$$\mathbf{X} = \theta \mathbf{X}'$$

其中

$$\mathbf{X} = \begin{bmatrix} x & y & z \end{bmatrix}^{\mathbf{T}}$$

$$\mathbf{X}' = \begin{bmatrix} x' & y' & z' \end{bmatrix}^{\mathbf{T}}$$

$$\theta = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$$

局部坐标系下局部应力与局部应变为:

$$
\begin{aligned}
\boldsymbol{\varepsilon}' &= \begin{bmatrix} \varepsilon_{x'} & \varepsilon_{y'} & \gamma_{x'y'} & \gamma_{y'z'} & \gamma_{z'x'} \end{bmatrix}^T \\
&= \begin{bmatrix} \frac{\partial u'}{\partial x'} & \frac{\partial v'}{\partial y'} & \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} & \frac{\partial v'}{\partial z'} + \frac{\partial w'}{\partial y'} & \frac{\partial u'}{\partial z'} + \frac{\partial w'}{\partial x'} \end{bmatrix}^T
\end{aligned}
$$

$$
\boldsymbol{\sigma}' = \begin{bmatrix} \sigma_{x'} & \sigma_{y'} & \tau_{x'y'} & \tau_{y'z'} & \tau_{z'x'} \end{bmatrix}^T = \bar{\mathbf{D}}\boldsymbol{\varepsilon}'
$$

其中

$$
\bar{\mathbf{D}} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\nu}{2k} & 0 \\ 0 & 0 & 0 & 0 & \frac{1-\nu}{2k} \end{bmatrix}
$$

全局坐标系和局部坐标系位移导数关系：

$$\begin{bmatrix} \frac{\partial u'}{\partial x'} & \frac{\partial v'}{\partial x'} & \frac{\partial w'}{\partial x'} \\ \frac{\partial u'}{\partial y'} & \frac{\partial v'}{\partial y'} & \frac{\partial w'}{\partial y'} \\ \frac{\partial u'}{\partial z'} & \frac{\partial v'}{\partial z'} & \frac{\partial w'}{\partial z'} \end{bmatrix} = \boldsymbol{\theta}^T \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix} \boldsymbol{\theta}$$

其中

$$\boldsymbol{\theta} = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$$

全局和局部导数算子之间关系:

$$\begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{pmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \mathbf{J} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \mathbf{J}^{-1} \begin{pmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{pmatrix}$$

Isogeometric Shell Analysis

全局应变可表示为

$$\varepsilon = \mathbf{L}_g \mathbf{u}_g = \mathbf{L}_g \mathbf{u}_g = \mathbf{L}_g \mathbf{N} \mathbf{a}^{\mathbf{e}}$$

其中

$$\mathbf{L}_g \mathbf{N}_i = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ & \frac{\partial}{\partial y} & \\ & & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & \\ & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_i(\xi,\eta) & 0 & 0 & \frac{\zeta}{2} h_i l_{1i} N_i(\xi,\eta) & -\frac{\zeta}{2} h_i l_{2i} N_i(\xi,\eta) \\ 0 & N_i(\xi,\eta) & 0 & \frac{\zeta}{2} h_i m_{1i} N_i(\xi,\eta) & -\frac{\zeta}{2} h_i m_{2i} N_i(\xi,\eta) \\ 0 & 0 & N_i(\xi,\eta) & \frac{\zeta}{2} h_i n_{1i} N_i(\xi,\eta) & -\frac{\zeta}{2} h_i n_{2i} N_i(\xi,\eta) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial N_i(\xi,\eta)}{\partial x} & 0 & 0 & \frac{1}{2} h_i l_{1i} \frac{\partial}{\partial x}(\zeta \cdot N_i(\xi,\eta)) & -\frac{1}{2} h_i l_{2i} \frac{\partial}{\partial x}(\zeta \cdot N_i(\xi,\eta)) \\ 0 & \frac{\partial N_i(\xi,\eta)}{\partial y} & 0 & \frac{1}{2} h_i m_{1i} \frac{\partial}{\partial y}(\zeta \cdot N_i(\xi,\eta)) & -\frac{1}{2} h_i m_{2i} \frac{\partial}{\partial y}(\zeta \cdot N_i(\xi,\eta)) \\ 0 & 0 & \frac{\partial N_i(\xi,\eta)}{\partial z} & \frac{1}{2} h_i n_{1i} \frac{\partial}{\partial z}(\zeta \cdot N_i(\xi,\eta)) & -\frac{1}{2} h_i n_{2i} \frac{\partial}{\partial z}(\zeta \cdot N_i(\xi,\eta)) \\ \frac{\partial N_i(\xi,\eta)}{\partial y} & \frac{\partial N_i(\xi,\eta)}{\partial x} & 0 & b_{44} & b_{45} \\ 0 & \frac{\partial N_i(\xi,\eta)}{\partial z} & \frac{\partial N_i(\xi,\eta)}{\partial y} & b_{54} & b_{55} \\ \frac{\partial N_i(\xi,\eta)}{\partial z} & 0 & \frac{\partial N_i(\xi,\eta)}{\partial x} & b_{64} & b_{65} \end{bmatrix}$$

Isogeometric Shell Analysis

全局坐标系和局部坐标系应变关系：

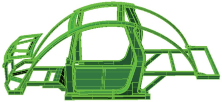
$$\varepsilon' = \mathbf{Q}_g \varepsilon_g = \mathbf{Q}_g \mathbf{L}_g \mathbf{u}_g = \mathbf{Q}_g \mathbf{L}_g \mathbf{N} \mathbf{a}^e = \bar{\mathbf{B}} \mathbf{a}^e$$

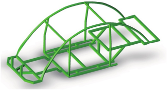
每个NURBS壳元的刚度阵可表示为：

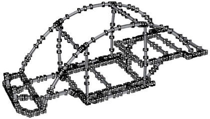
$$\mathbf{K}_e = \int_{V^e} \mathbf{H} dxdydz = \int_0^1 \int_0^1 \int_0^1 \bar{\mathbf{B}}^T \bar{D} \bar{\mathbf{B}} \, |J| d\xi d\eta d\zeta$$

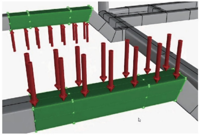
Isogeometric Shell Analysis

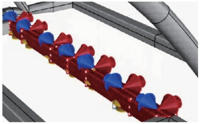
自主研发彩虹车钢骨架结构等几何分析（包括182个壳体）

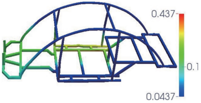












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1. IGA for Linear Elastic Problems
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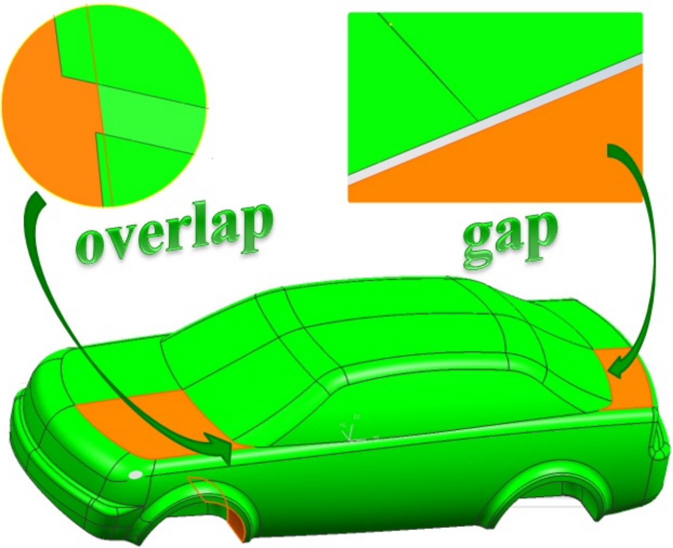
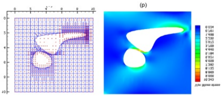
3. Trimmed Isogeometric Analysis
4. B++ Splines with Applications to IGA

Trimmed Isogeometric Analysis

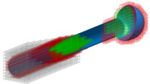


Trimmed Isogeometric Analysis



❖ **Isogeometric Analysis for Trimmed CAD Surfaces (by Hyun-Jung Kim, Yu-Deok Seo, Sung-Kie Youn 2009)**

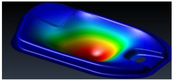
Hyun-Jung Kim, Yu-Deok Seo, Sung-Kie Youn, Isogeometric analysis for trimmed CAD surfaces , Comput. Methods Appl. Mech. Engrg.



❖ **Web-splines (by K. Höllig, U. Reif, J. Wipper 2001)**

K. Höllig, U. Reif, J. Wipper: *Weighted extended b-spline approximation of Dirichlet problems.* SIAM Journal on Numerical Analysis, Volume 39, Number 2, pp. 442-462, 2001.

❖ **Isogeometric analysis for B-rep models (by M. Breitenberger, A. Apostolatos, B. Philipp, R. Wüchner, K.-U.Bletzinger 2015)**

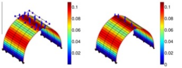


M. Breitenberger, A. Apostolatos, B. Philipp, R. Wüchner, K.-U.Bletzinger, Analysis in computer aided design: Nonlinear isogeometric B-Rep analysis of shell structures, Comput. Methods Appl. Mech. Engrg. (2014)

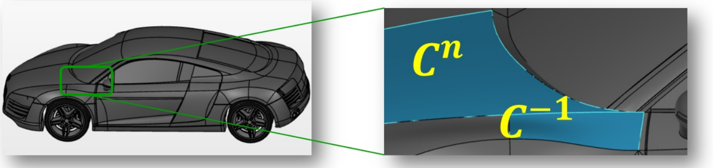
❖ **Weak coupling for trimmed surfaces using Nitsche method. (by Ruess, M., Schillinger, D., Özcan, A. I., & Rank, E. 2014)**

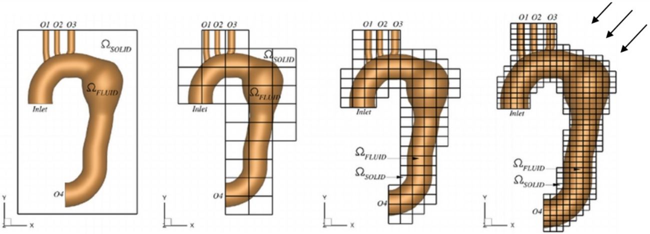
Ruess, M., Schillinger, D., Özcan, A. I., & Rank, E. (2014). Weak coupling for isogeometric analysis of non-matching and trimmed multi-patch geometries. Computer Methods in Applied Mechanics and Engineering, 269, 46-71.

❖**Isogeometric analysis of trimmed NURBS geometries. (by Robert Schmidt, , Roland Wüchner , Kai-Uwe Bletzinge)**

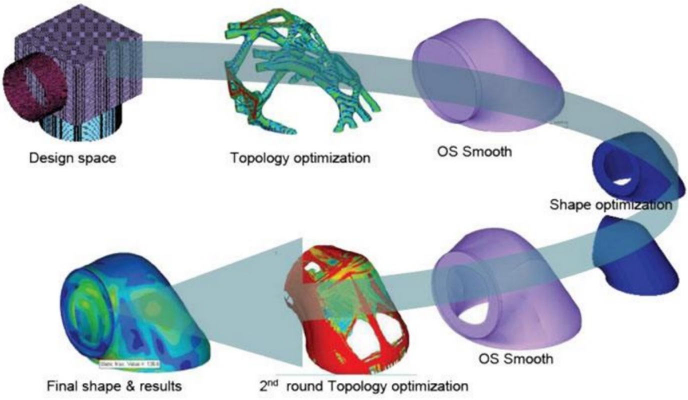


Trimmed Isogeometric Analysis

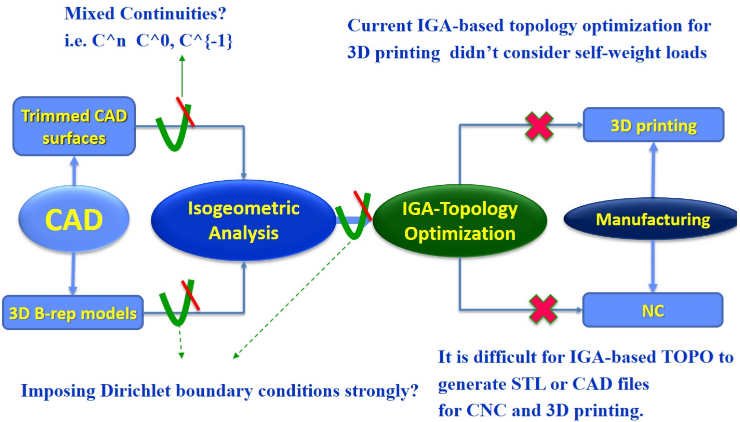




Trimmed Isogeometric Analysis



Trimmed Isogeometric Analysis



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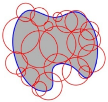
Immersed methods and non-body fitted methods

- **IGA-based immersed methods include immersed IGA, FCM, XIGA et al.**
- **Traditional non-body fitted methods include such as IBM, XFEM, EFGM, RBF-based meshless method et al.**

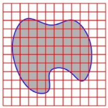
Issue I

It is difficult for these methods to **strongly** impose Dirichlet boundary conditions because they do not satisfy **Kronecker delta property**.

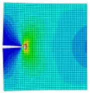
RBF



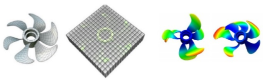
EFGM



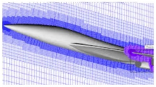
XFEM



Finite Cell Method (by D Schillinger et al.)

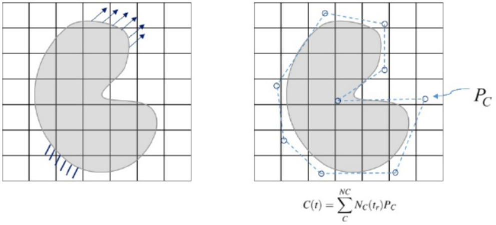


Embedded boundary method



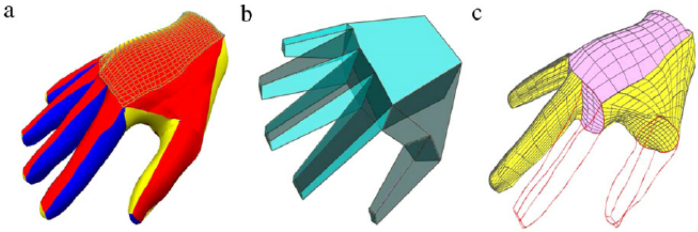
Issue II

Another disadvantage of immersed methods is that they are difficult to incorporate boundary collocation points or boundary curves/surfaces as boundary representation.



Issue III

Volume parameterization for isogeometric analysis is still a big challenge, especially for the CSG models with complex topology structures. Stress functions are usually discontinuous!



B++ spline method

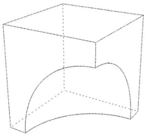
B++ Splines = Boundary Plus Plus Splines

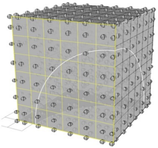
Advantages of B++ splines

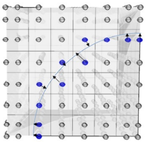
- Strongly impose Dirichlet boundary conditions
- Incorporate boundary collocation points as the boundary representation.
- Say bye to multi-blocks and stress functions are continuous.

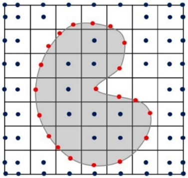
B++ splines

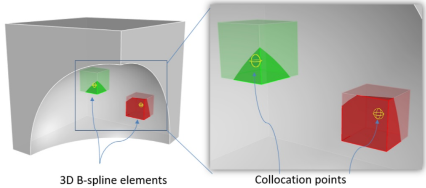
Its idea is replacing the control points of the background spline mesh using the boundary collocation points or the boundary spline curves/surfaces.

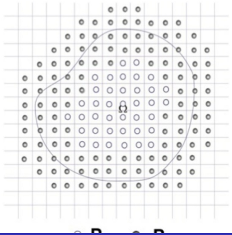






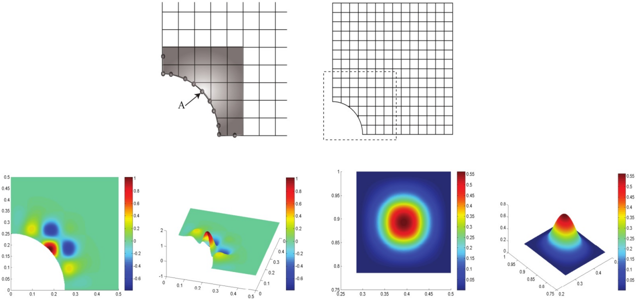






B++ spline basis functions

B++ spline basis functions satisfy the Kronecker delta property, build the partition of unity. They are also linearly independent.



Mutually orthogonal components

By $\mathbf{BP}_B = \mathbf{P}_C$, we can rewrite $\boldsymbol{P}_B$, in terms of two mutually orthogonal components, one in the row space of $\mathbf{B}$ and the other in the null space of $\mathbf{B}$, that is

$$\mathbf{P}_B = \mathbf{B}^T\boldsymbol{\Phi} + \mathbf{Z} = \mathbf{B}^T\boldsymbol{\Phi} + \mathbf{R}\boldsymbol{\Psi} \tag{7}$$

where

$$\mathbf{BR} = \mathbf{0} \tag{8}$$

Combining these equations, we get

$$\mathbf{P}_B = \mathbf{B}^T(\mathbf{BB}^T)^{-1}\mathbf{P}_C + \mathbf{RP}_E \tag{9}$$

Thus, $\mathbf{S}$ is reformulated

$$\mathbf{S} = \mathbf{N}_A\mathbf{P}_A + \mathbf{N}_B\mathbf{B}^T(\mathbf{B}\mathbf{B}^T)^{-1}\mathbf{P}_C + \mathbf{N}_B\mathbf{R}\mathbf{P}_E \tag{10}$$

For simplicity

$$\mathbf{S} = \mathbf{N}_A\mathbf{P}_A + \widetilde{\mathbf{N}}_C\mathbf{P}_C + \widetilde{\mathbf{N}}_E\mathbf{P}_E \tag{11}$$

- $\mathbf{P}_A$ Control points with complete basis functions.
- $\mathbf{P}_C$ Boundary collocation points or the control points of trim curves.
- $\mathbf{P}_E$ Enriched control points.
- $\mathbf{N}_A$ Complete basis functions.
- $\widetilde{\mathbf{N}}_C$ Basis functions of the boundary collocation points or the control points of trim curves.
- $\widetilde{\mathbf{N}}_E$ Basis functions of enriched control points.

The matrix **T** is

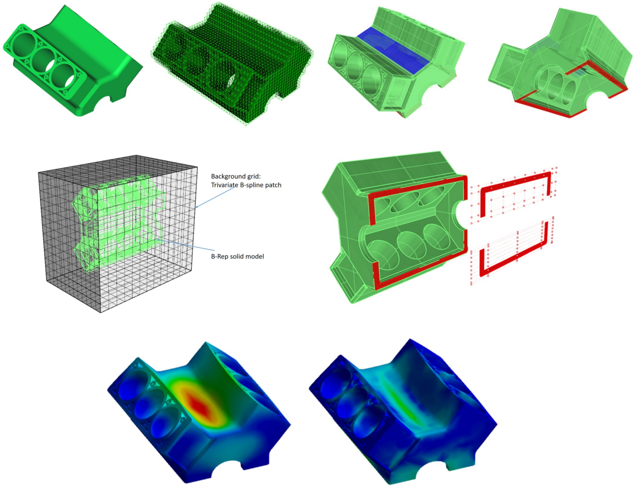
$$\mathbf{T} = [\mathbf{T}_1 \quad \mathbf{T}_2] = \left[\ \mathbf{B}^T(\mathbf{B}\mathbf{B}^T)^{-1} \quad \mathbf{R}\ \right]. \tag{12}$$

where $\mathbf{T}_1 = \mathbf{B}^T(\mathbf{B}\mathbf{B}^T)^{-1}$ and $\mathbf{T}_2 = \mathbf{R}$.

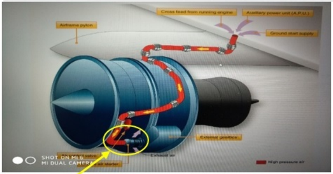
Advantages

- **Linear independence. Yes**
- **Kronecker delta property. Yes**
- **Partition of unity. Yes.**

Applications-Structural analysis

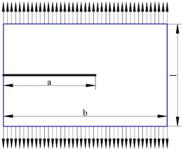


Applications-Structural analysis

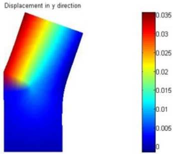


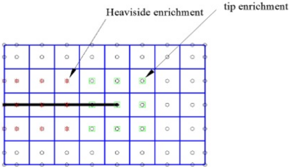
起动机带动涡轮发动机风扇

Applications-XIGA

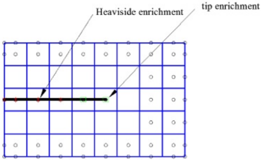


Edge cracked model





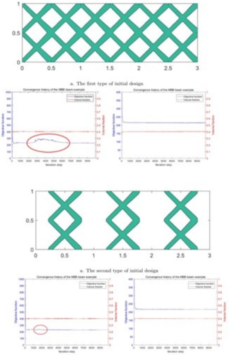
NURBS-based XIGA



B++ spline-based XIGA

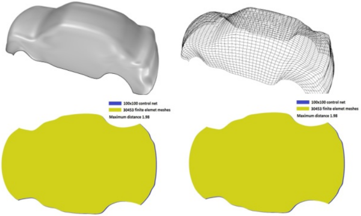
Applications-Topology optimization

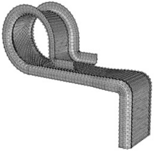
Approach	MMC188	TOP-IGA-MMC	MMC188	TOP-IGA-MMC
Elements	120×40	12×4 ($refine$ 4 $times$) $[192 \times 64]$	120×40	12×4 ($refine$ 4 $times$) $[192 \times 64]$
Initial MMCs				
Iterative step 50				
Iterative step 100				
Iterative step 200				
Iterative step 300				
Iterative step 500				
Iterative step 1000				
Iterative step 3000				
Iterative step 4000				
Iterative step 8000				
Iterative step 10000				

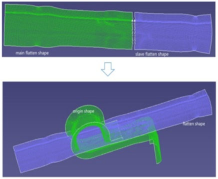


基于IGA的可变多组件拓扑优化方法

Applications-Onestep inverse isogeometric analysis







Processing and potential applications on B++ splines

In processing:

- Fluid-solid interactions using B++ splines.
- Topology optimization using B++ spline-based isogeometric analysis.
- B++ spline-based XIGA.

Potential applications:

- Simulations of failure and crack using B++ spline-based XIGA.
- Multigrid method using B++ splines
- Hierarchical B++ splines with applications to adaptive isogeometric analysis.

Thank you!