Numerical Coarsening using Discontinuous Shape Functions

Jiong Chen[1], Hujun Bao[1], Tianyu Wang[1], Mathieu Desbrun[2], Jin Huang[1]

[1] State Key Lab of CAD&CG, Zhejiang University   [2] Caltech

Challenge

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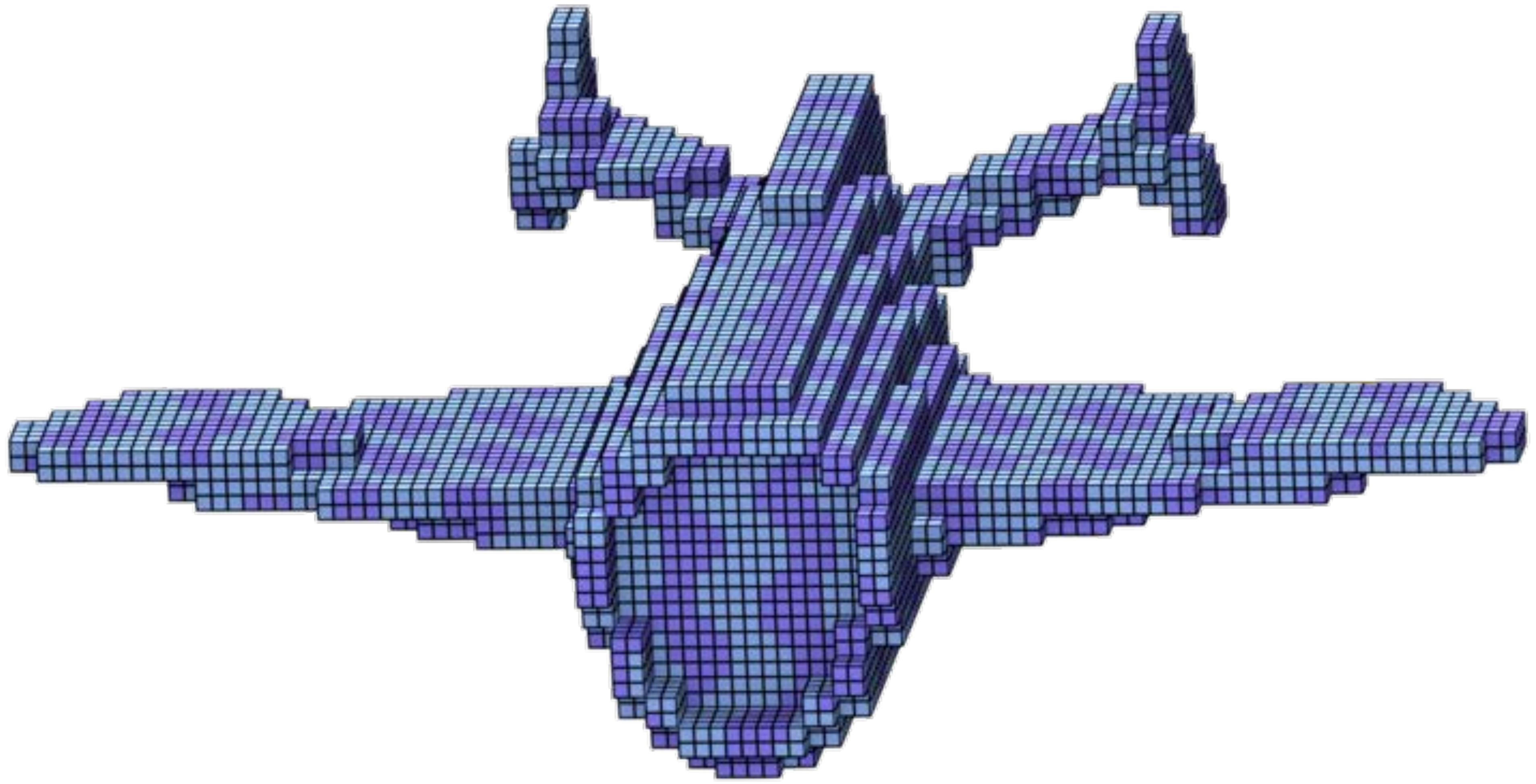
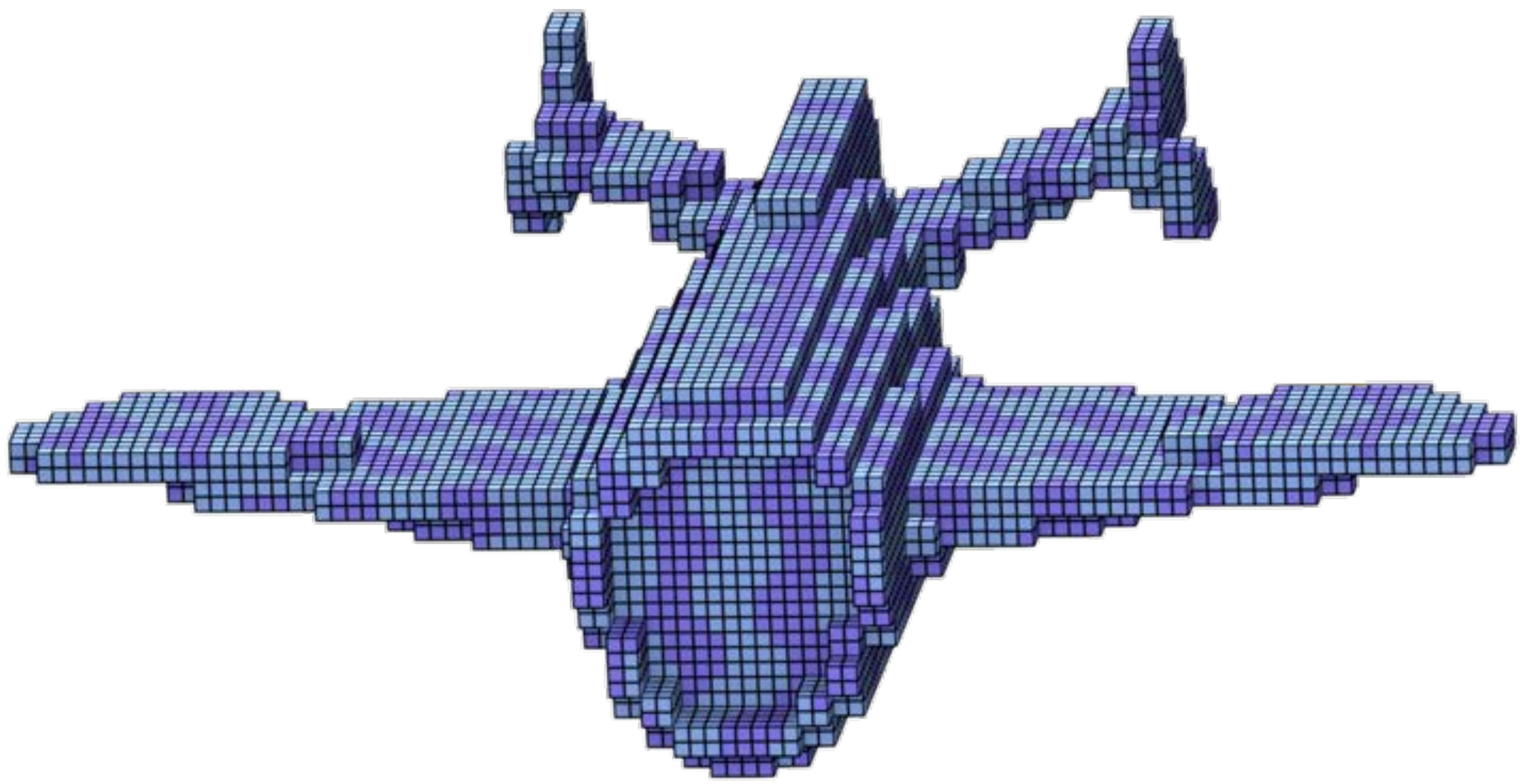


Inhomogeneous

nonlinear

Challenge



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Require fine mesh

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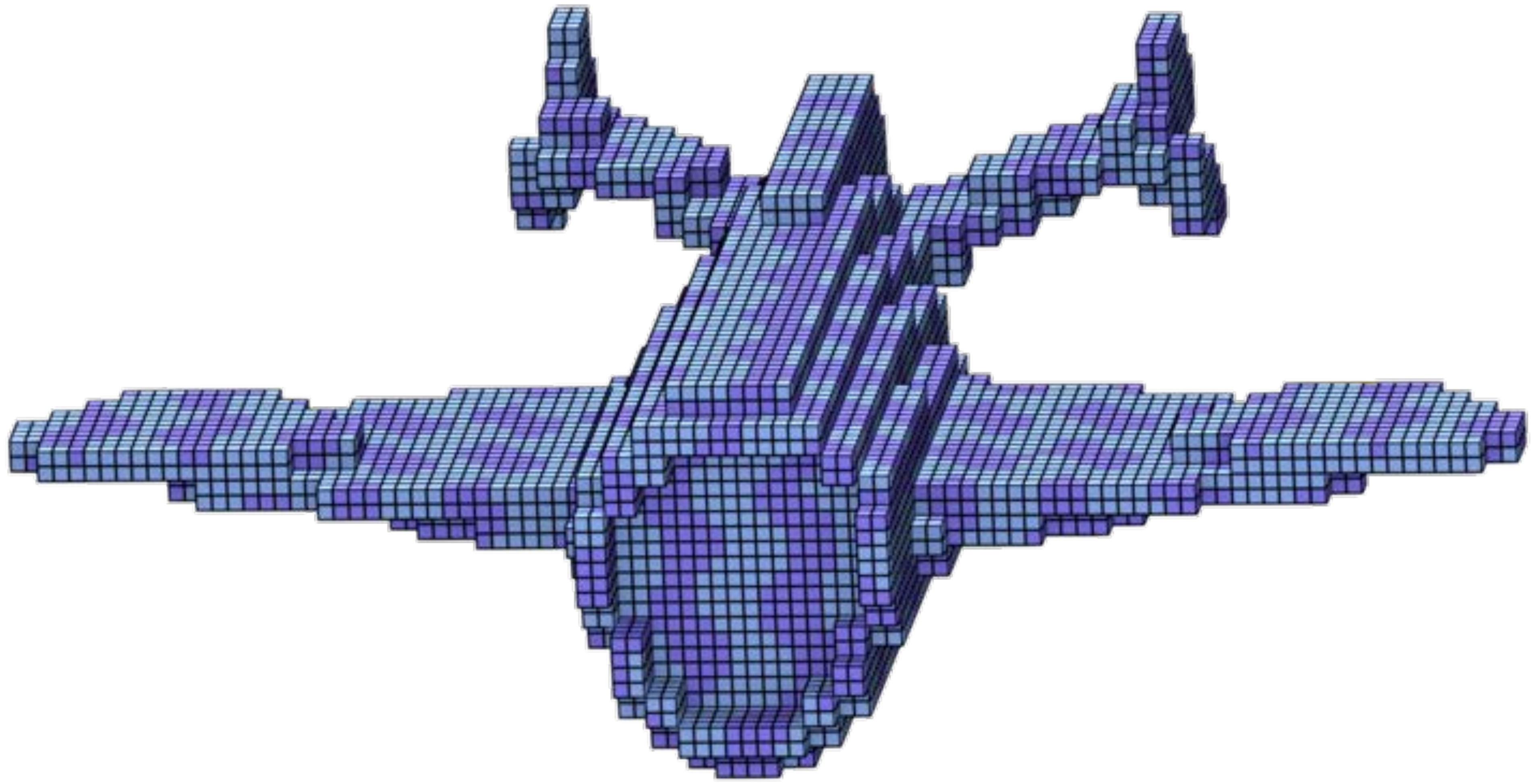

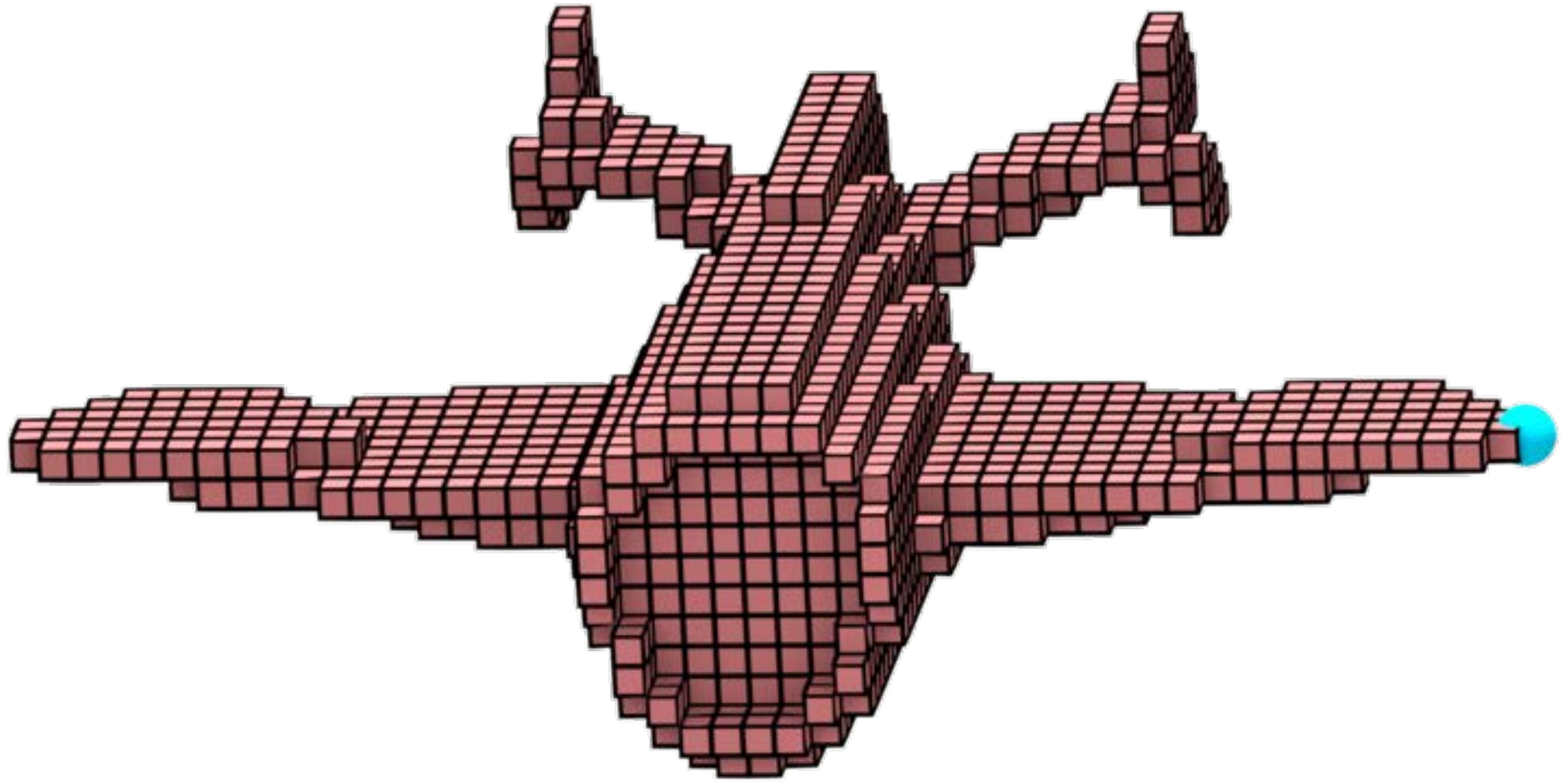

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Coarse mesh

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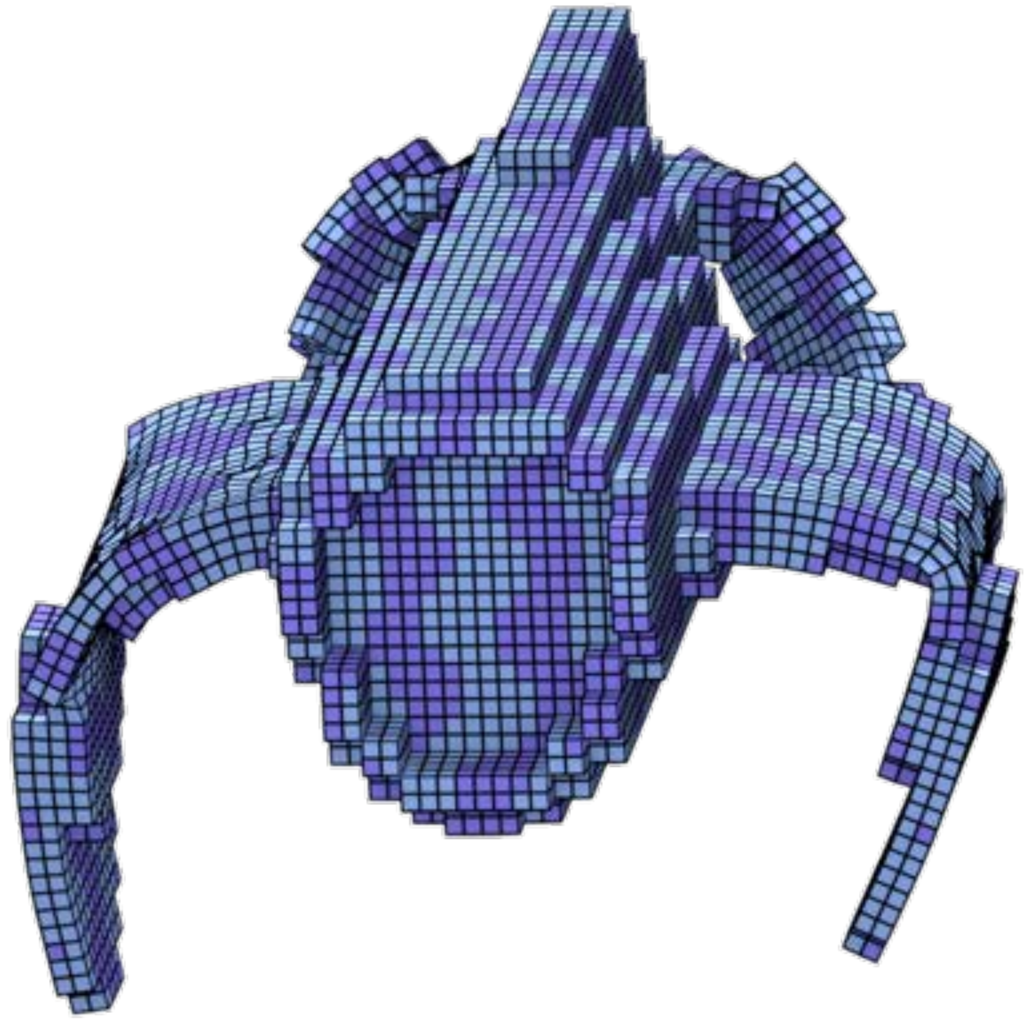
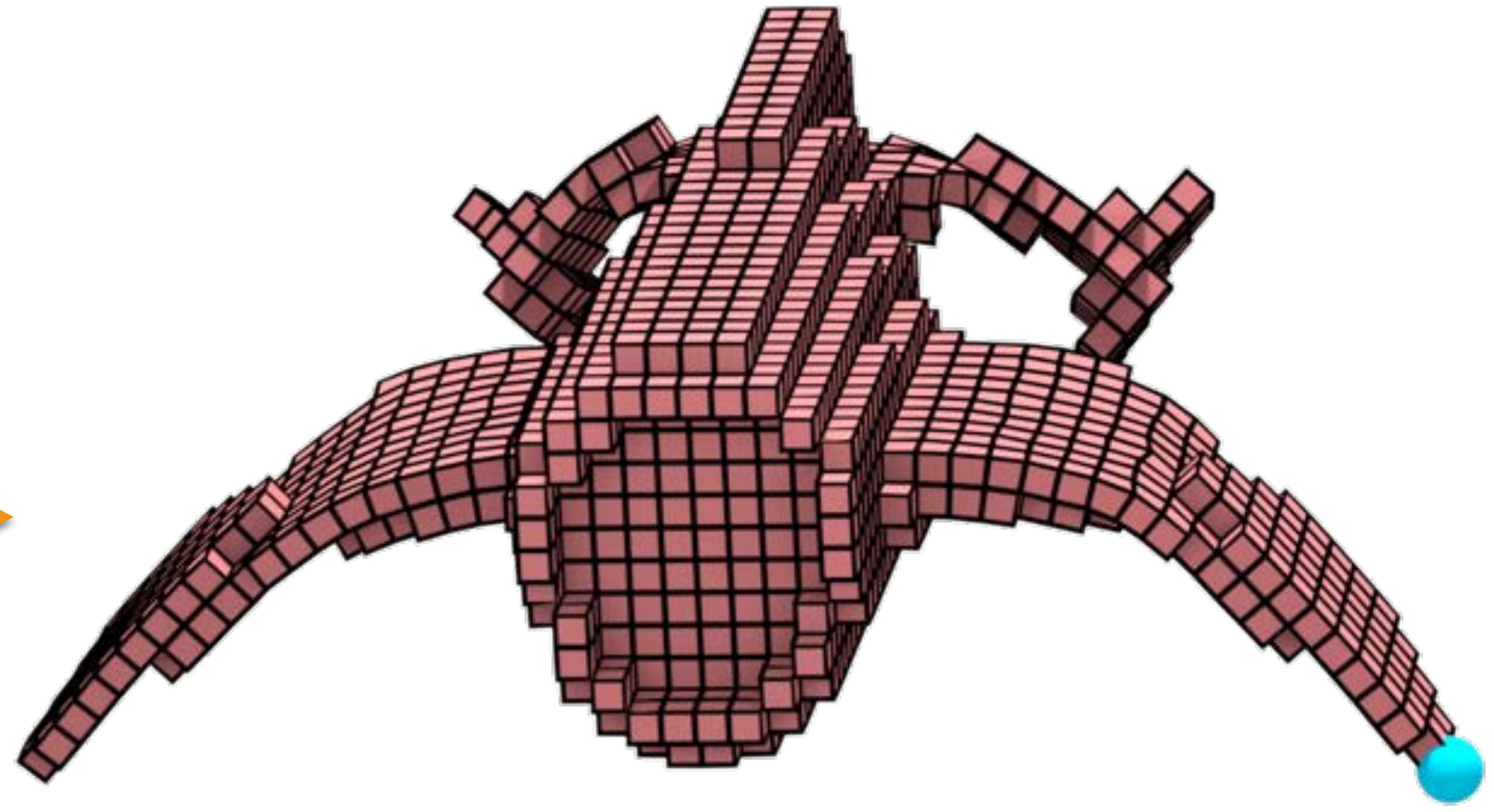


linear bases



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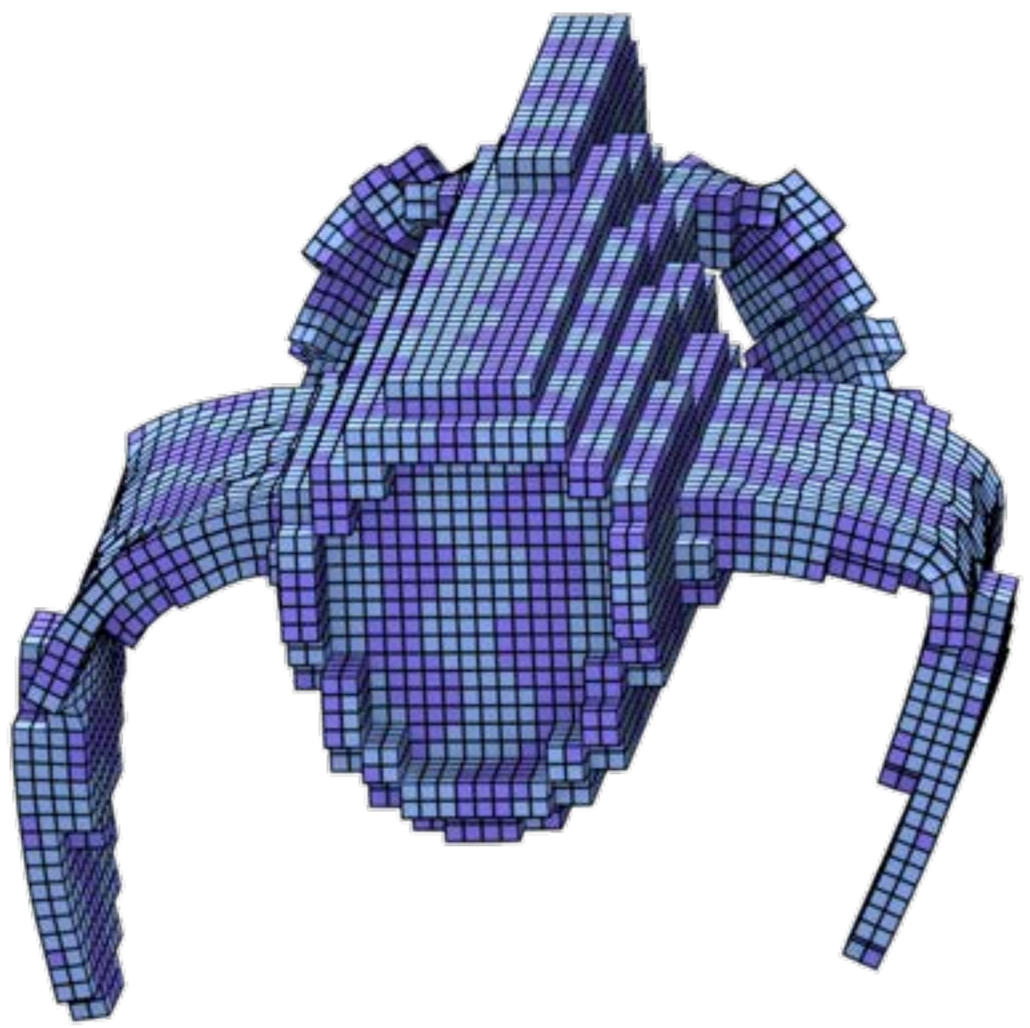

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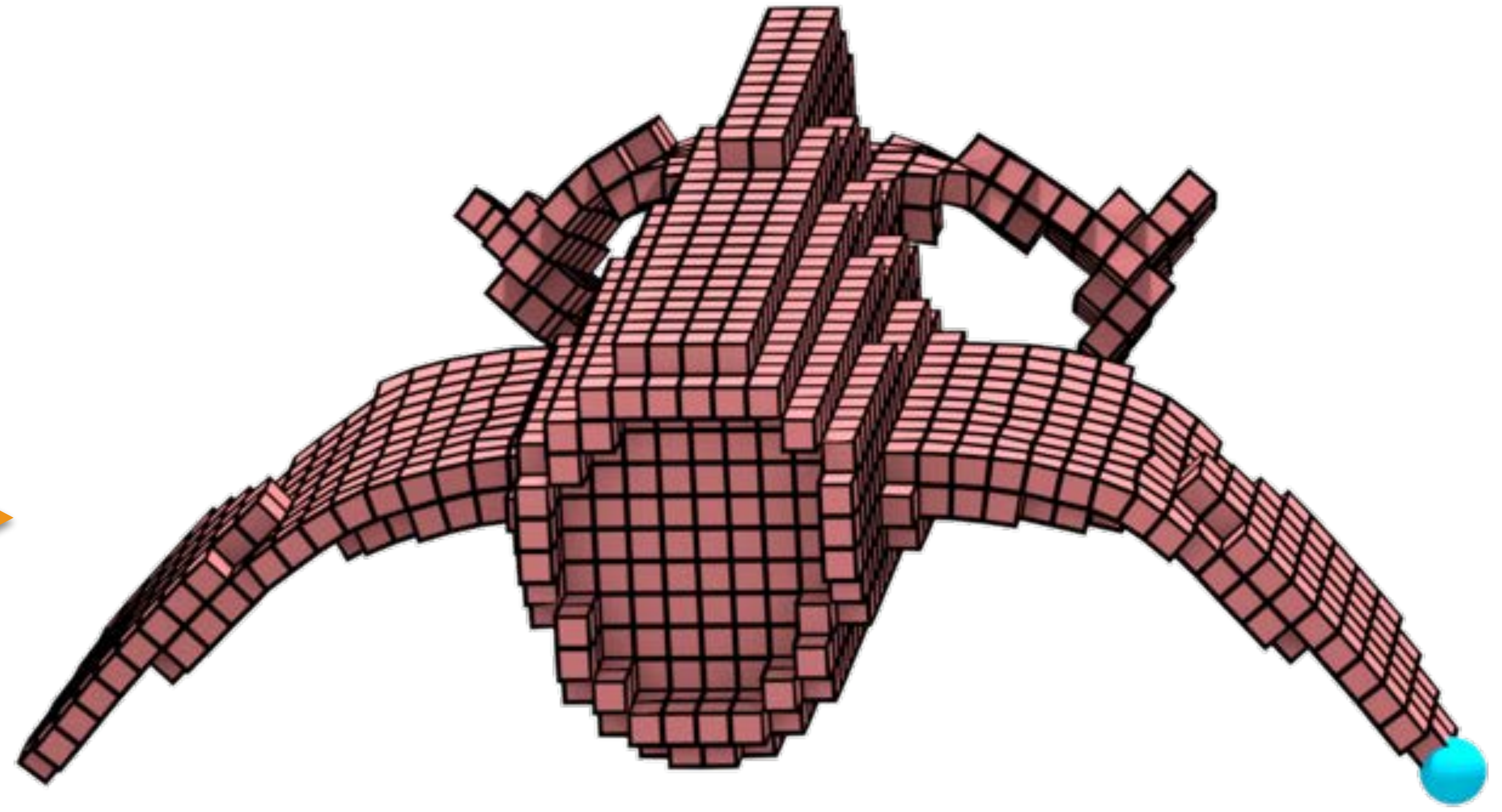
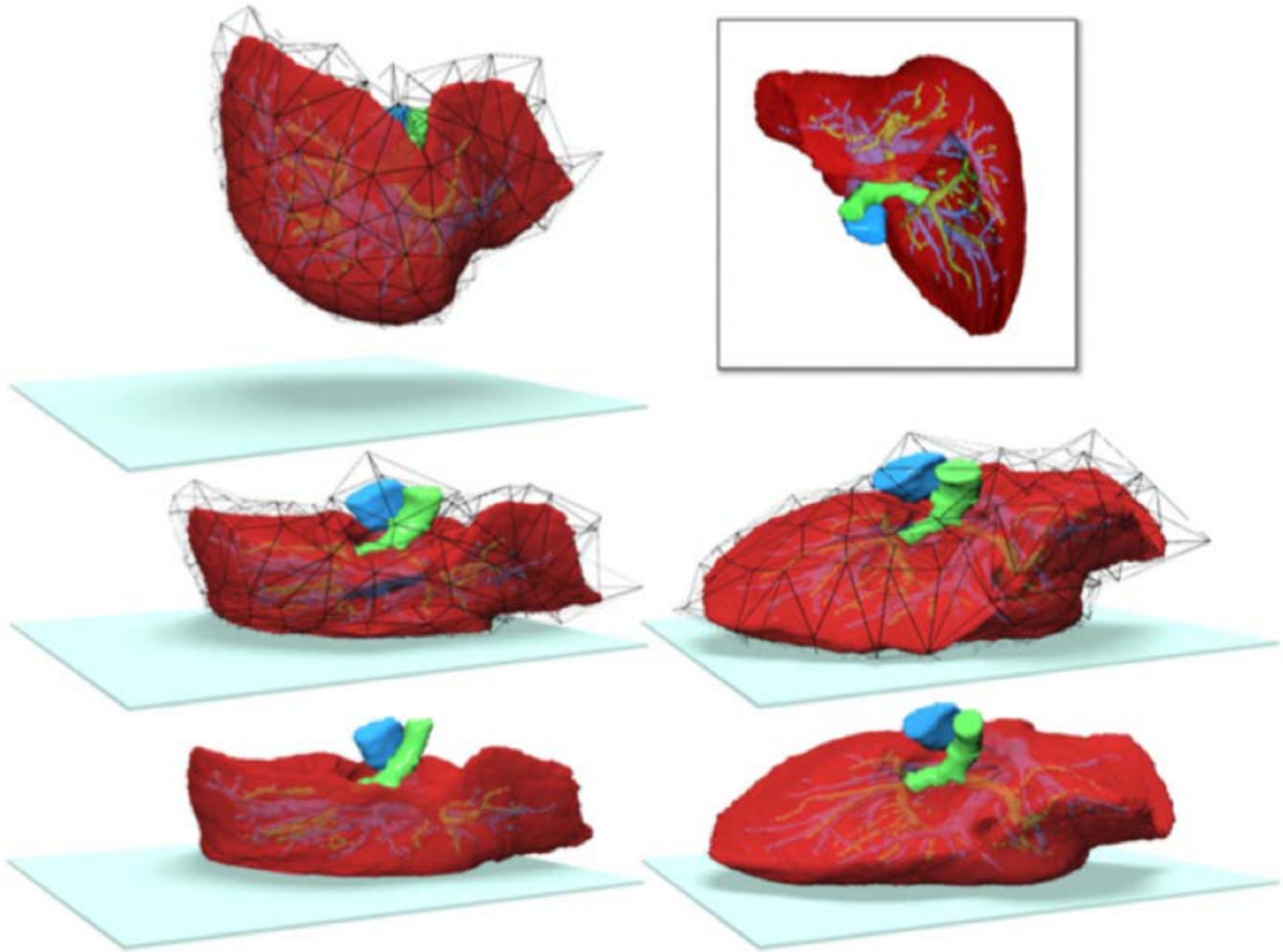
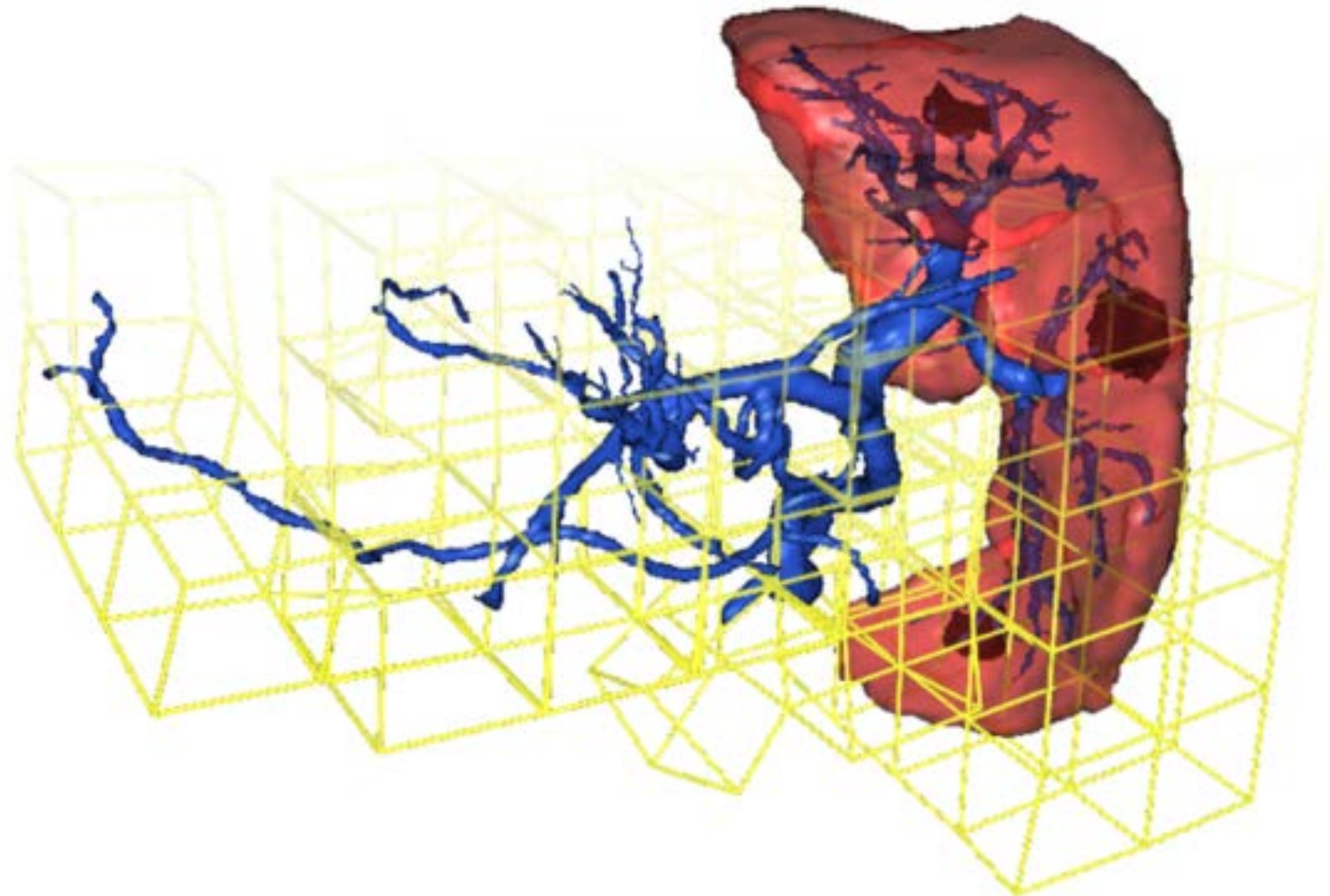


Coarse mesh

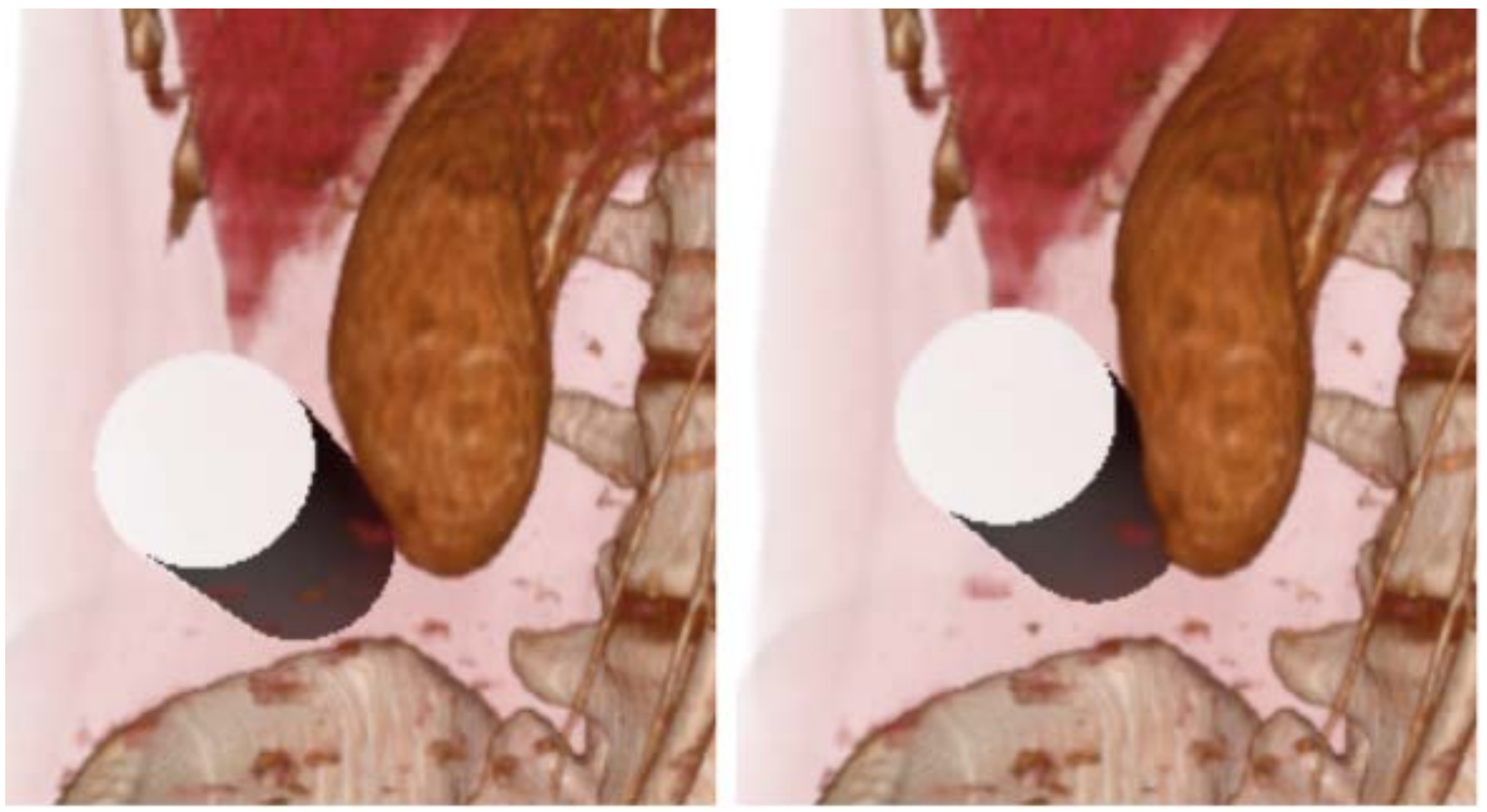
Previous works



[Kharevych 2009]

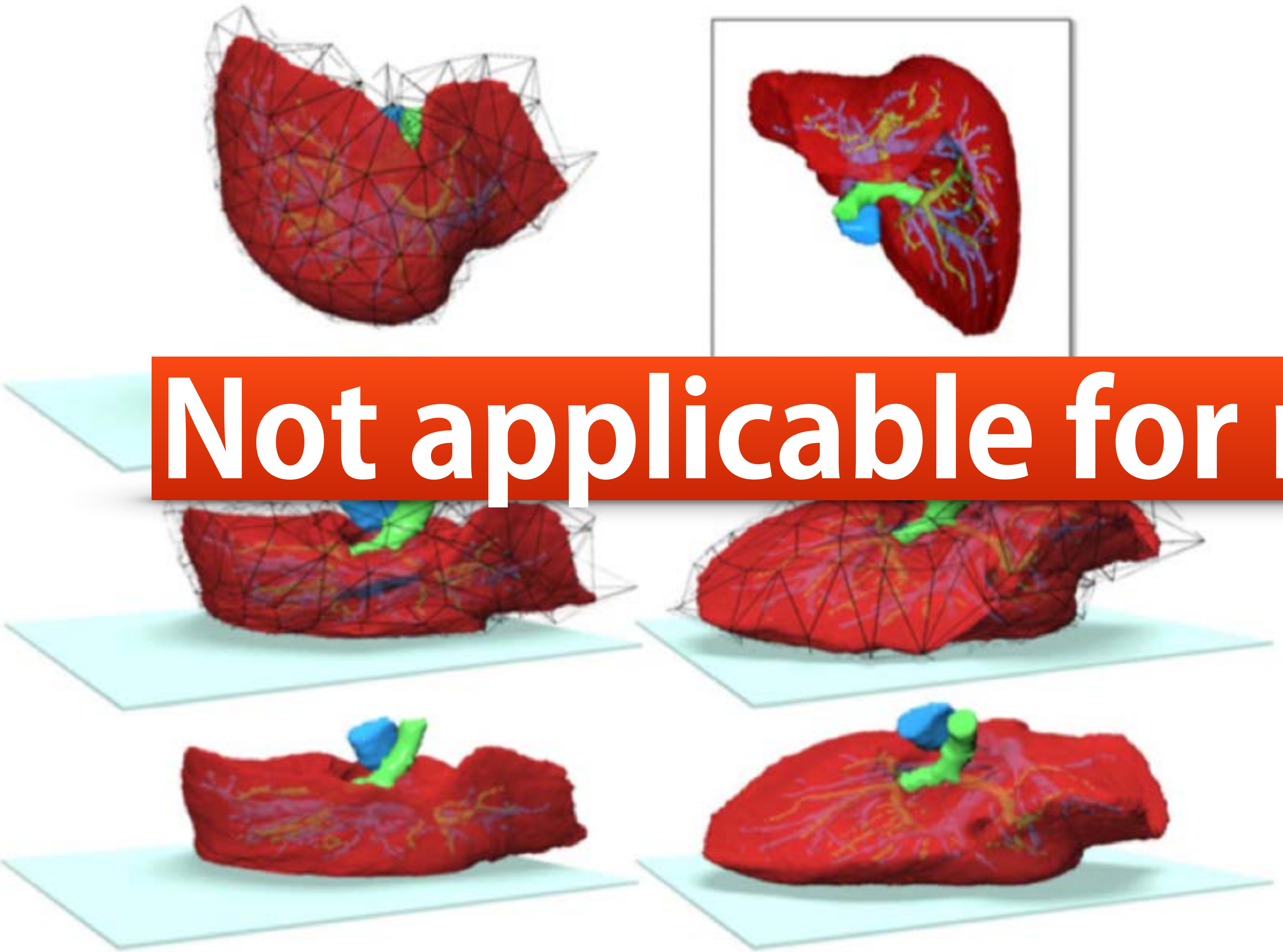


[Nesme 2009]



[Torres 2016]

Previous works



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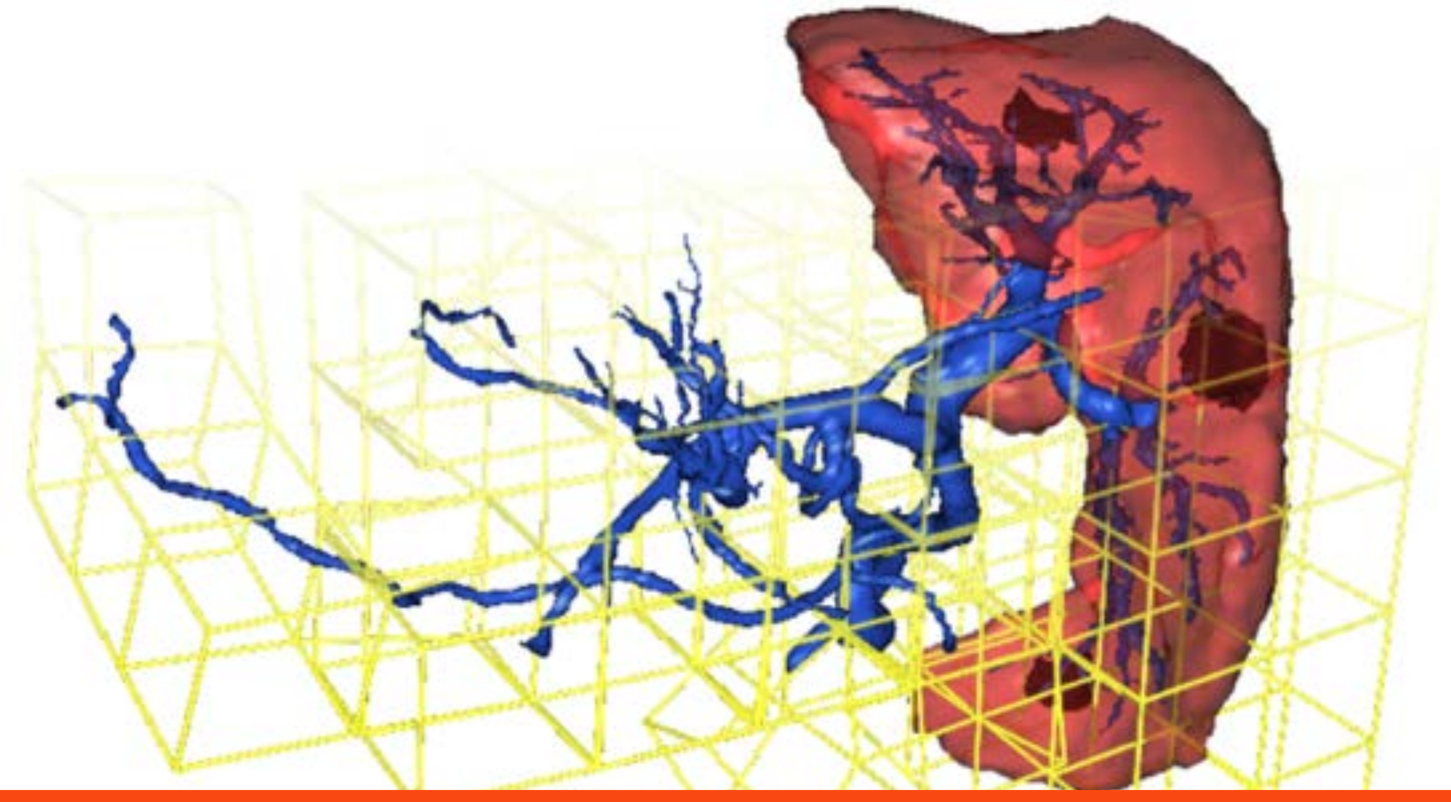
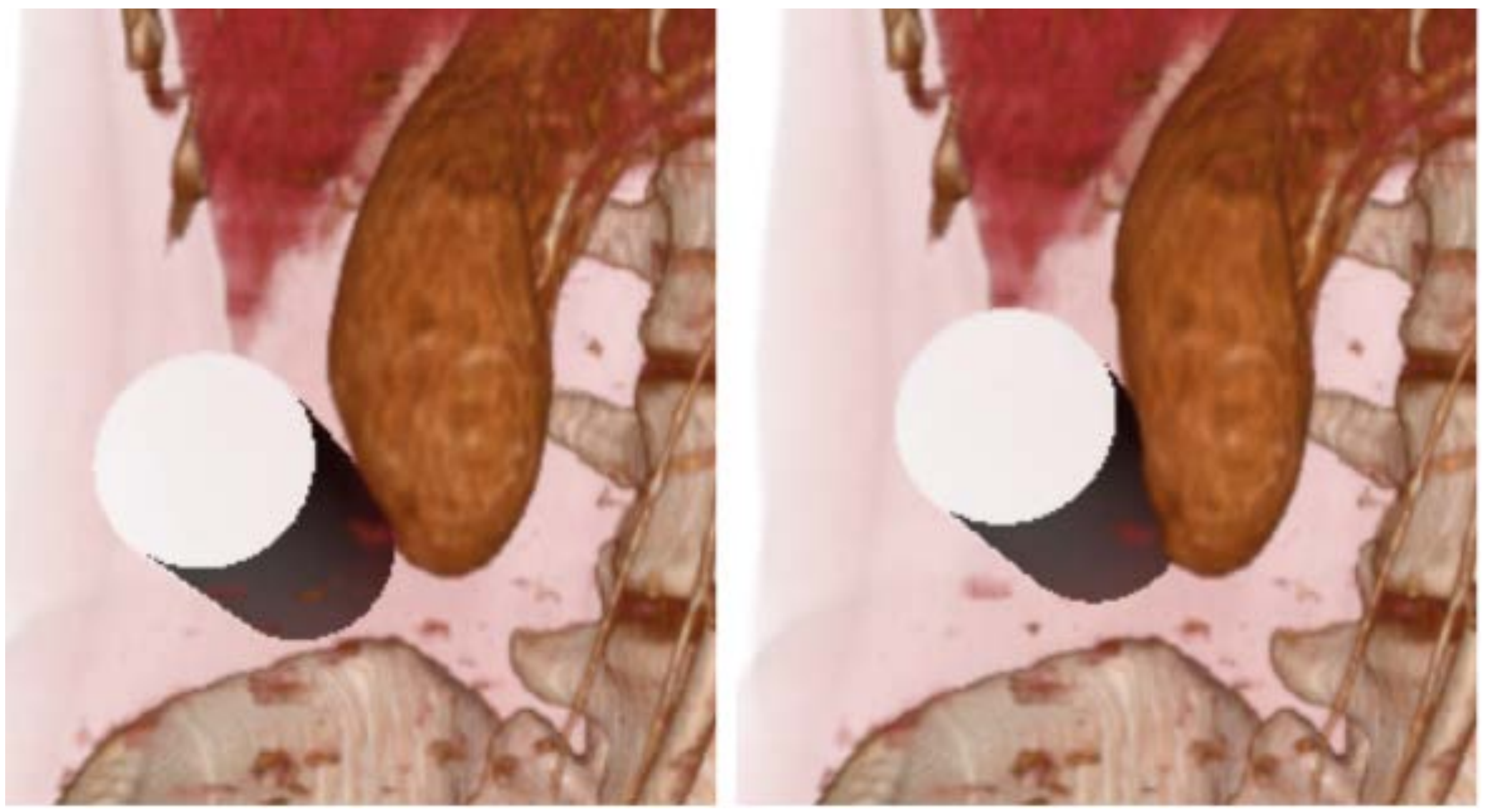


Not applicable for nonlinear elasticity

[Nesme 2009]



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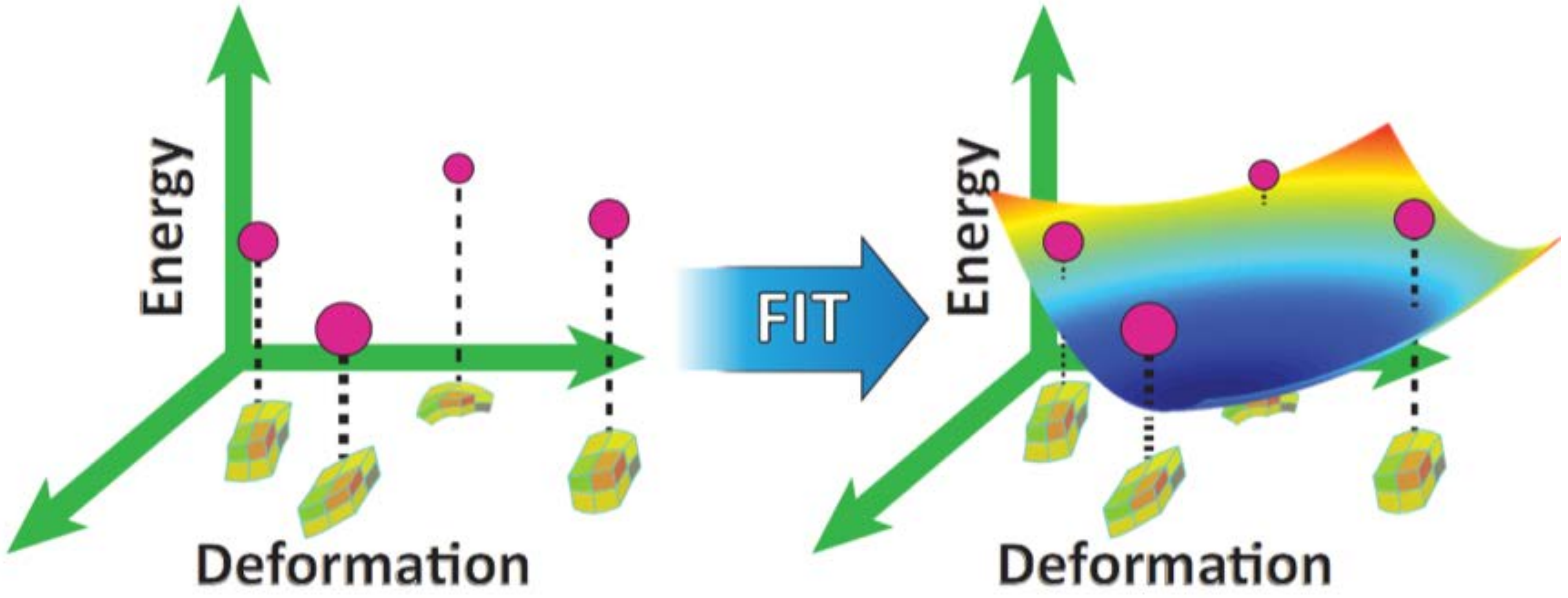
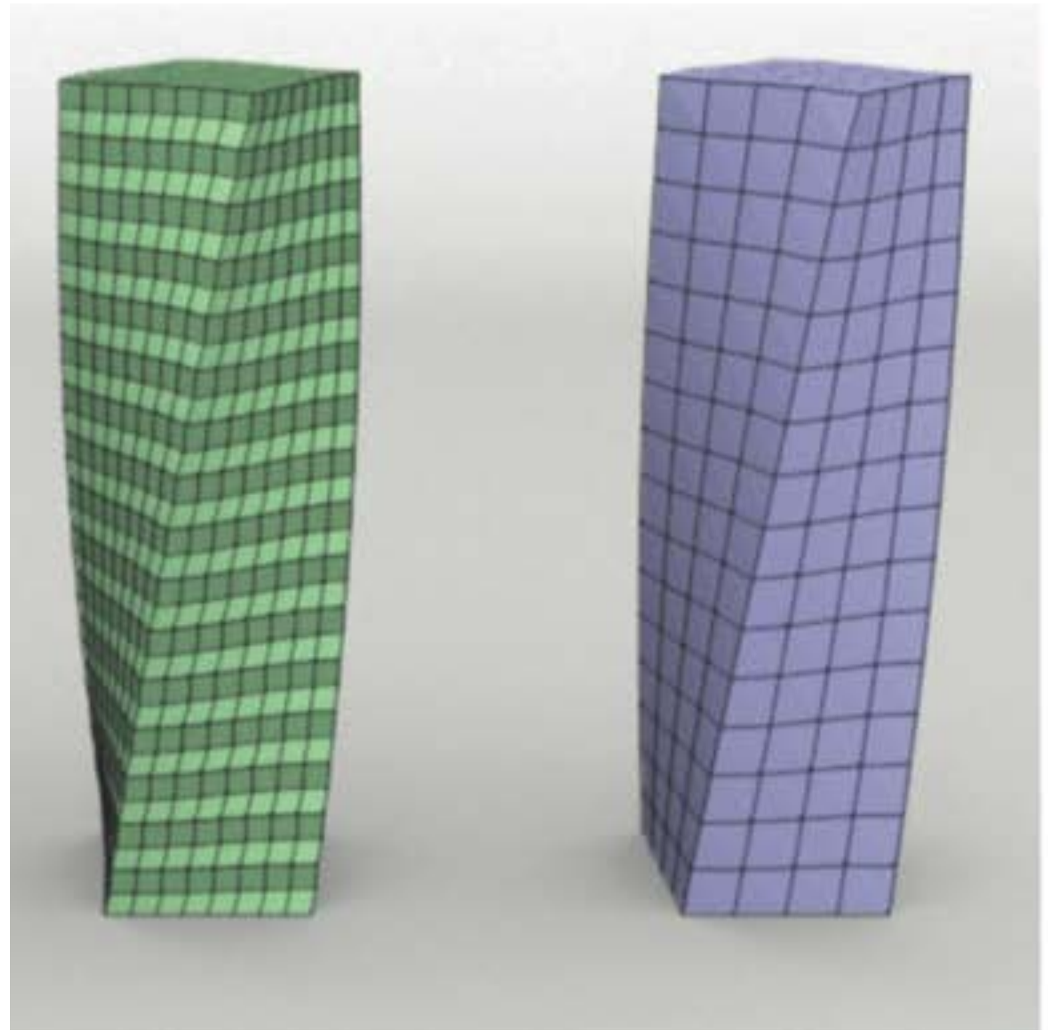
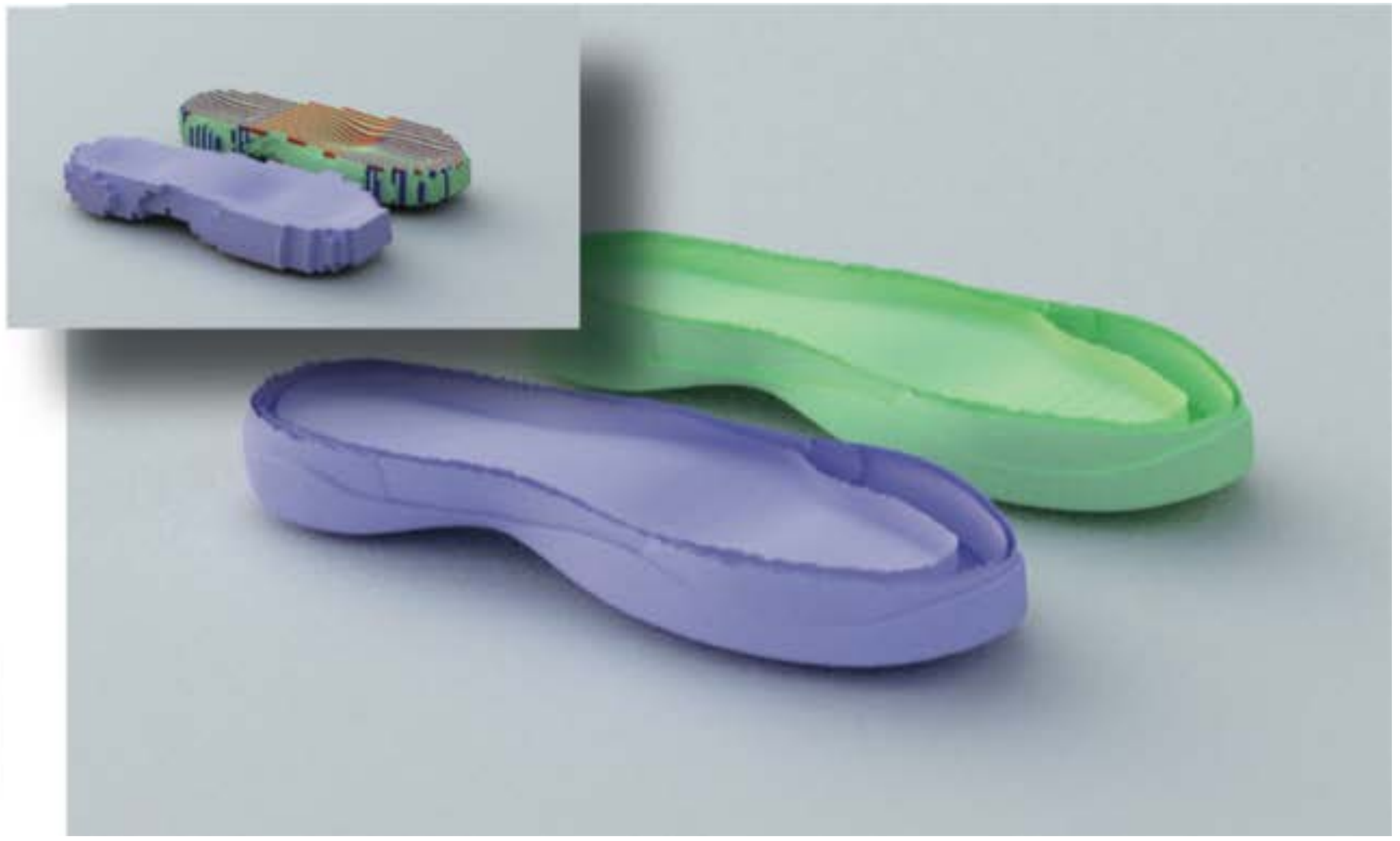


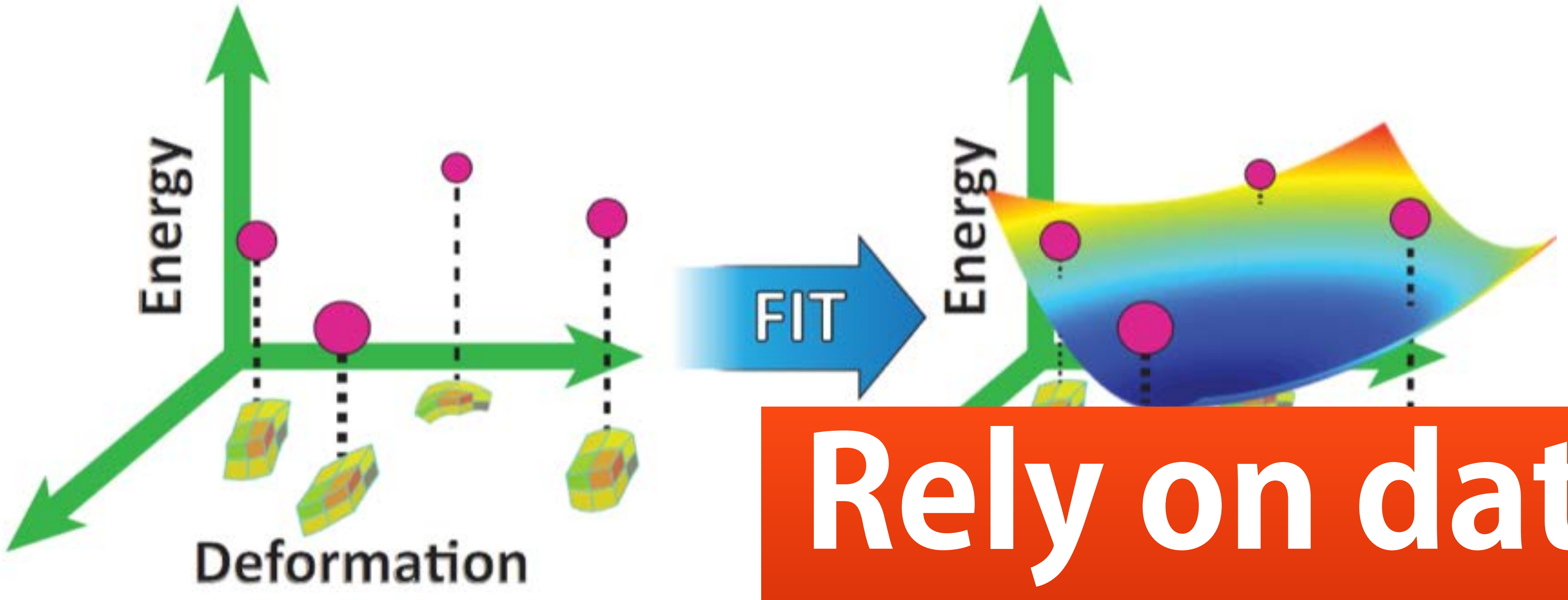
[Chen 2015]

Data-driven approach to regress the coarse elastic model





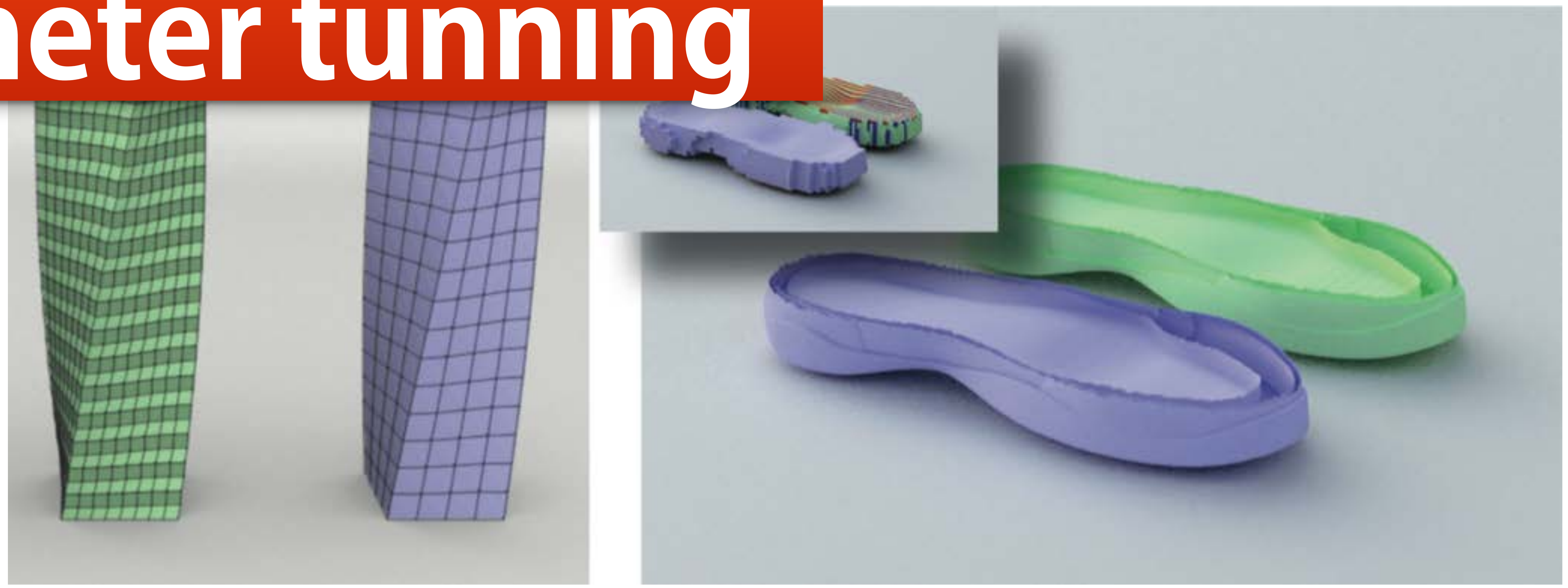
Previous work



Data-driven approach to regress the coarse elastic model

Rely on data set and parameter tunning

[Chen



Our solution

$$E[u] = \int_\Omega \Psi(\nabla u) dX$$

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$$\boxed{K_{ij}(u) = \int \nabla N_i^T : \frac{\partial^2 \Psi}{\partial \nabla u^2} : \nabla N_j}$$

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$$\cancel{E[u] = \int_\Omega \Psi(\nabla u)\, dX} \qquad \nabla u = \sum_i \nabla N_i(X) u_i$$

 **Homogenize the constitutive model**

Our solution

$$\cancel{E[u] = \int_\Omega L(\nabla u)\, dX}$$

$$\nabla u = \sum_i \boxed{\nabla N_i}(X) u_i$$

 **Homogenize the constitutive model**

 **Approximate the solution space better**

Our solution

- ***Matrix-valued*** shape functions

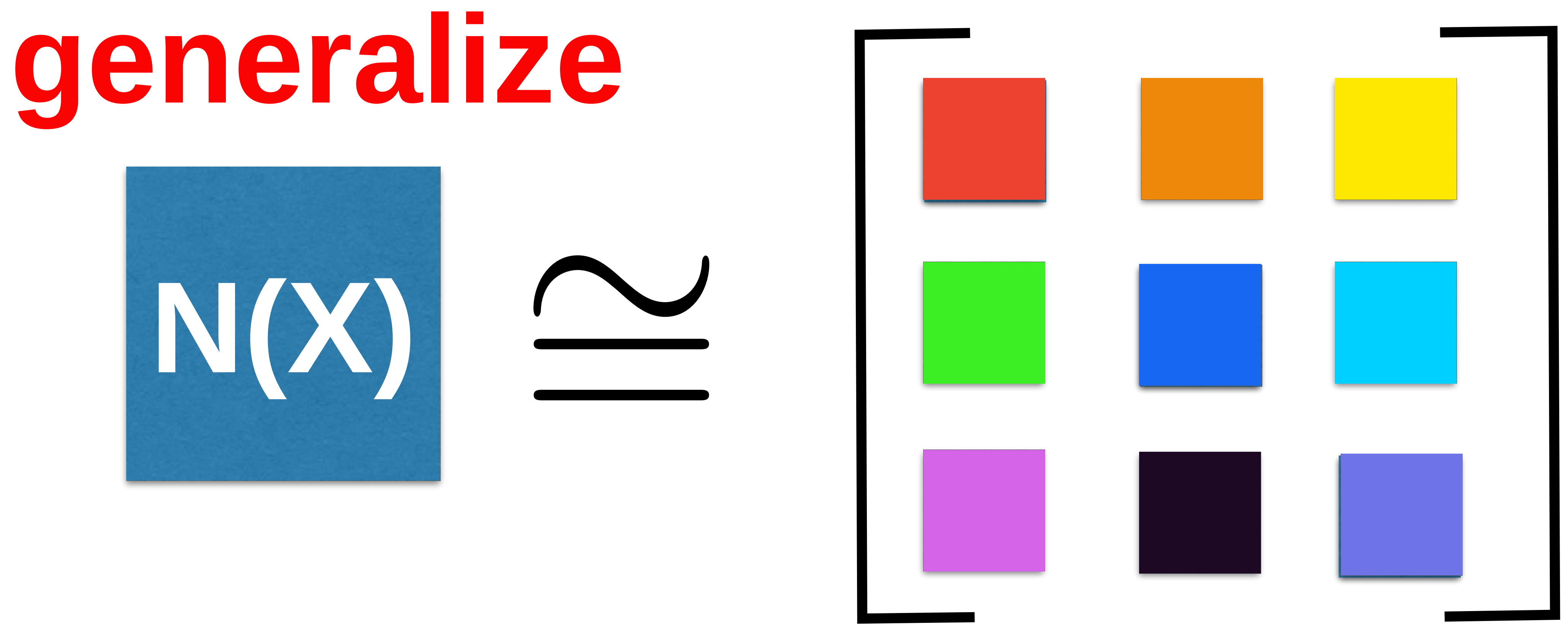
scalar basis

$$\boxed{N(X)} \cong \begin{bmatrix} \blacksquare & 0 & 0 \\ 0 & \blacksquare & 0 \\ 0 & 0 & \blacksquare \end{bmatrix}$$

Our solution

- ***Matrix-valued*** shape functions

$$N_i^H : \Omega \to \mathbb{R}^{d\times d}$$



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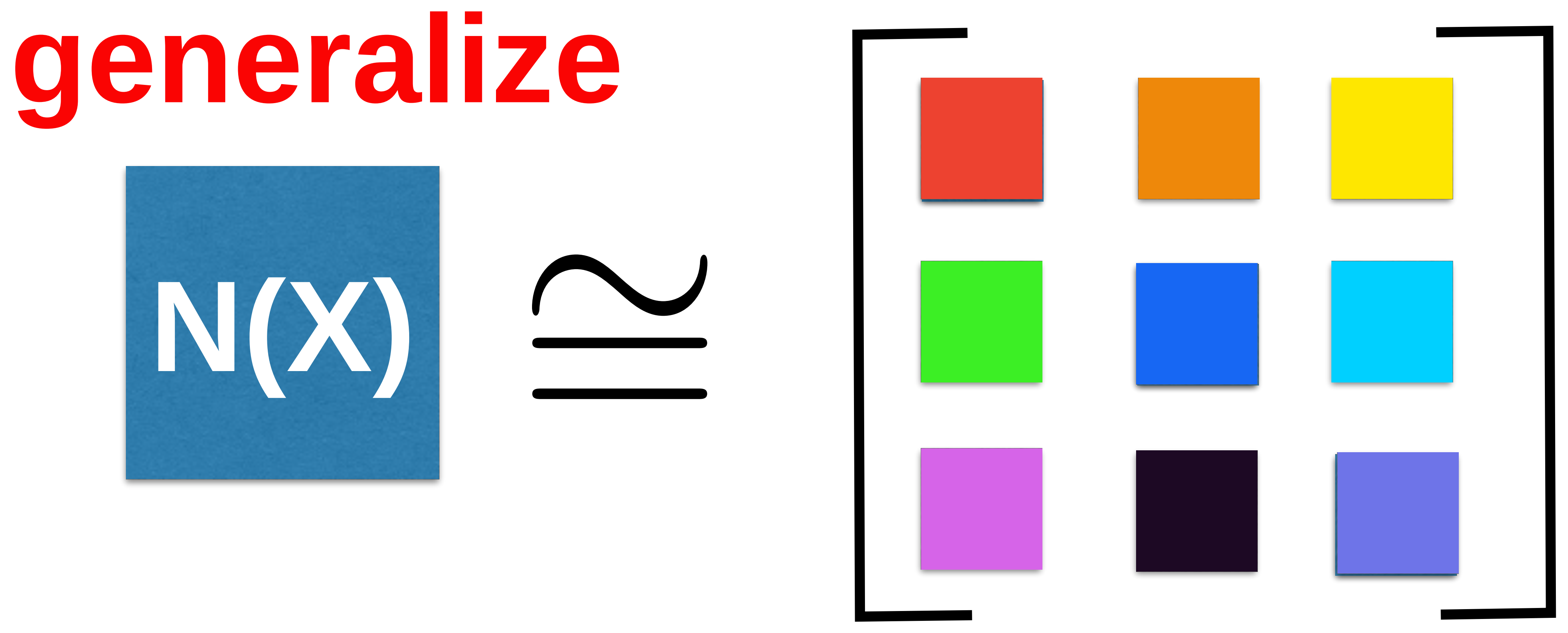
- Geometric & physical conditions



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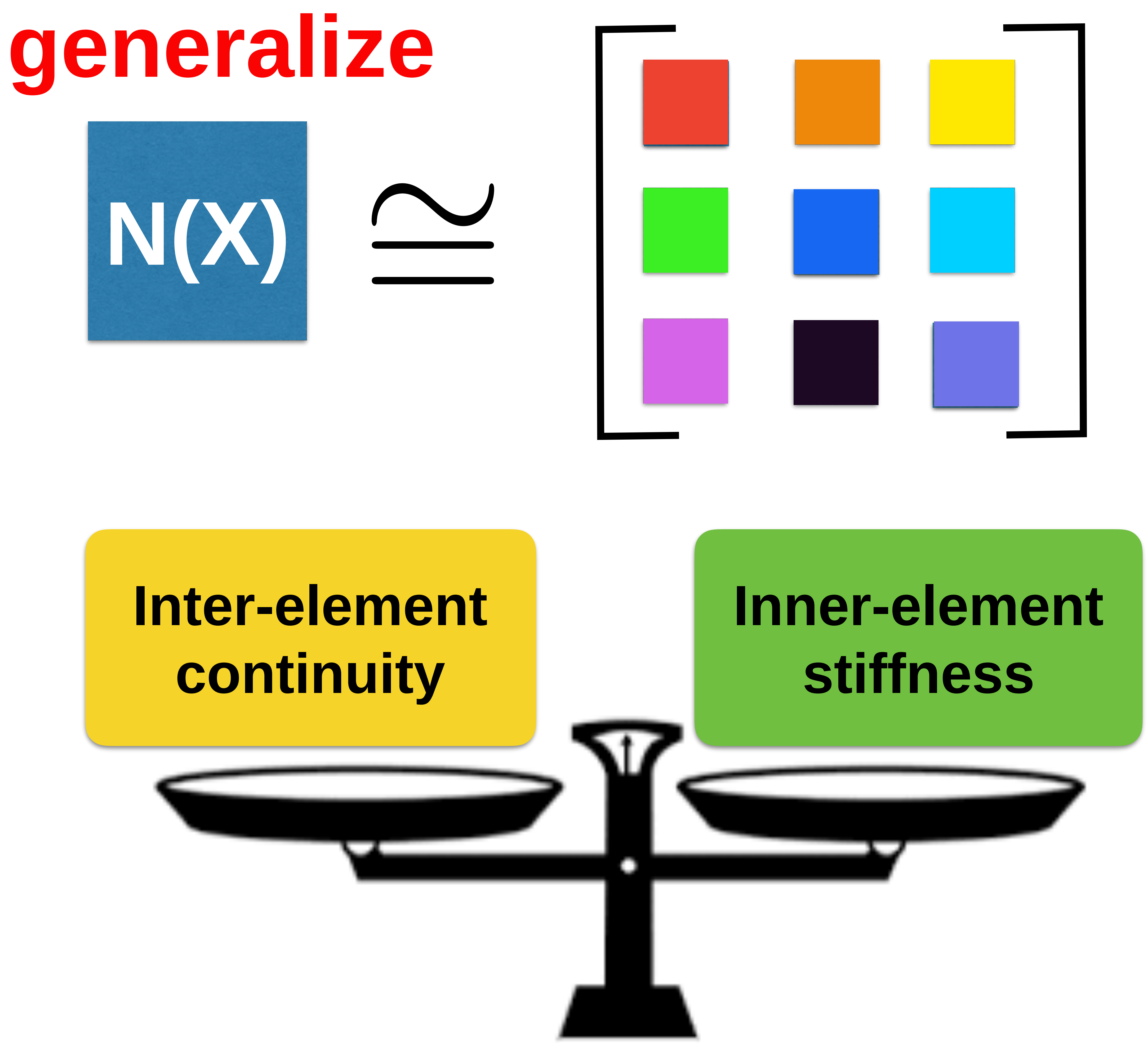
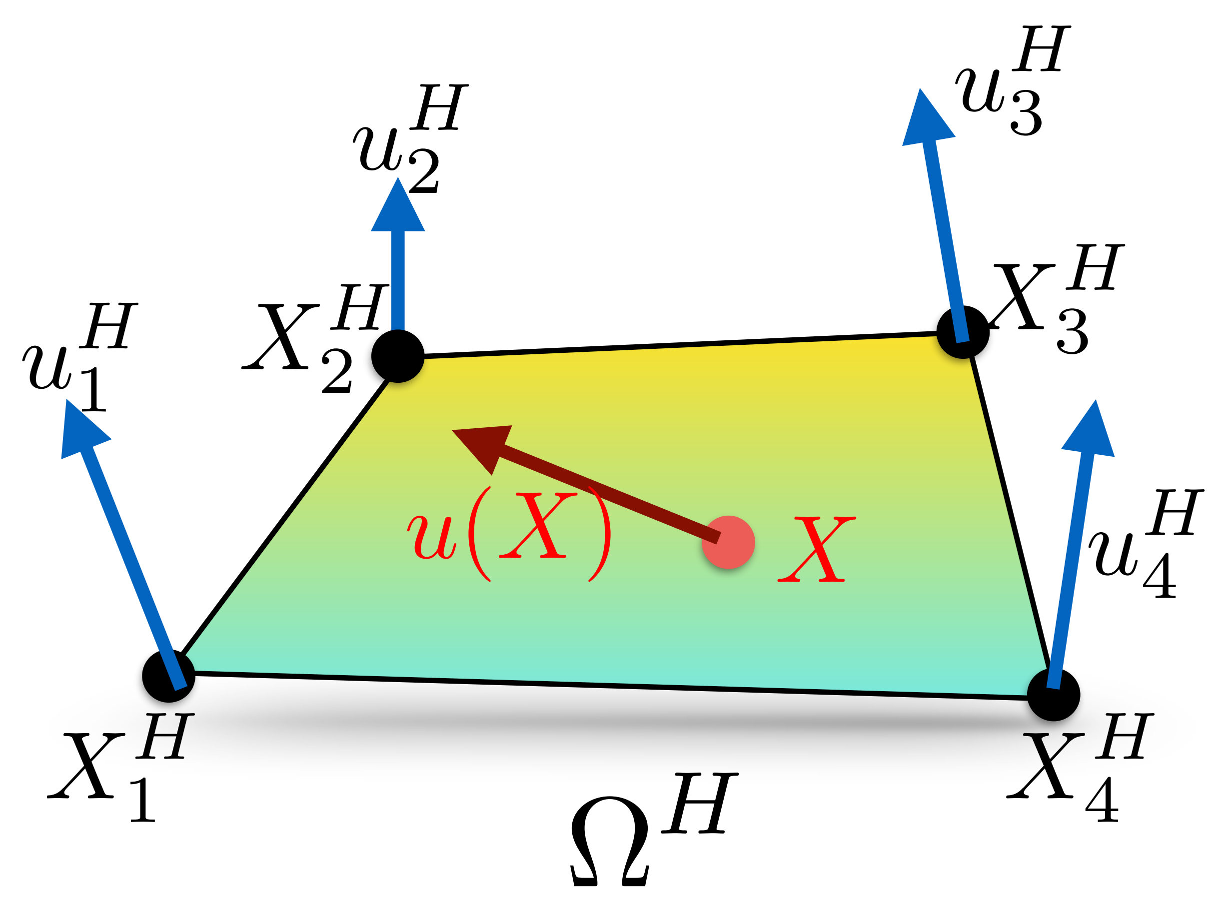


Matrix-valued shape function

- Element-wise interpolation

$$\forall X \in \Omega^H, u(X) = \sum_{X_i \in \Omega^H} N_i^H(X) u_i^X$$



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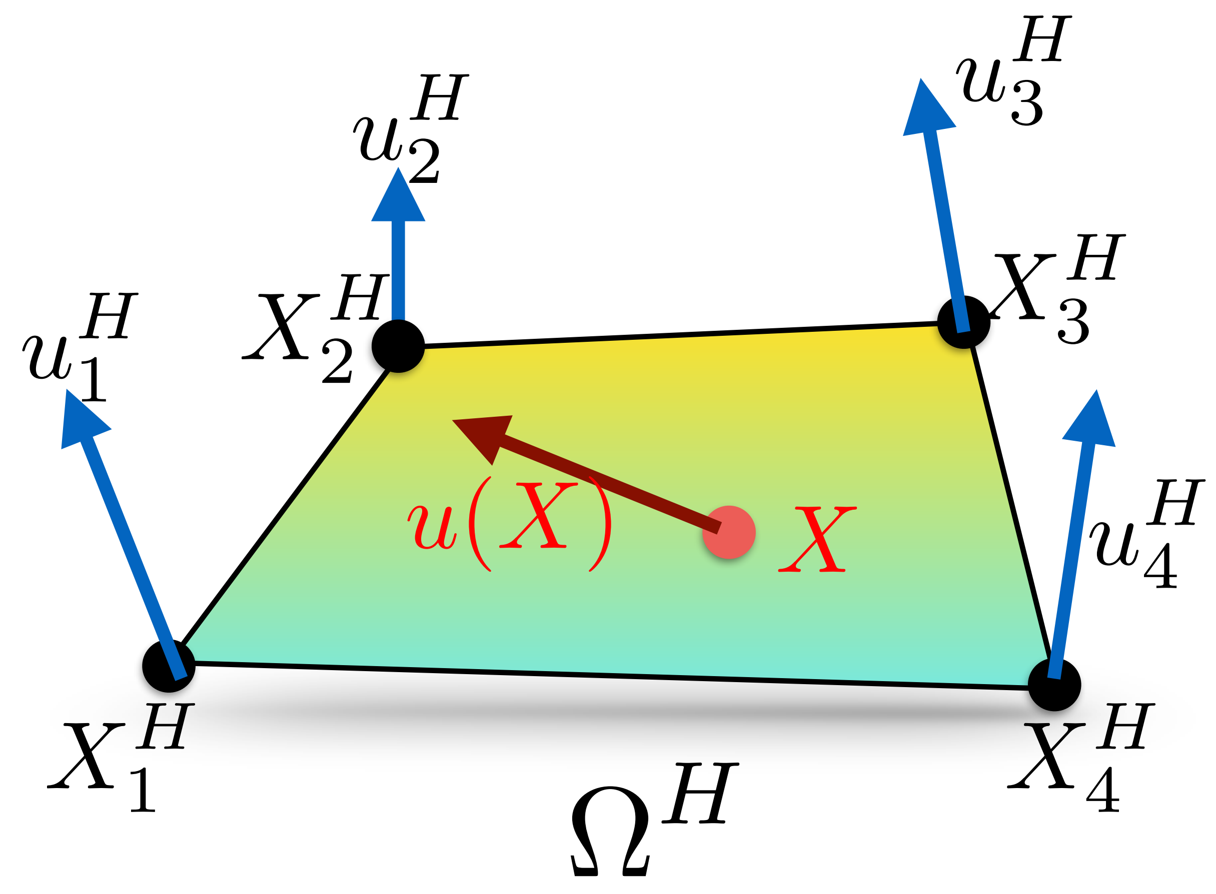
$$u(X) = R\left[X + \sum_i N_i^H(X)(R^T x_i^H - X_i^H)\right] - X$$



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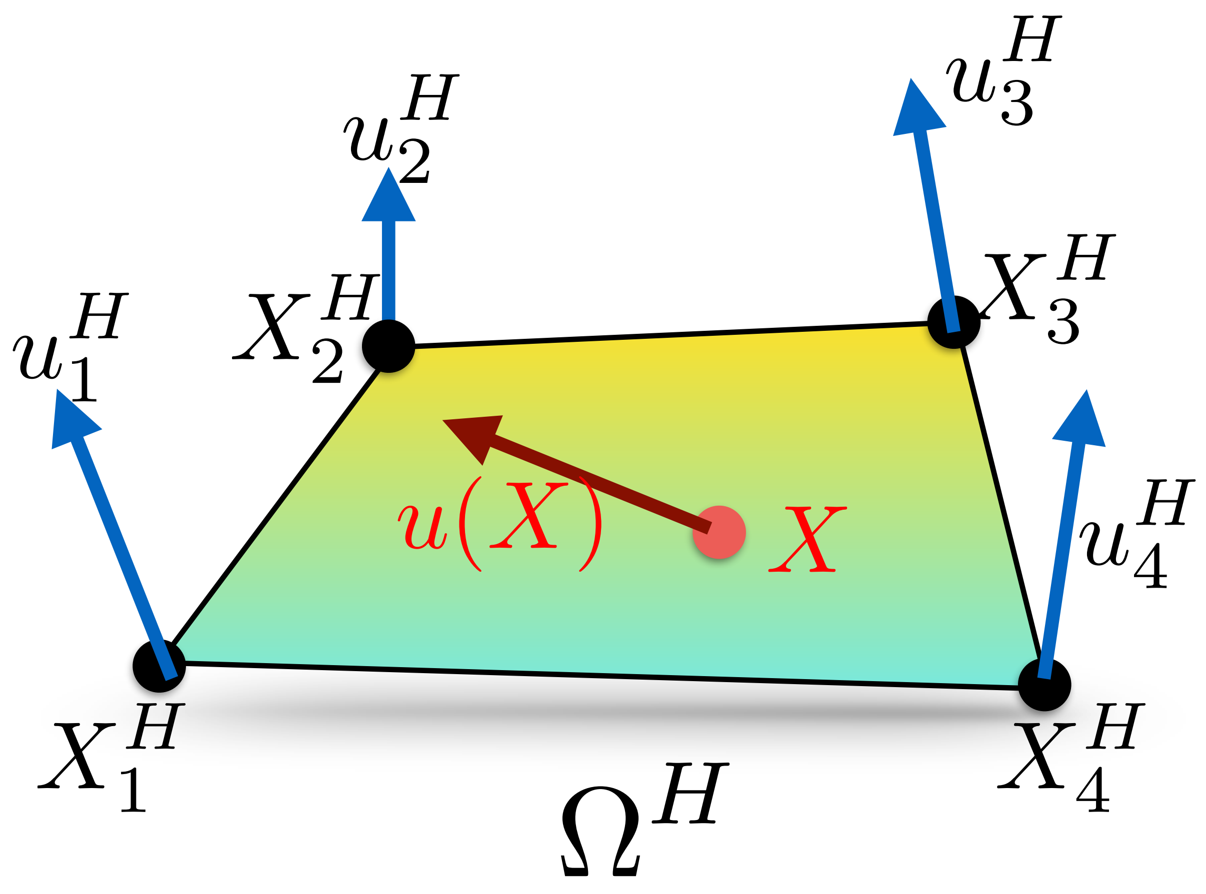

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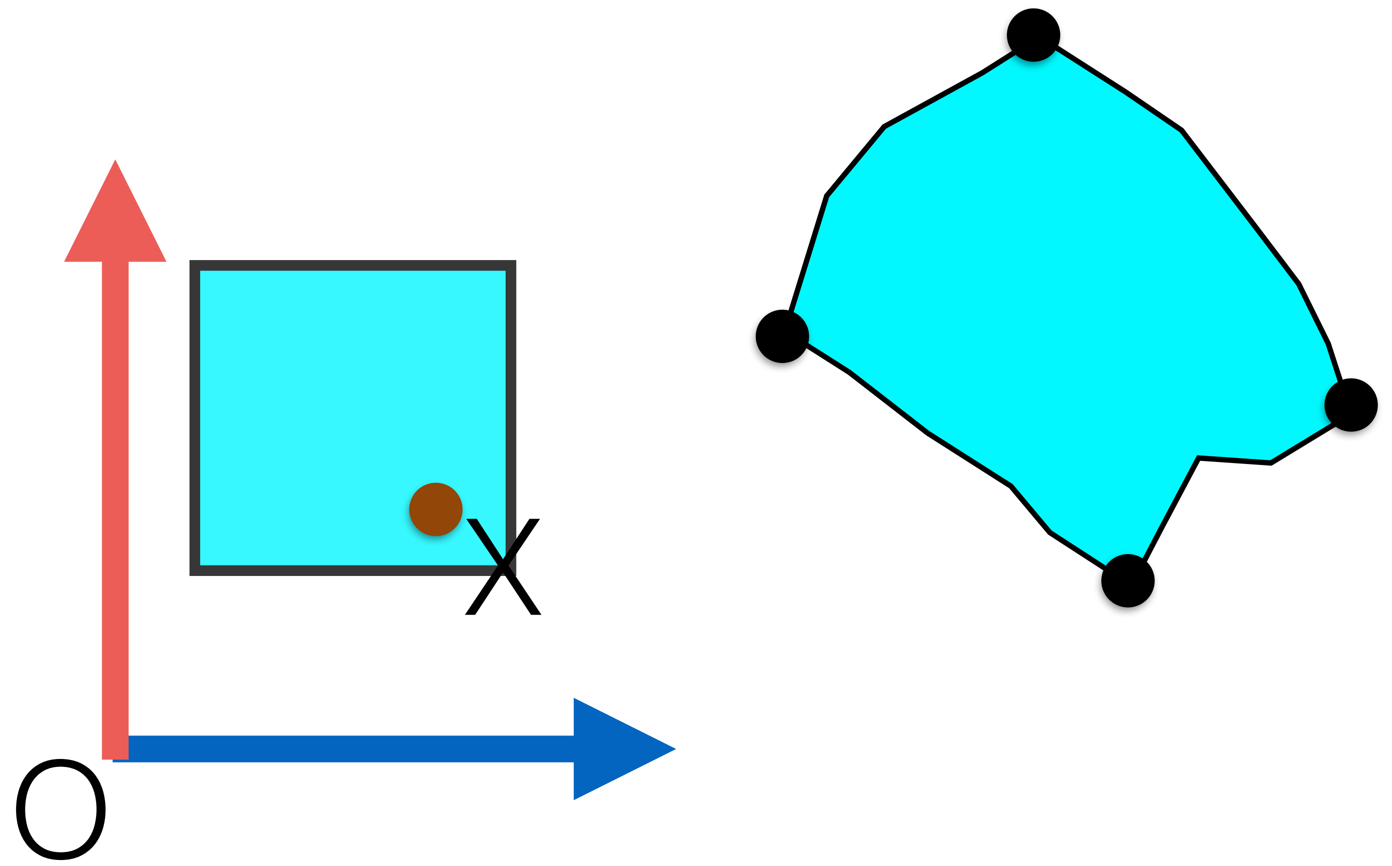

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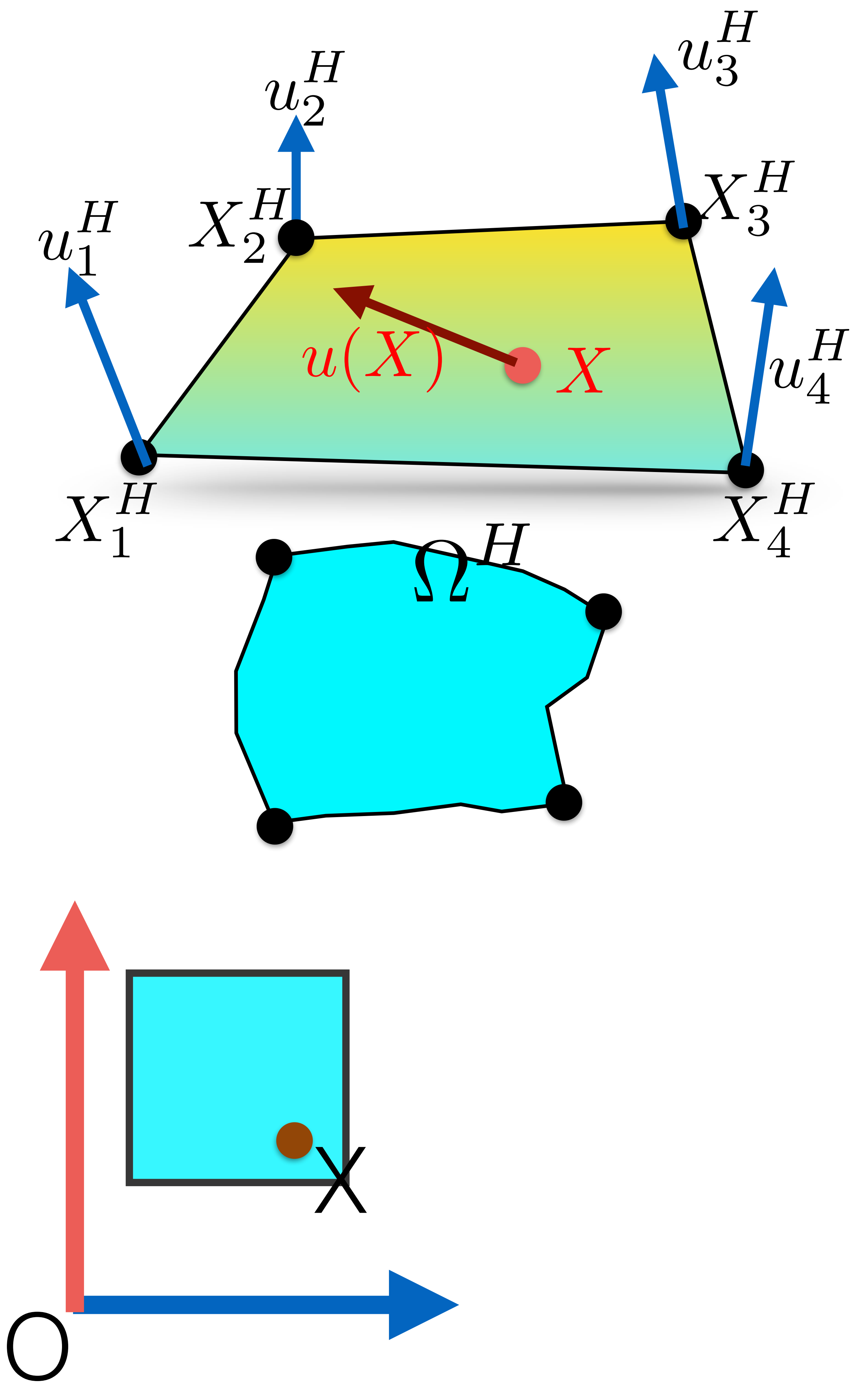

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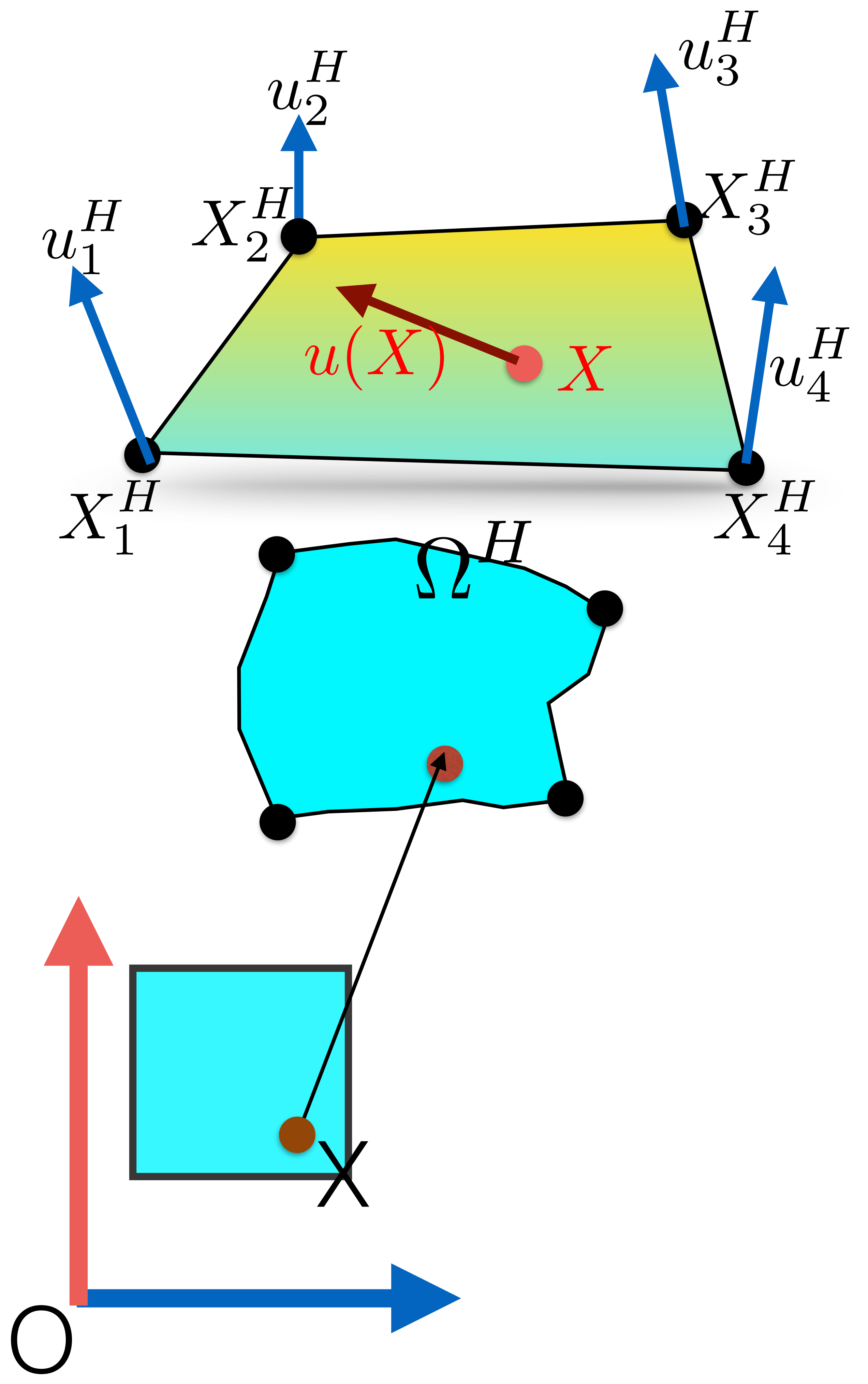

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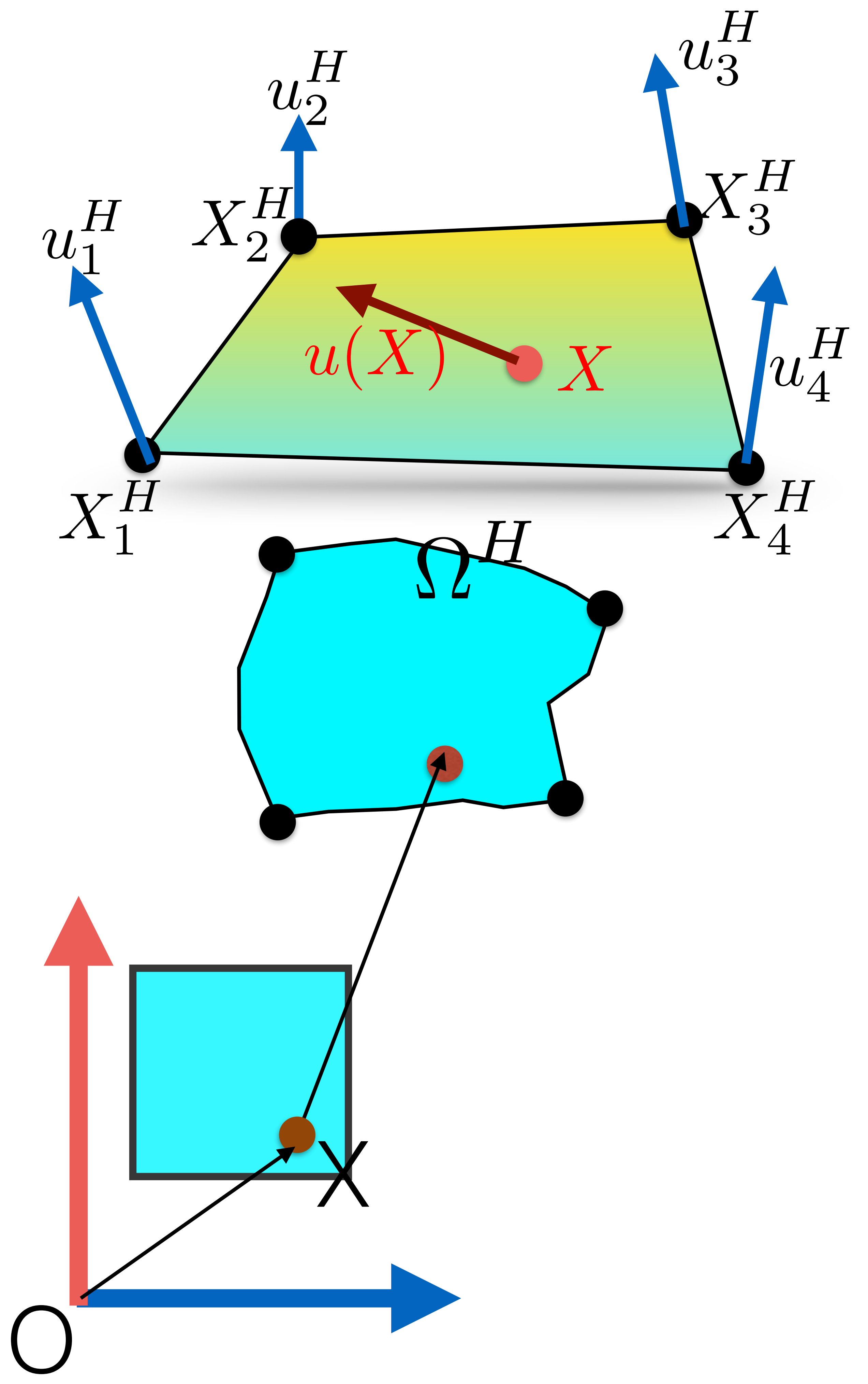

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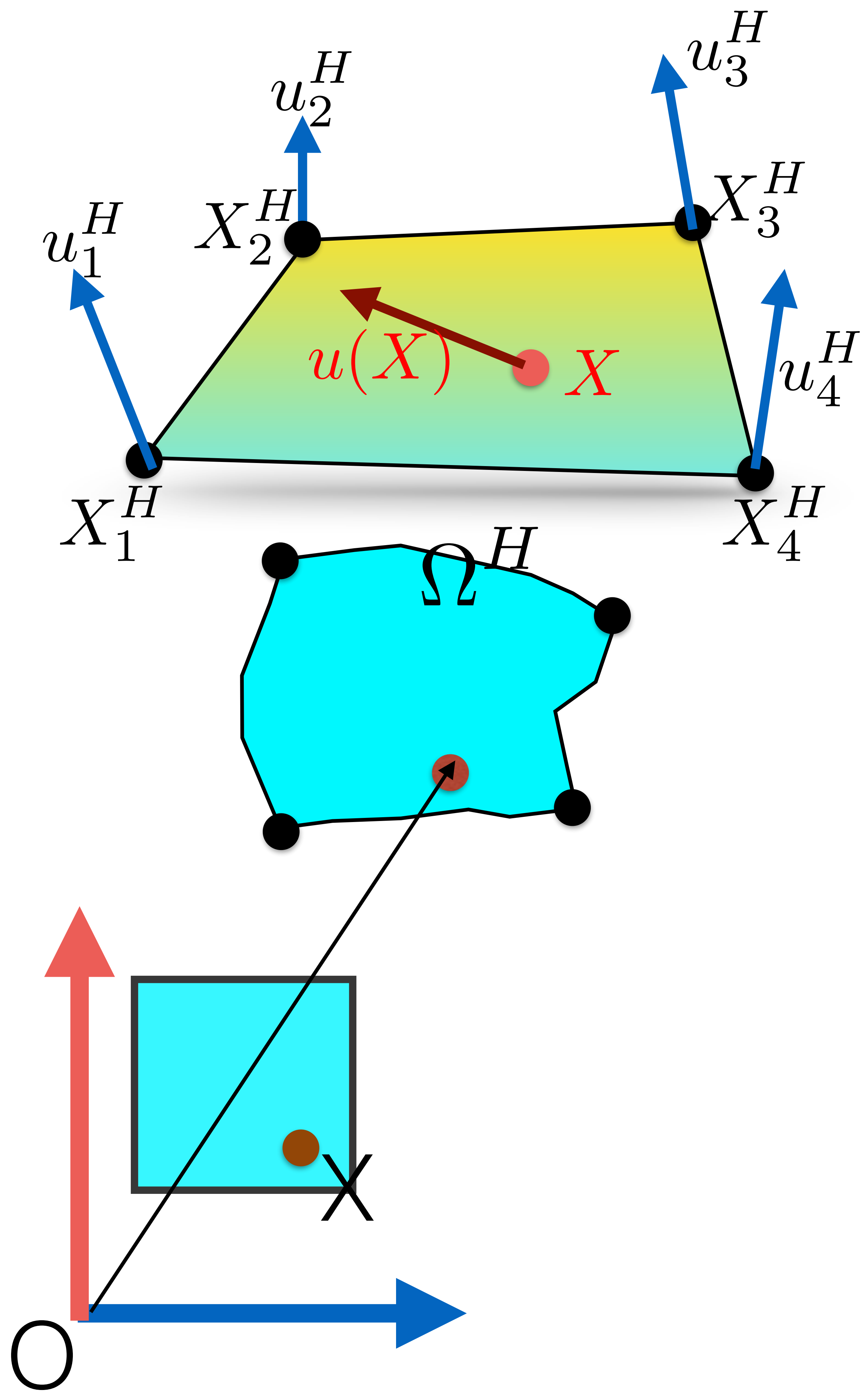

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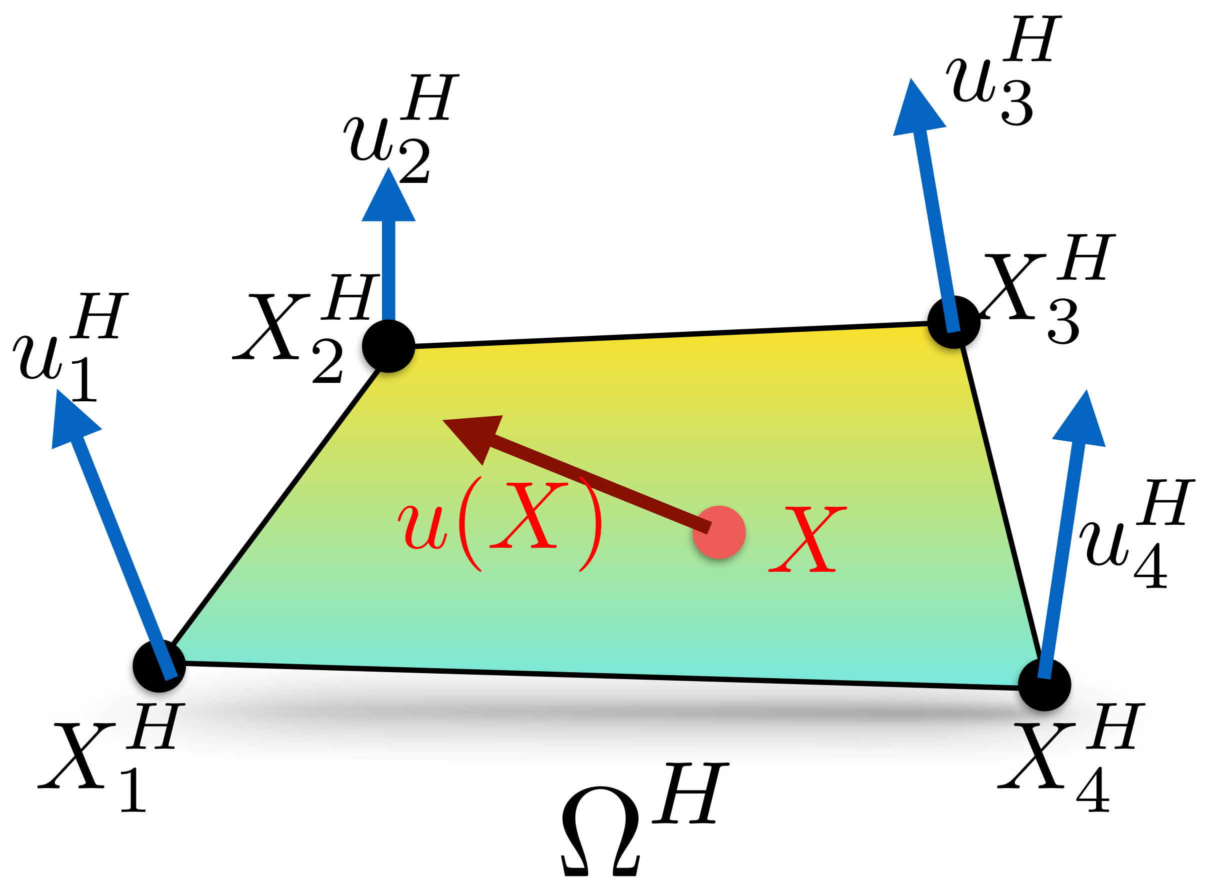

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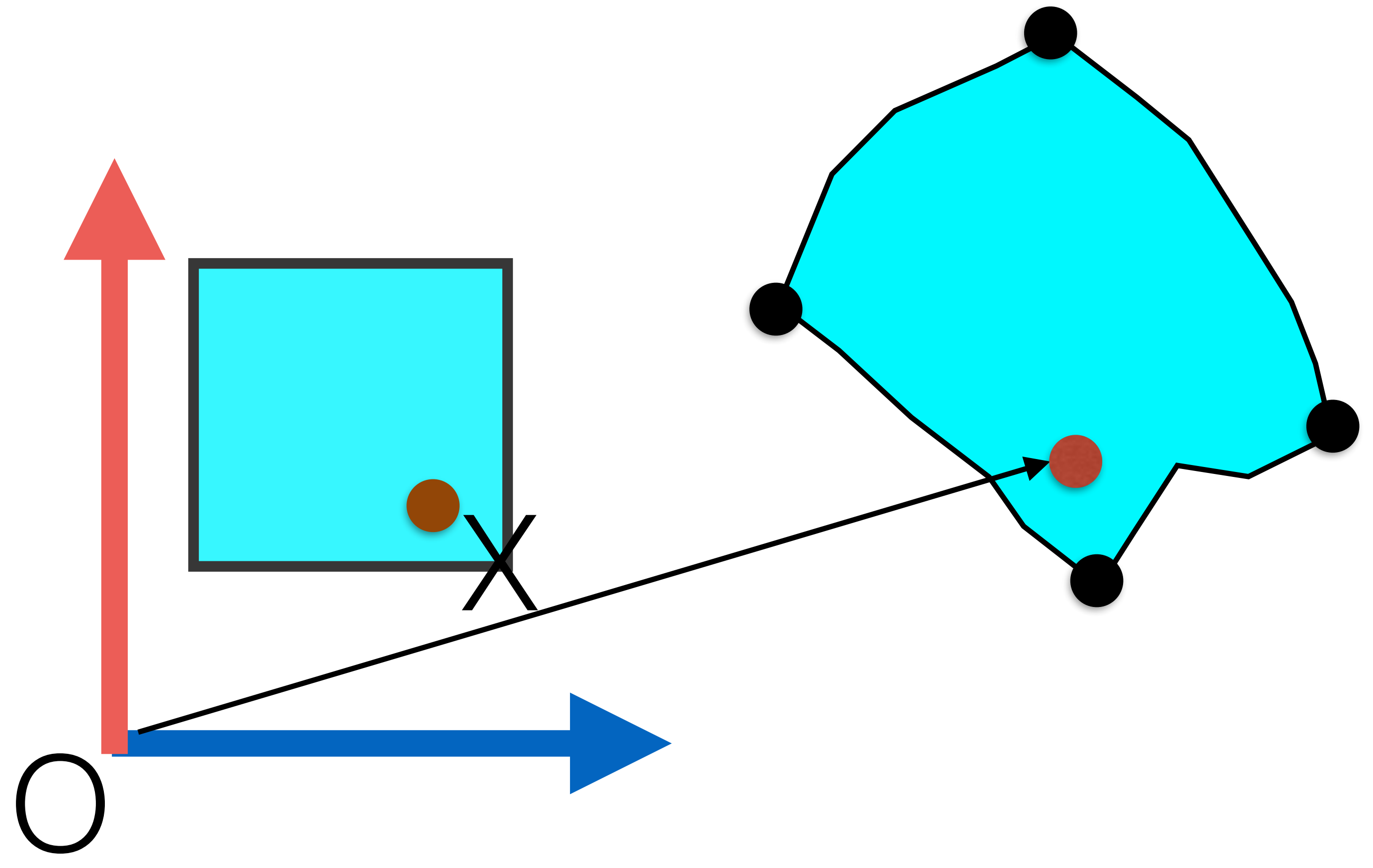

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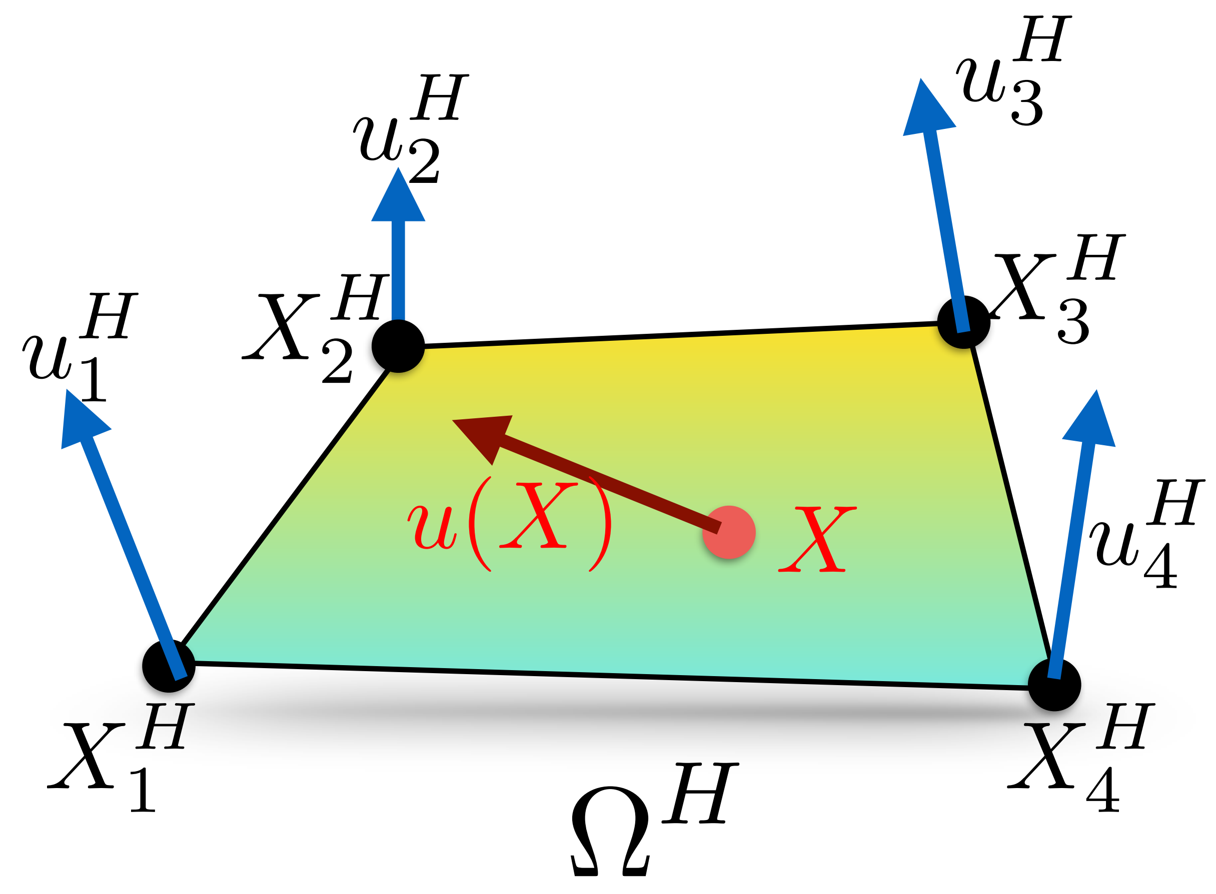

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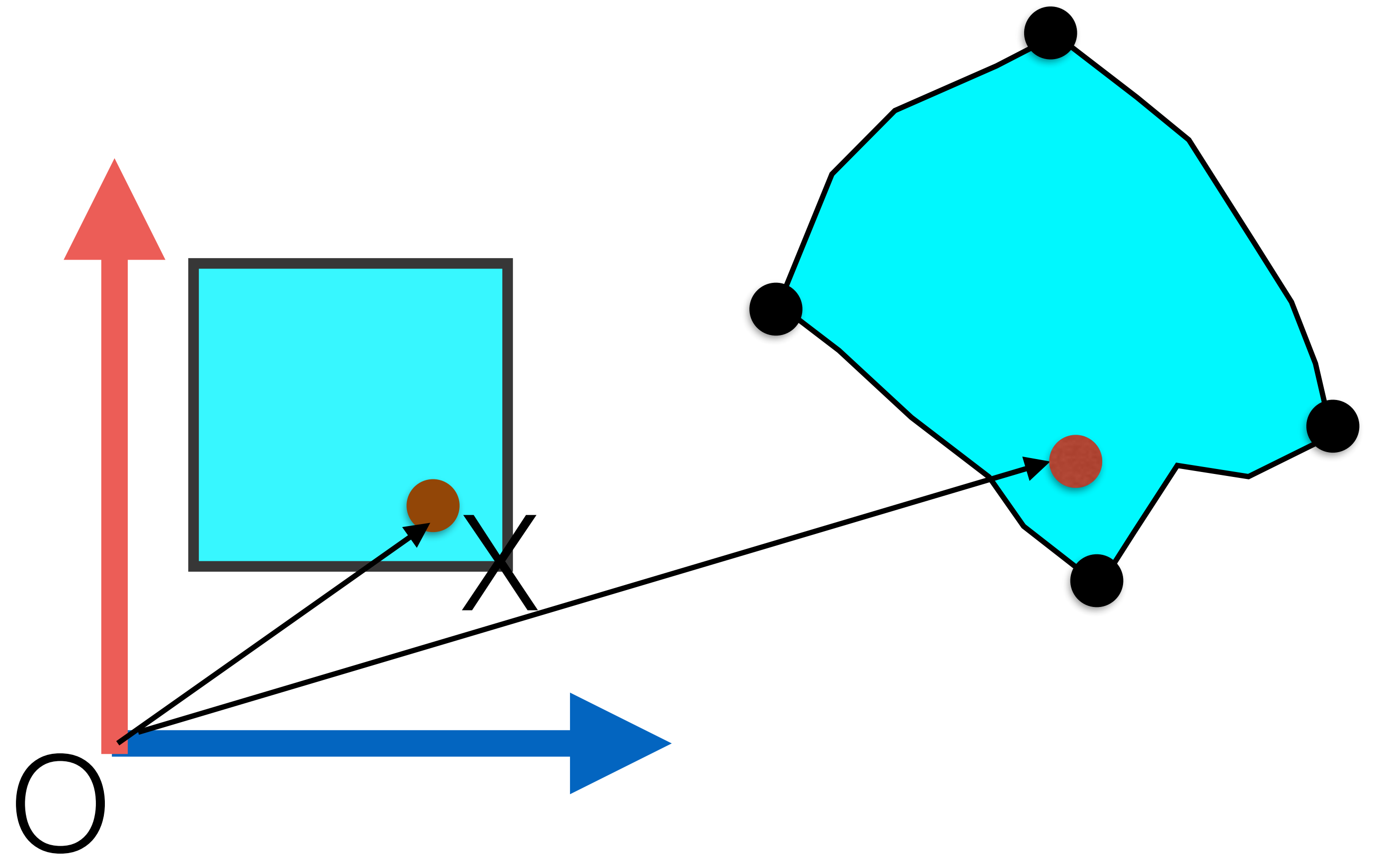

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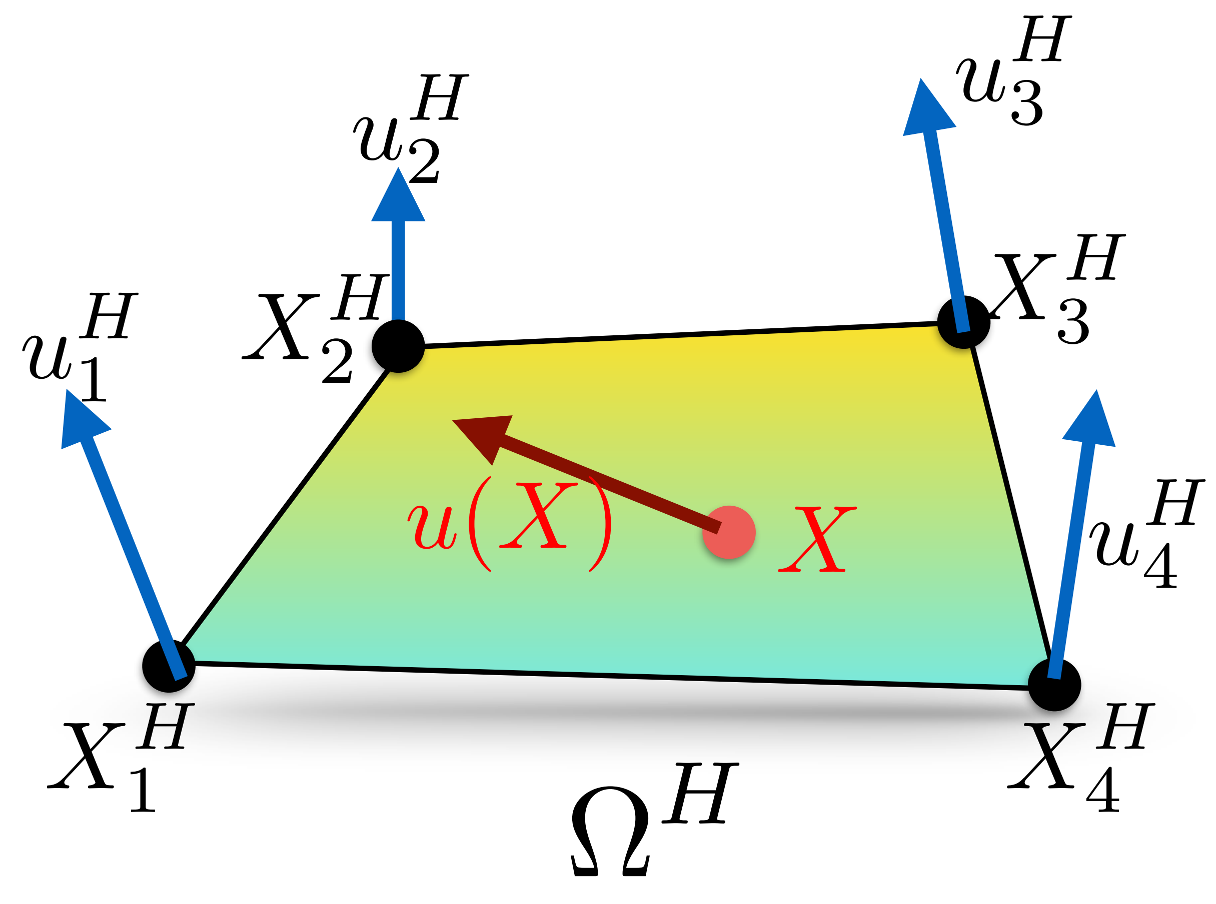

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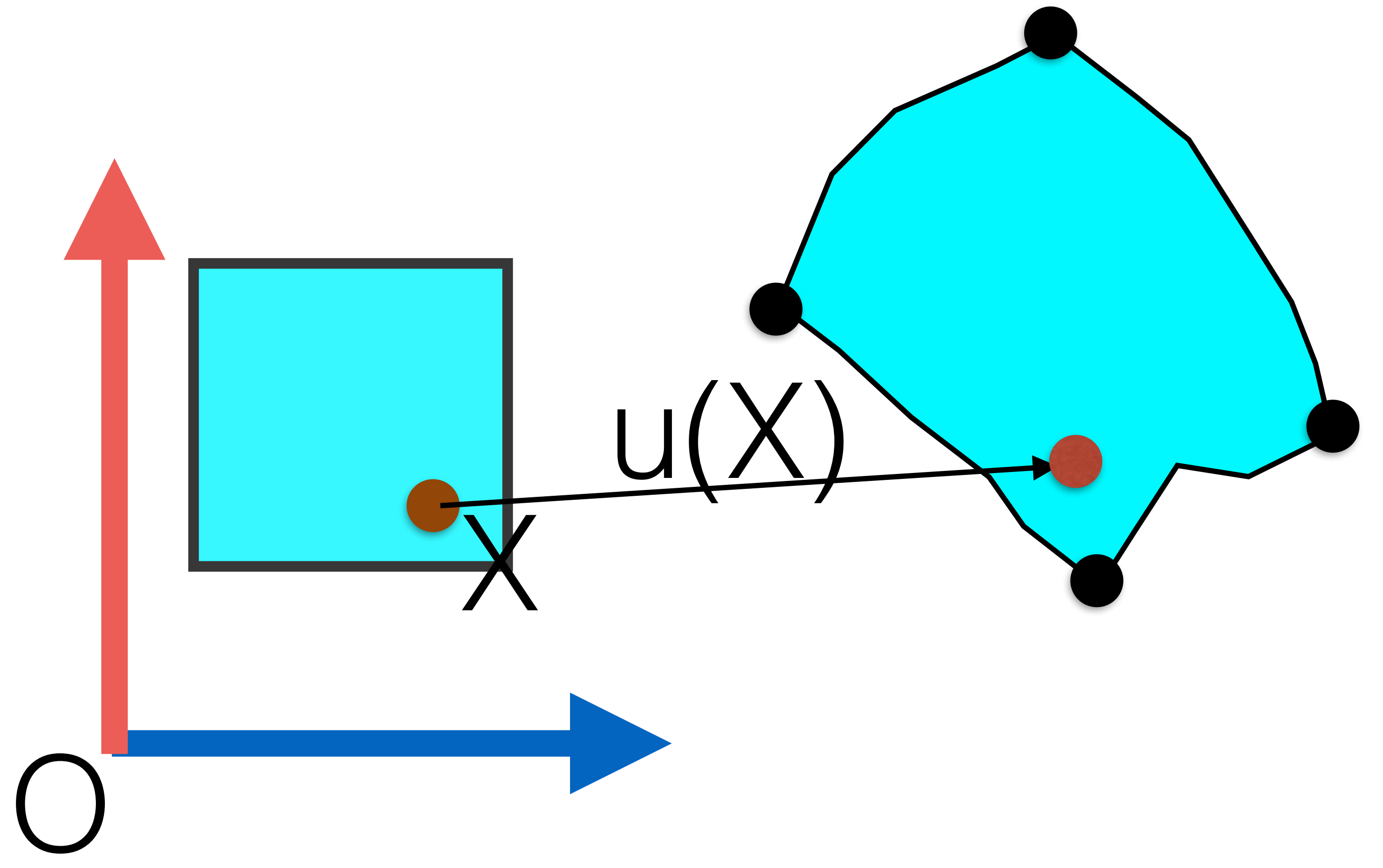
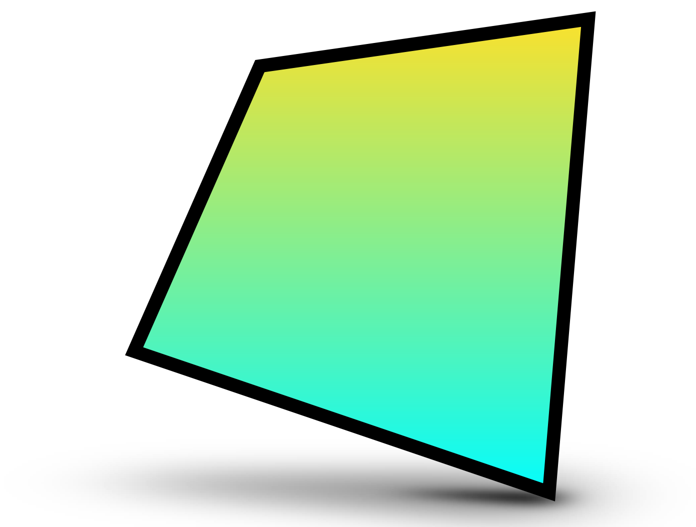


Conditions

Geometric conditions

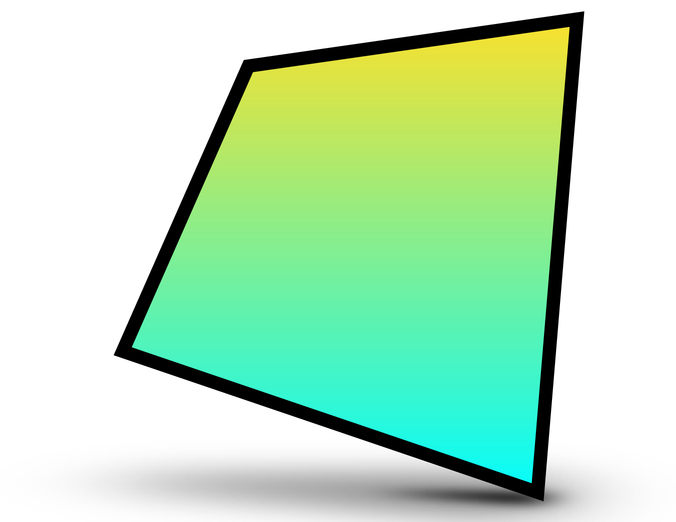


Conditions

Geometric conditions

- Translational invariance

$$\sum_i N_i^H(X) = \mathbb{I}$$



Conditions

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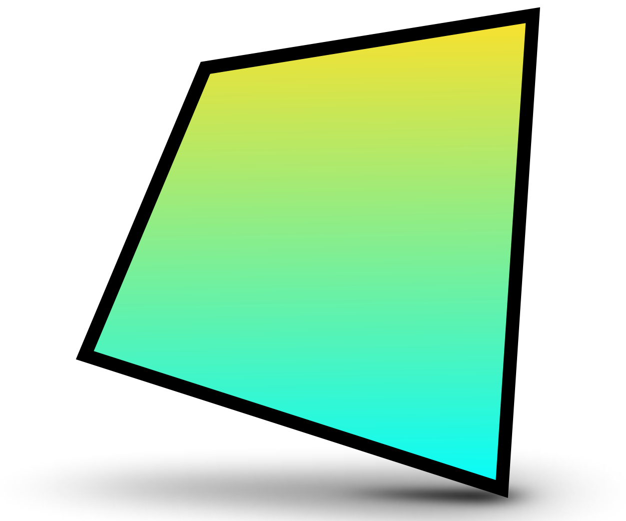
$$\sum_i N_i^H(X) = \mathbb{I}$$

- Rotational invariance

$$\sum_i N_i^H(X)[X_i^H]_\times = [X]_\times$$



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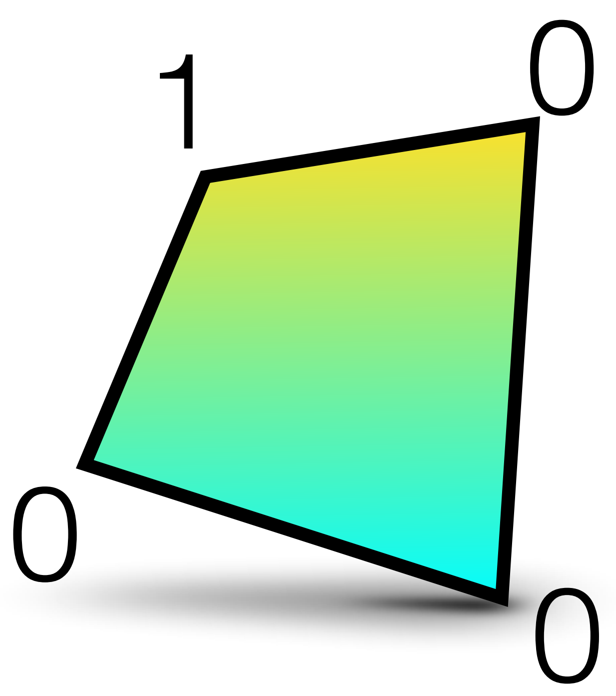
- Rotational invariance

$$\sum_i N_i^H(X)[X_i^H]_\times = [X]_\times$$

- Node interpolation

$$N_i^H(X_j^H) = \delta_{ij}\mathbb{I}$$



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- *Reconstruct global "representative" deformation*

$$h_{ab}(X) = \sum_i N_i^H(X) h_{ab}(X_i^H)$$

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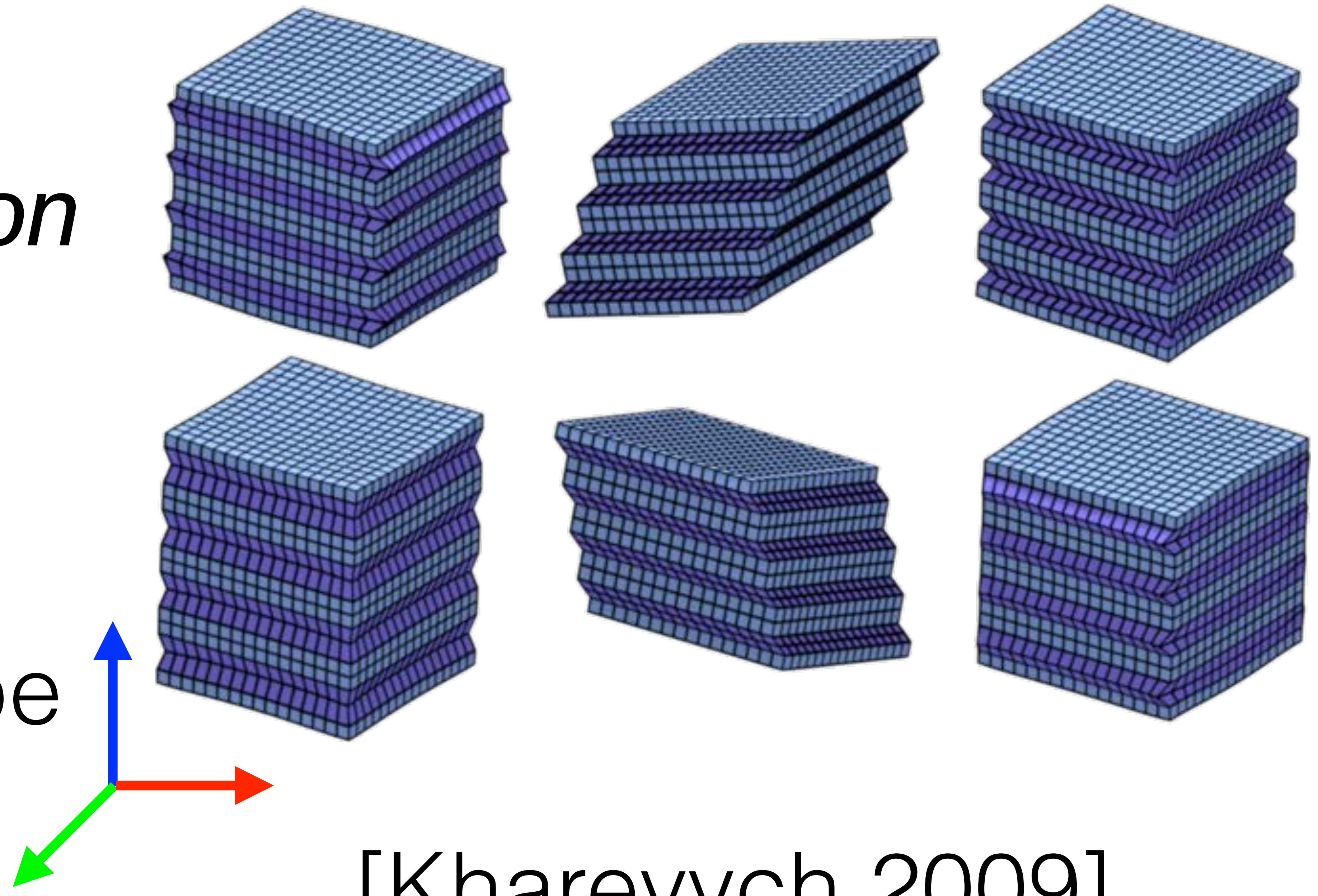
$$h_{ab}(X) = \sum_i N_i^H(X) h_{ab}(X_i^H)$$

- Global harmonic displacement at rest shape



[Kharevych 2009]

Conditions

Physical condition

- *Reconstruct global "representative" deformation*

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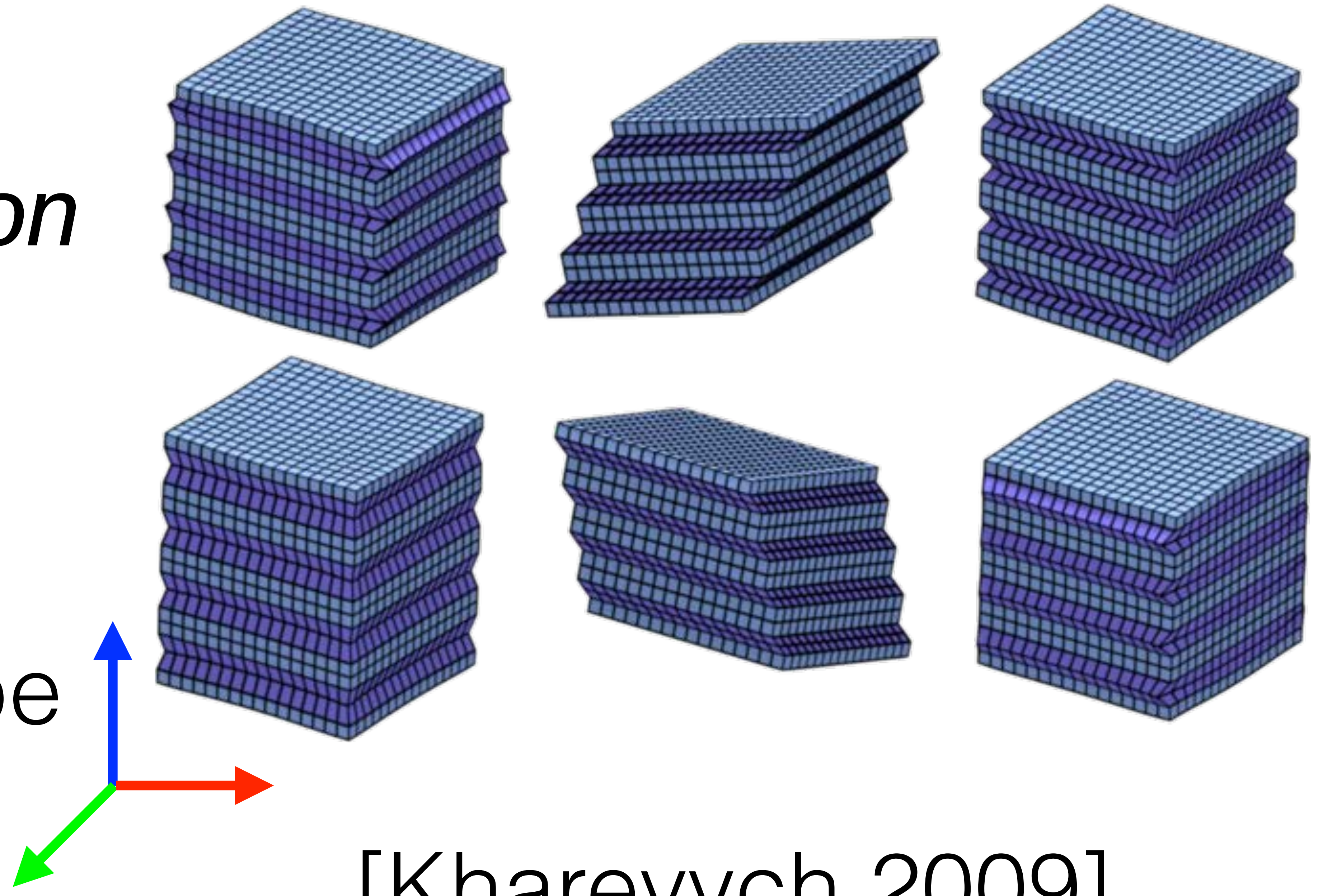
- Global harmonic displacement at rest shape

- Contribute 6 more constraints in 3D for each element



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Numerical conditioning

Smooth regularization

$$\int_{\Omega} \operatorname{tr}\left((\nabla N_i^H)^T : M : \nabla N_i^H\right) dX$$

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$$\int_\Omega \mathrm{tr}\left((\nabla N_i^H)^T : \boxed{M} : \nabla N_i^H \right) dX$$

rank-4 tensor

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Smooth regularization

$$\int_\Omega \mathrm{tr}\left((\nabla N_i^H)^T : \boxed{M} : \nabla N_i^H\right) dX$$

rank-4 tensor

- Two Options of metric

  - Harmonic: $M = \mathbb{I}$

  - Ψ-harmonic: $M = \partial^2\Psi/\partial F^2$

  (Ψ-constitutive model, F-deformation gradient)

Summary

- Finding basis ->

Summary

- Finding basis ->

Solve a constrained quadratic programming per element

$$\int_\Omega \mathrm{tr}\left((\nabla N_i^H)^T : M : \nabla N_i^H\right) dX$$

$$\text{s.t.} \sum_i N_i^H(X) = \mathbb{I}$$

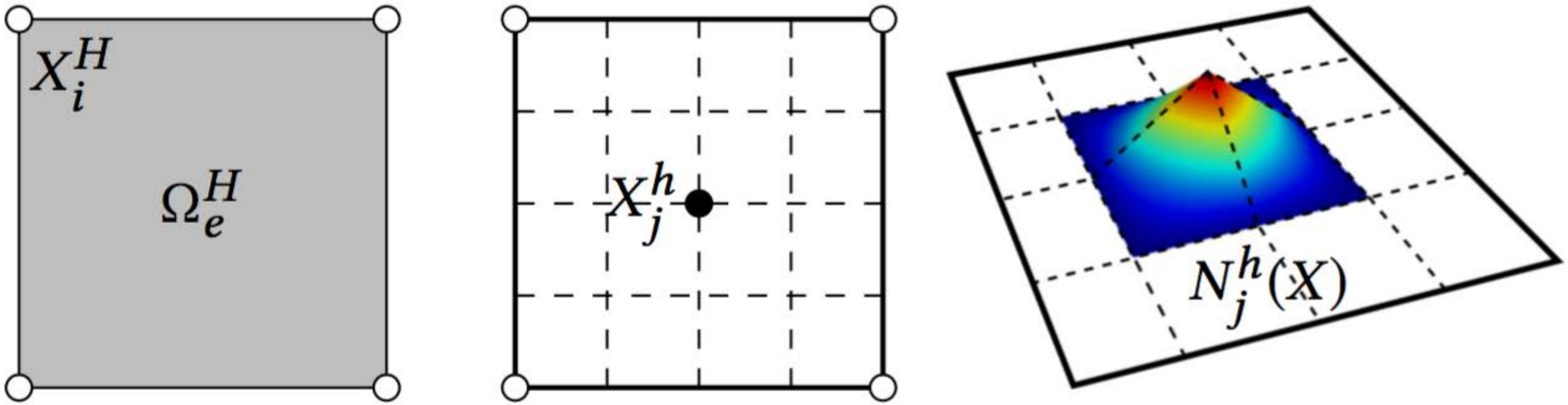
$$\sum_i N_i^H(X)[X_i^H]_\times = [X]_\times$$

$$\sum_i N_i^H(X) h_{ab}(X_i^H) = h_{ab}(X)$$

$$N_i^H(X_j^H) = \delta_{ij}\mathbb{I}$$

Basis discretization

- Our basis functions are discretely represented



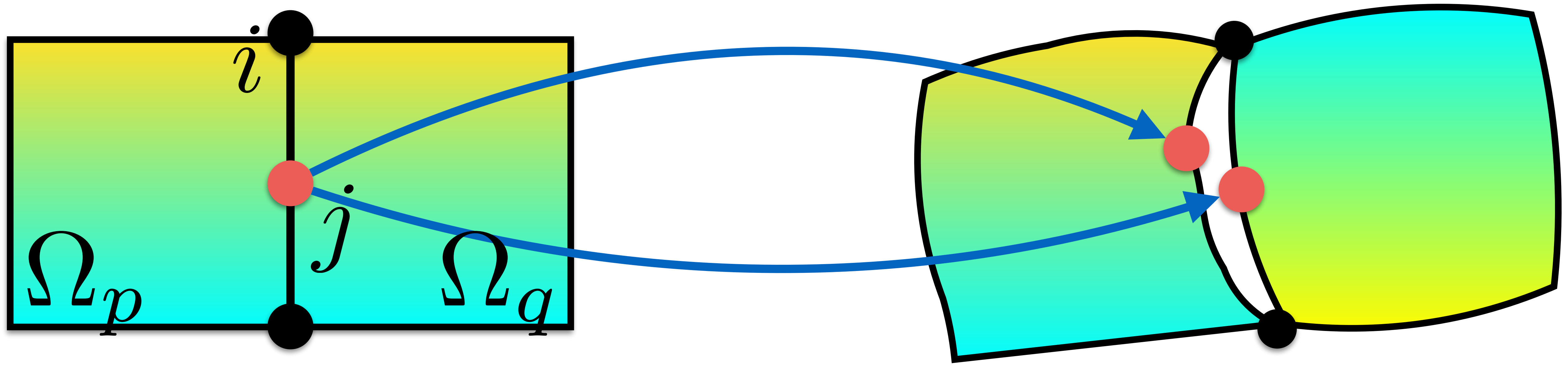
piecewise bilinear function

$$N_i^H(X) = \sum_j n_{ij} N_j^h(X)$$

Balance

- Our optimized basis function does not guarantee C^0-continuity

$$N_{p,i}(X_j^h) \neq N_{q,i}(X_j^h) \longrightarrow u_p(X_j^h) \neq u_q(X_j^h)$$



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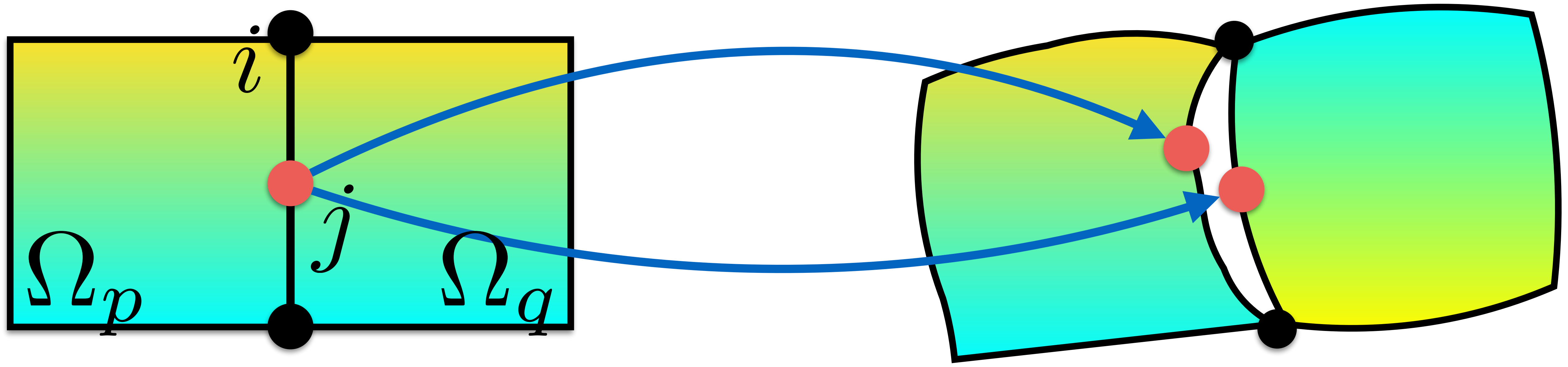
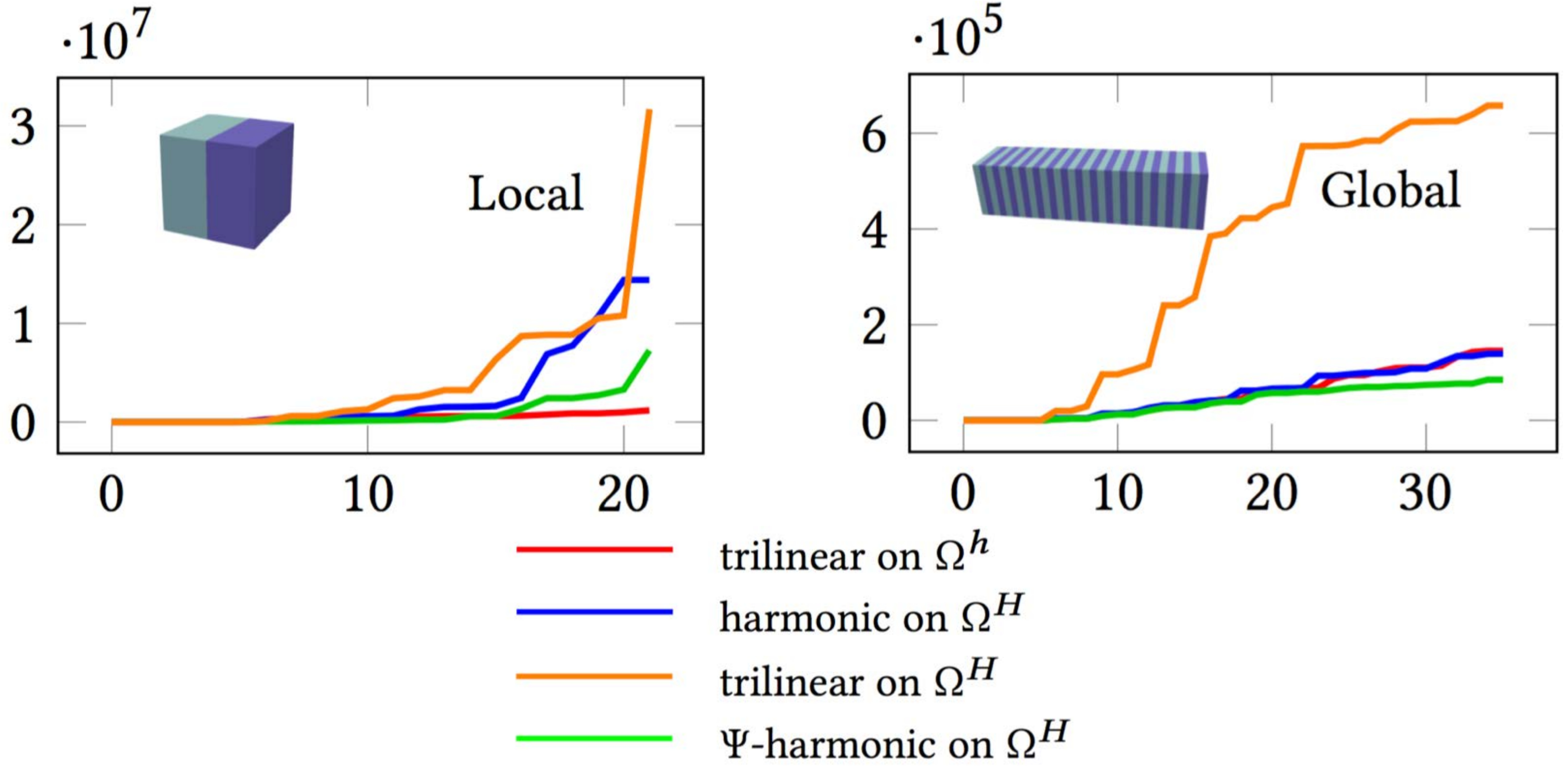


- **Coarse element** generally appears to be **“stiffer”**.
- **Discontinuous** basis functions make system **“softer”**.

Make balance



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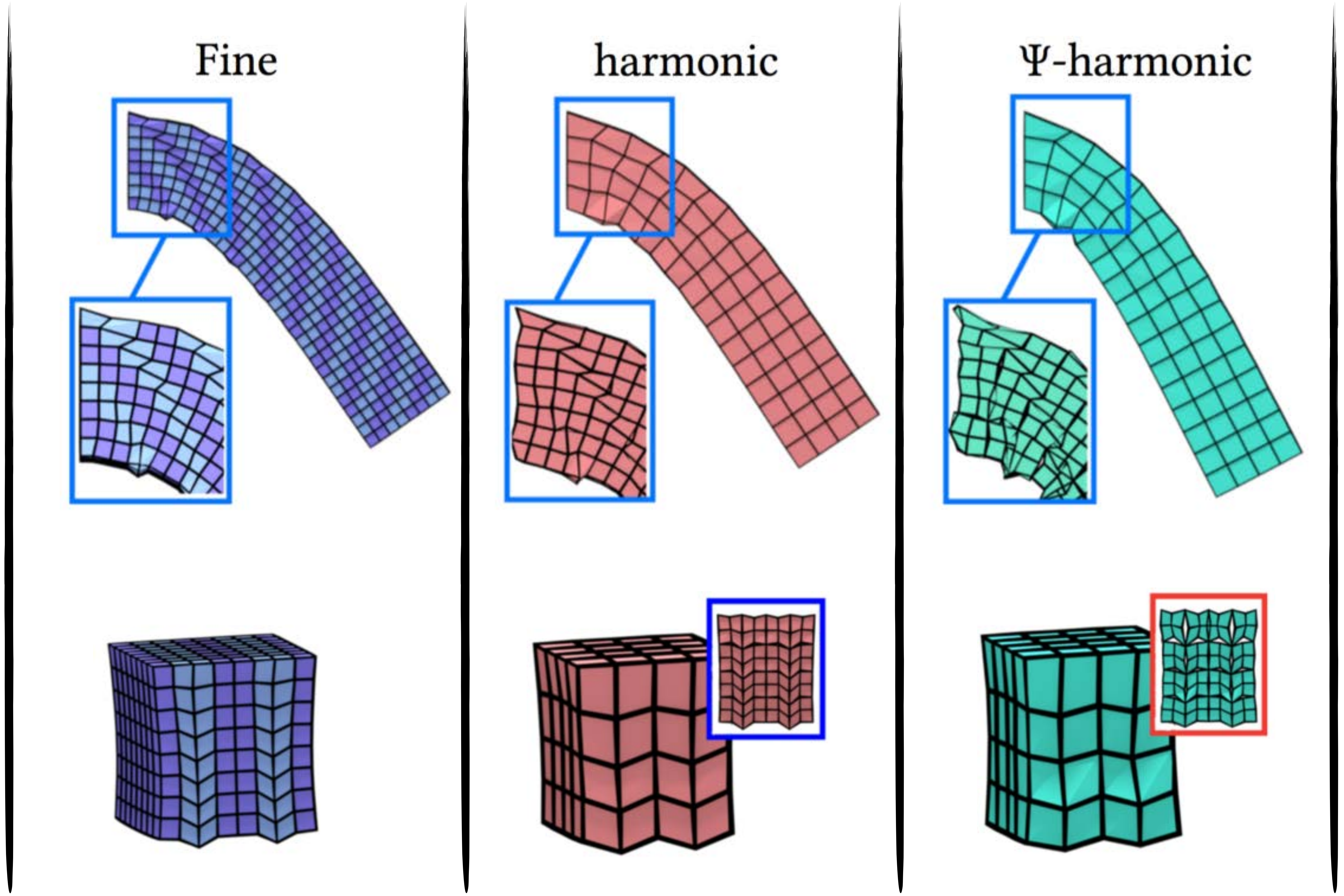
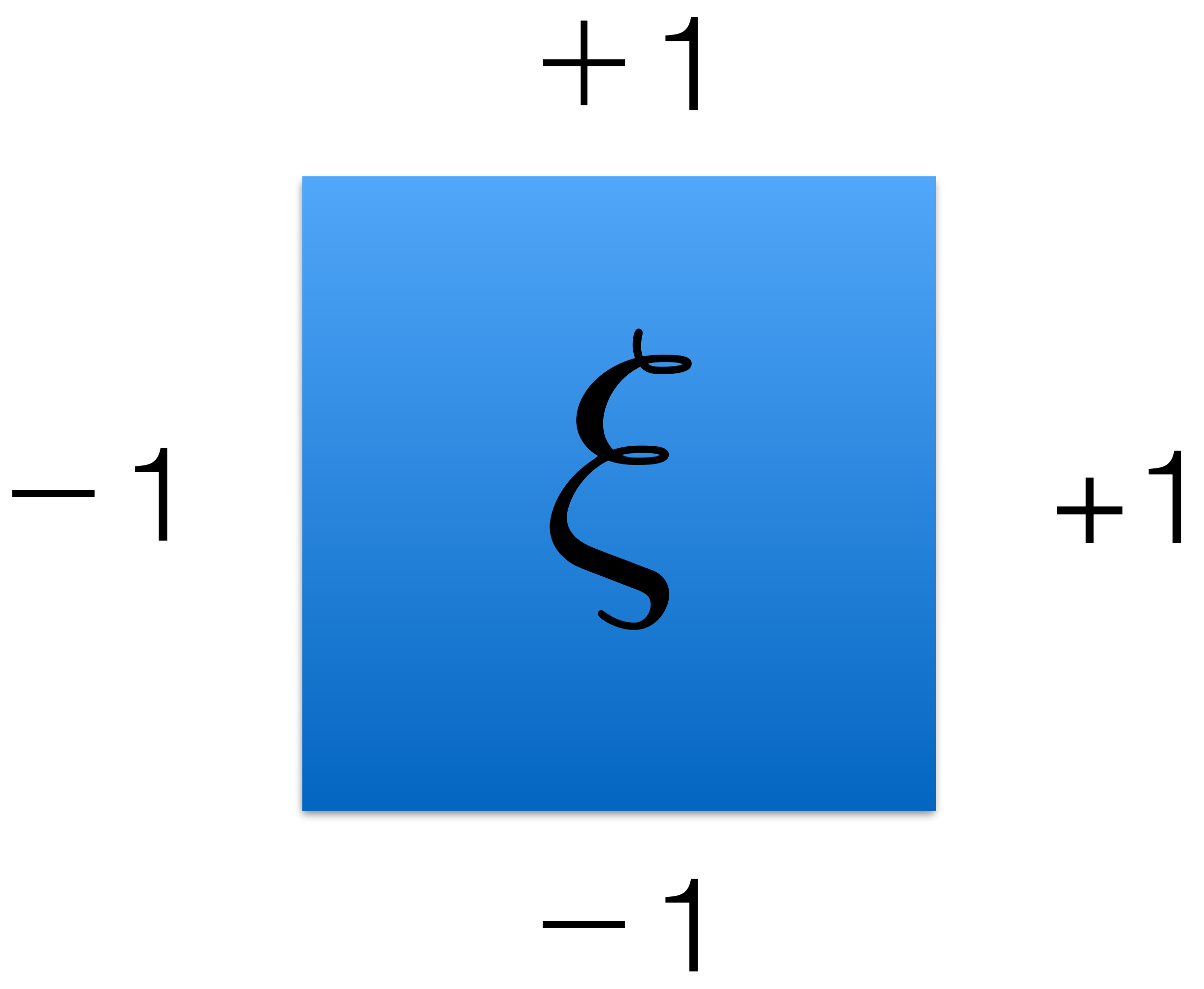


Simulation

- Calculation of deformation gradient

$$\nabla_X x = \nabla_X u + \mathrm{I} = (R_e - \mathrm{I}) + \sum_i R_e \otimes (R_e^T x_i - X_i) : \frac{\partial N_i^H}{\partial X} + \mathrm{I}$$

$$= R_e + \left(\sum_i R_e \otimes (R_e^T x_i - X_i) : \frac{\partial N_i^H}{\partial \xi} \right) \left(\sum_j \frac{\partial \overline{N}_j^H}{\partial \xi} \right)^{-1}$$



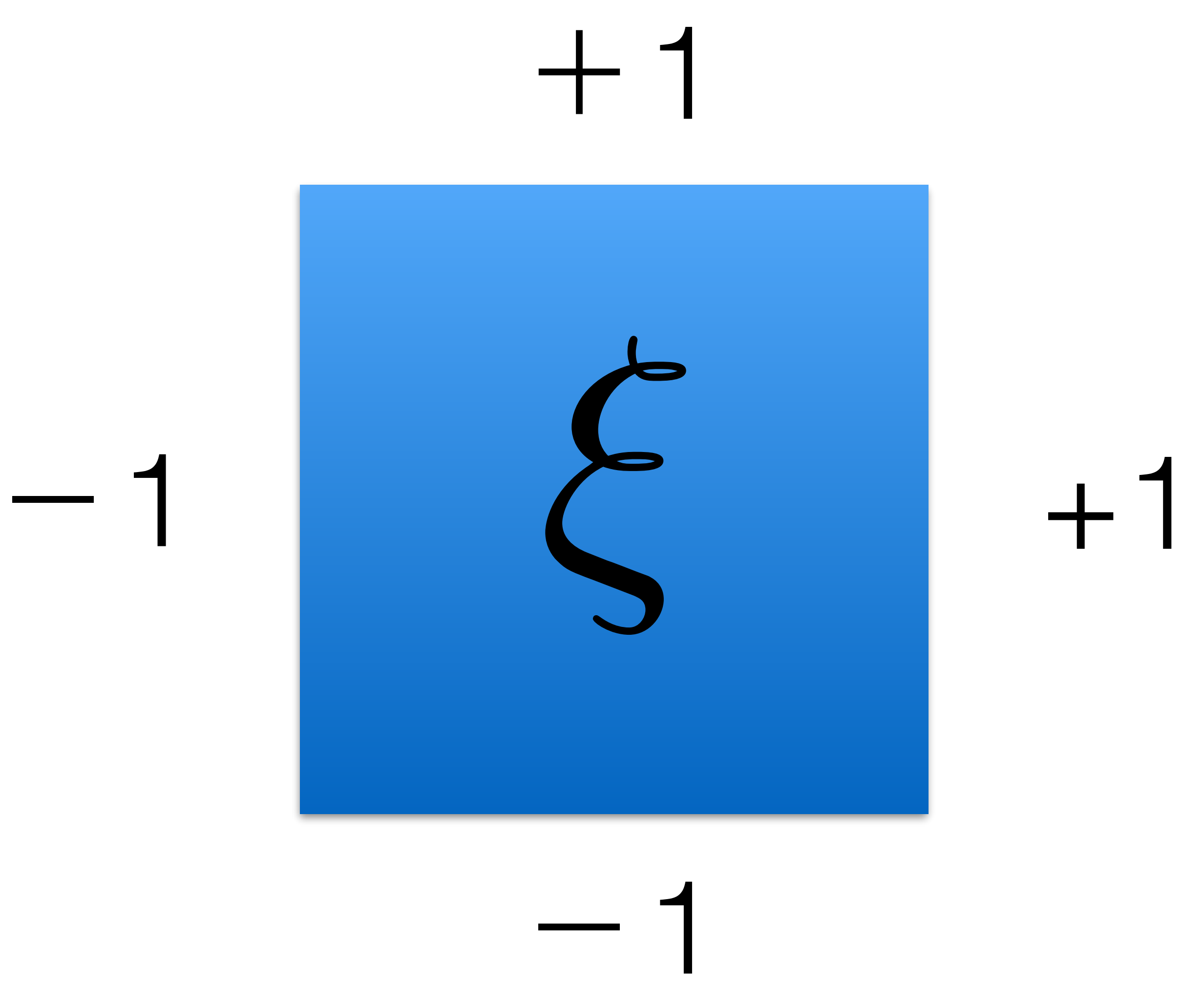
Simulation

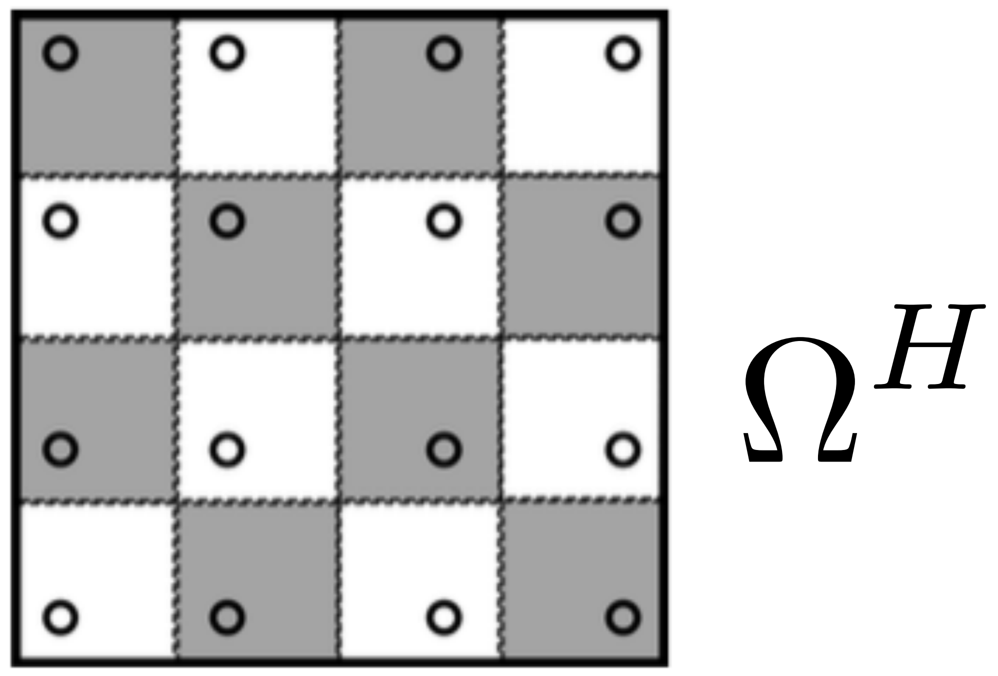
- Calculation of deformation gradient

$$\nabla_X x = \nabla_X u + \mathrm{I} = (R_e - \mathrm{I}) + \sum_i R_e \otimes (R_e^T x_i - X_i) : \frac{\partial N_i^H}{\partial X} + \mathrm{I}$$

$$= R_e + \left(\sum_i R_e \otimes (R_e^T x_i - X_i) : \frac{\partial N_i^H}{\partial \xi} \right) \left(\sum_j \frac{\partial \overline{N}_j^H}{\partial \xi} \right)^{-1}$$

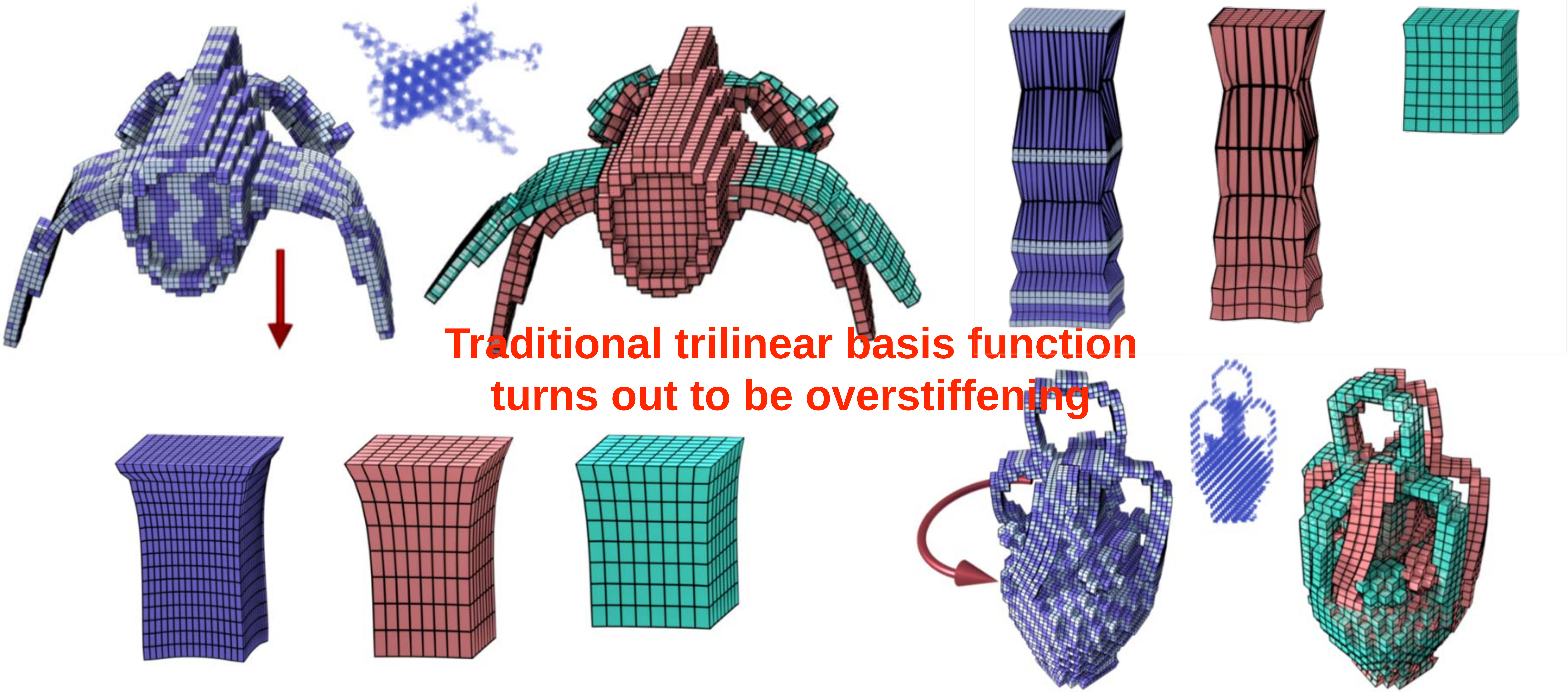
- Quadrature: standard Gaussian quadrature





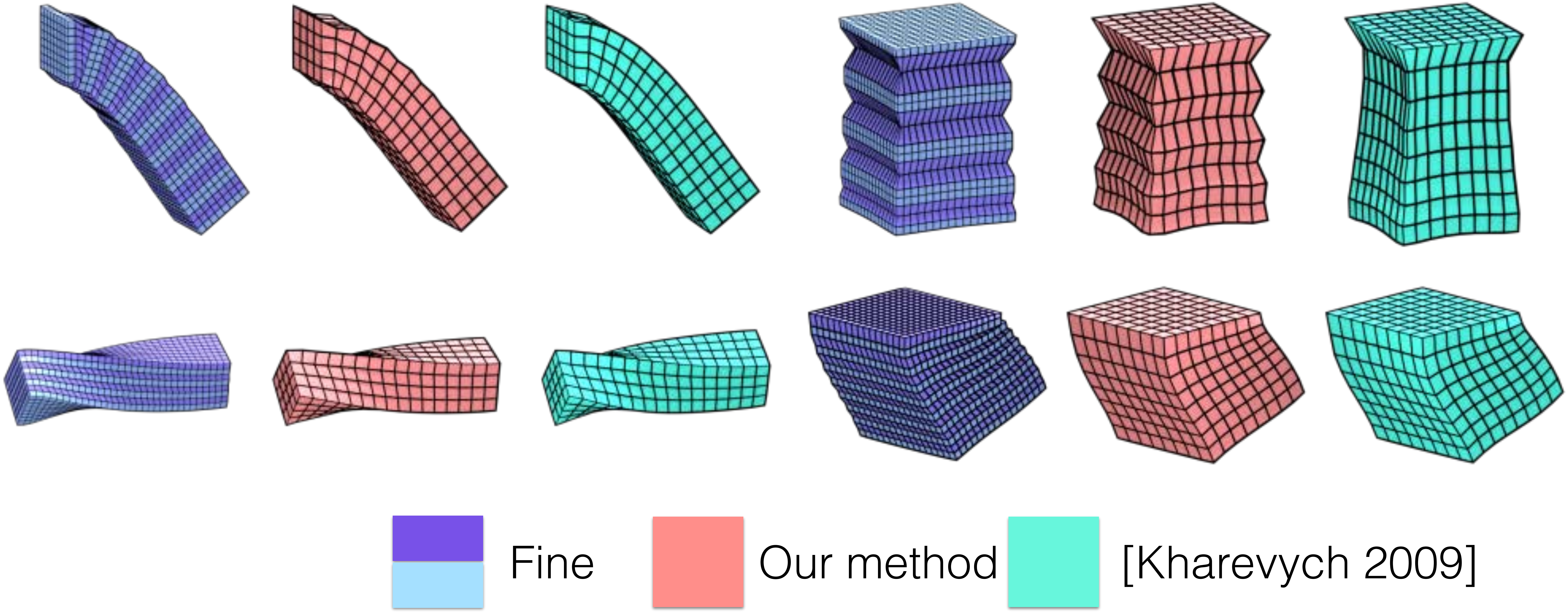
Results

Comparison with trilinear basis

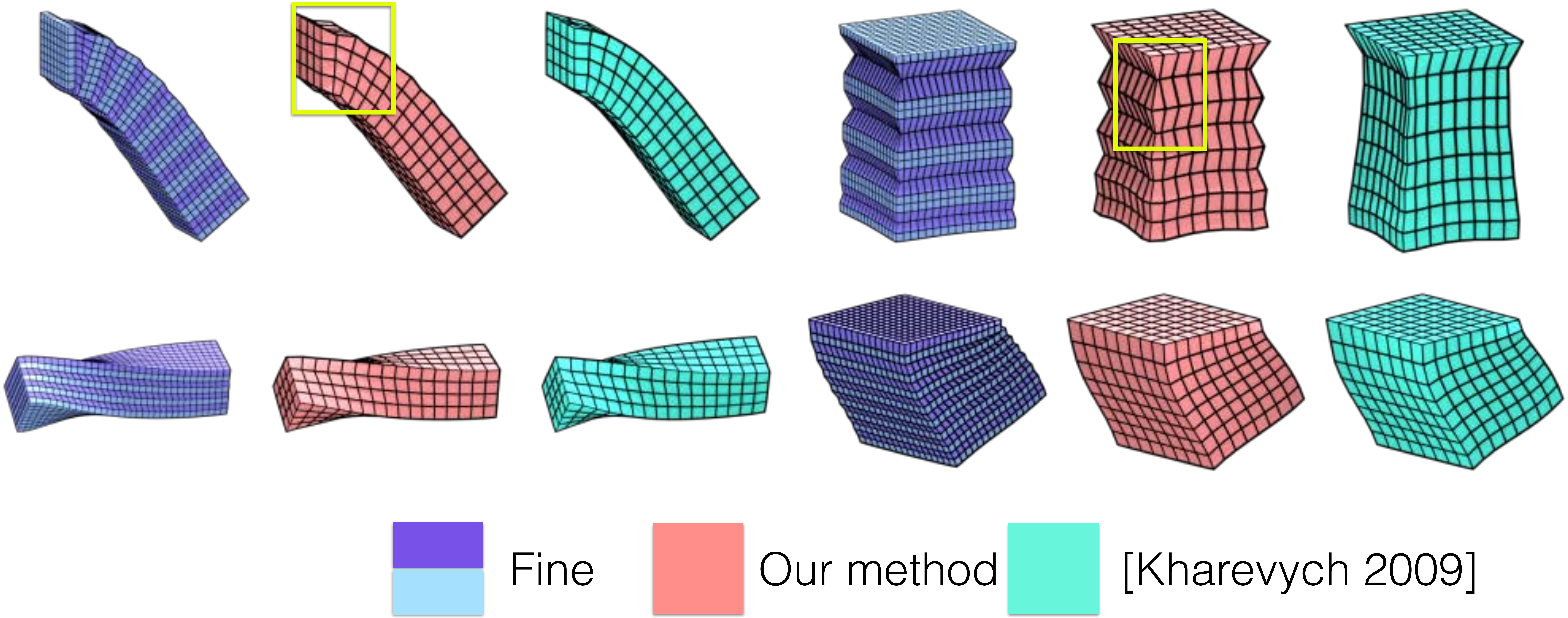


Traditional trilinear basis function turns out to be overstiffening

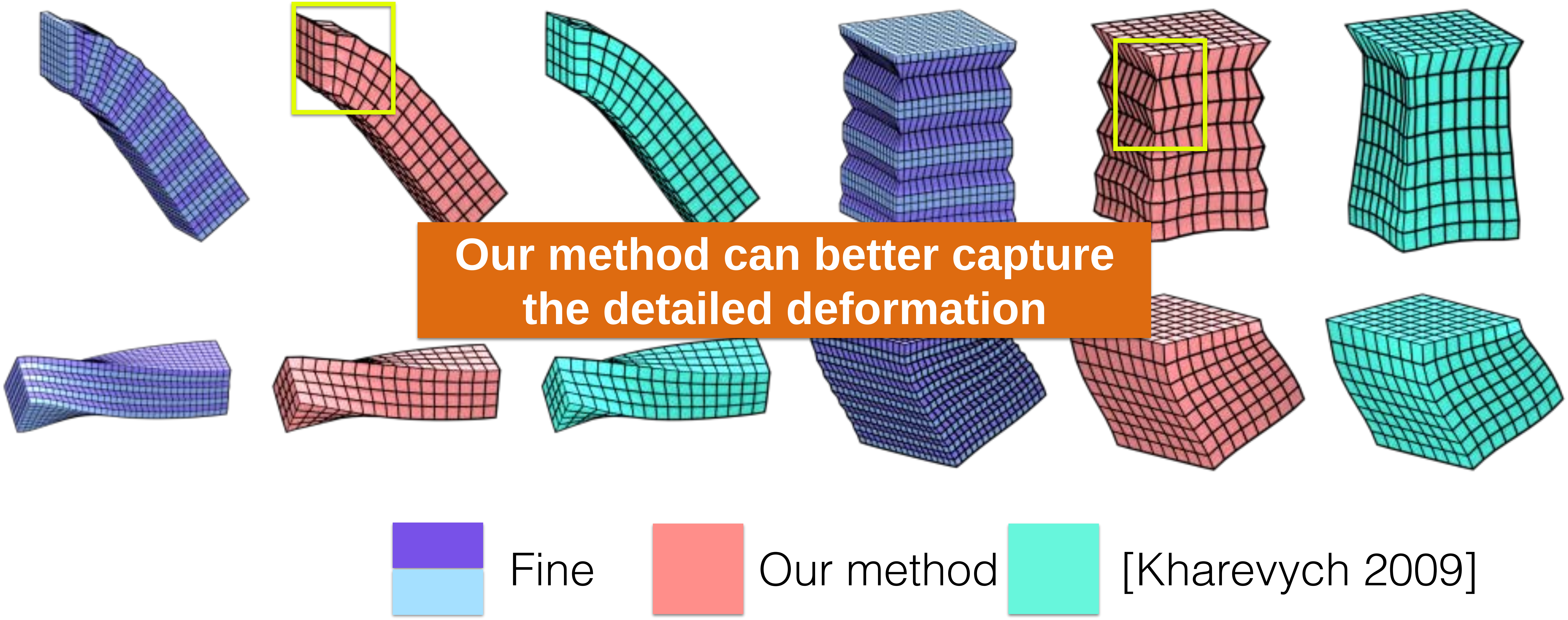
Relation to [Kharevych 2009]



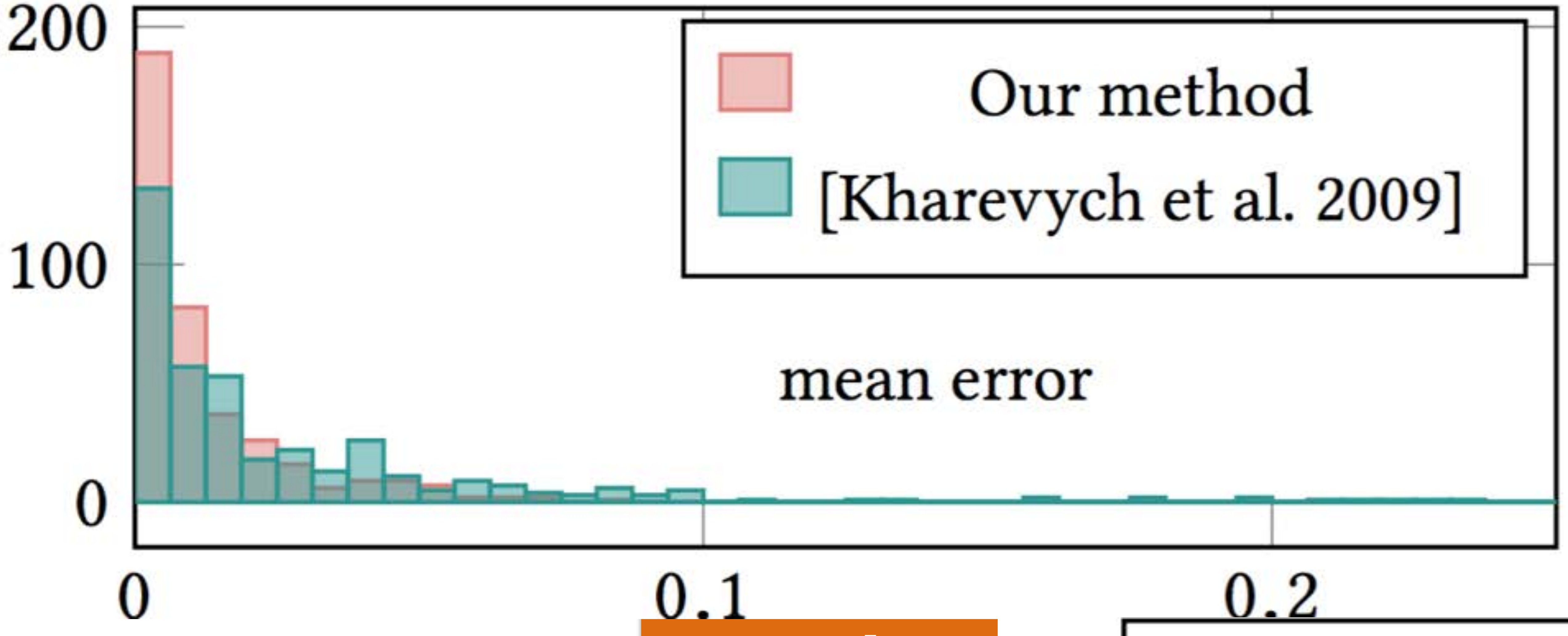
Relation to [Kharevych 2009]

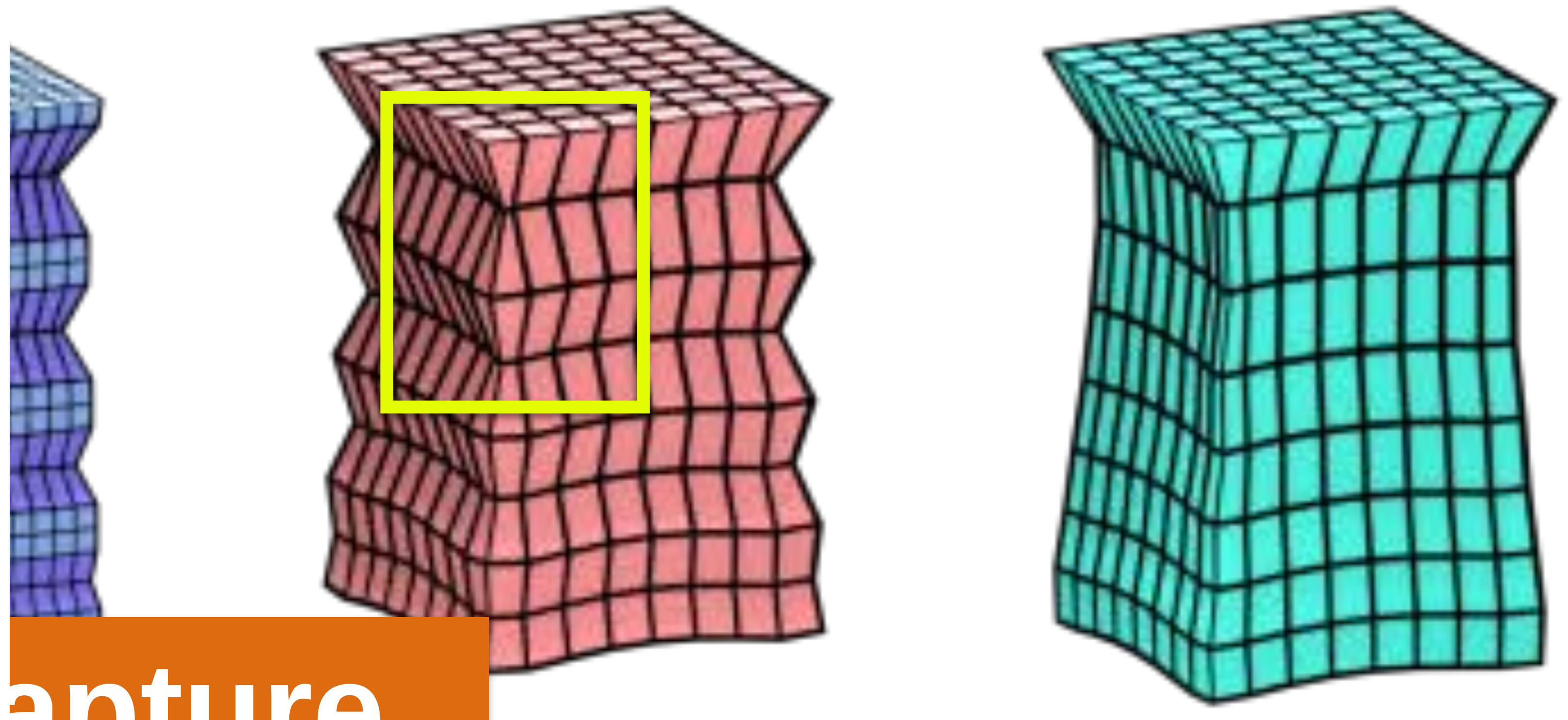


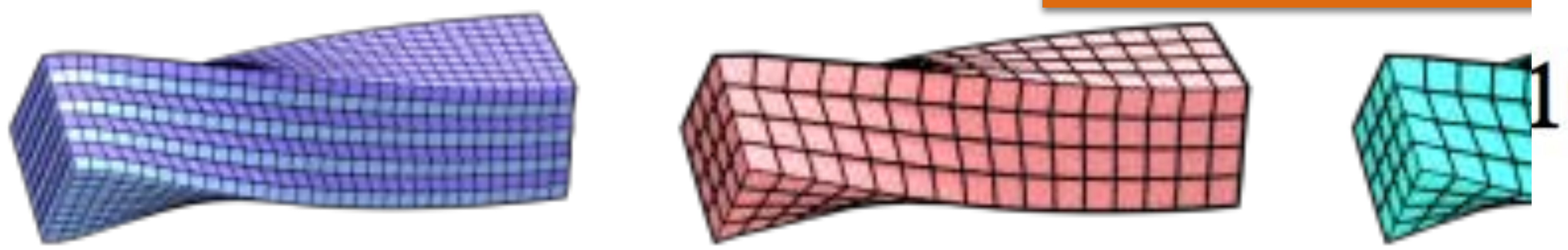
Relation to [Kharevych 2009]

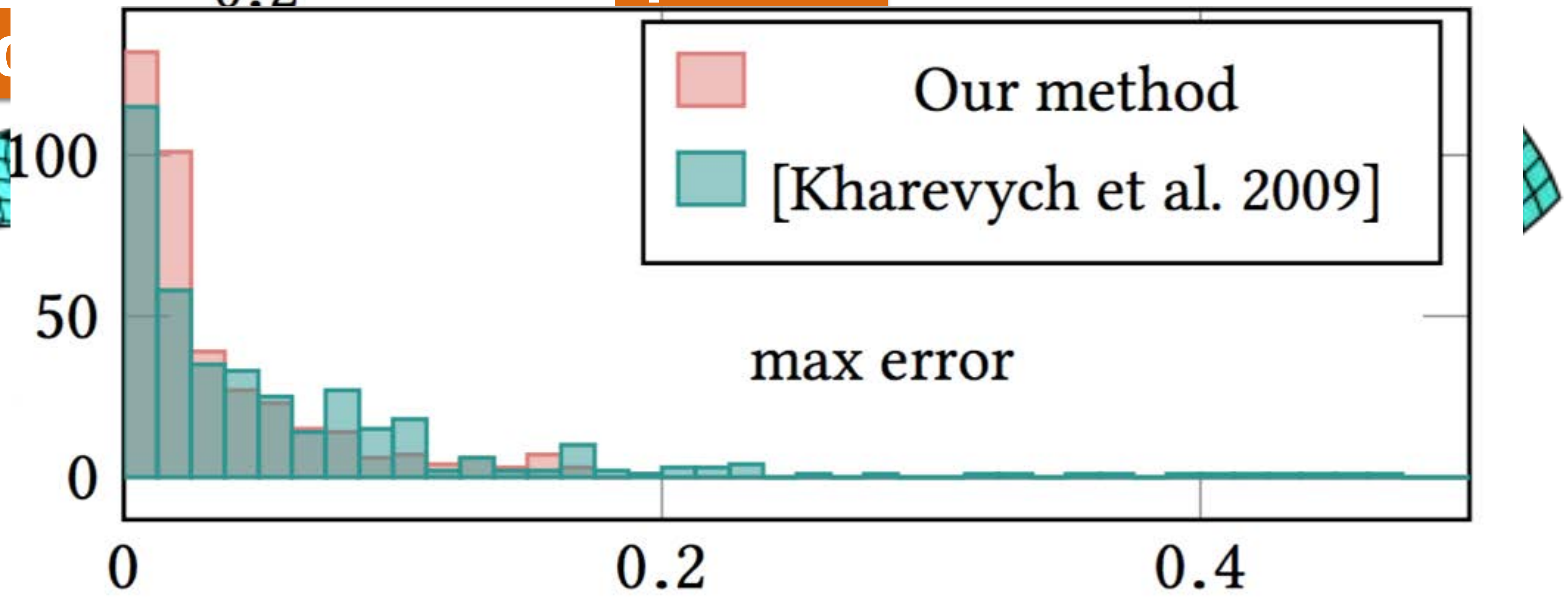


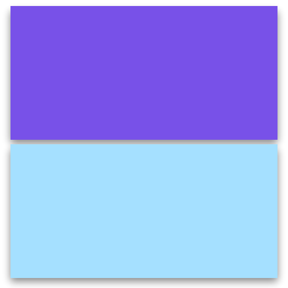
Relation to [Kharevych 2009]



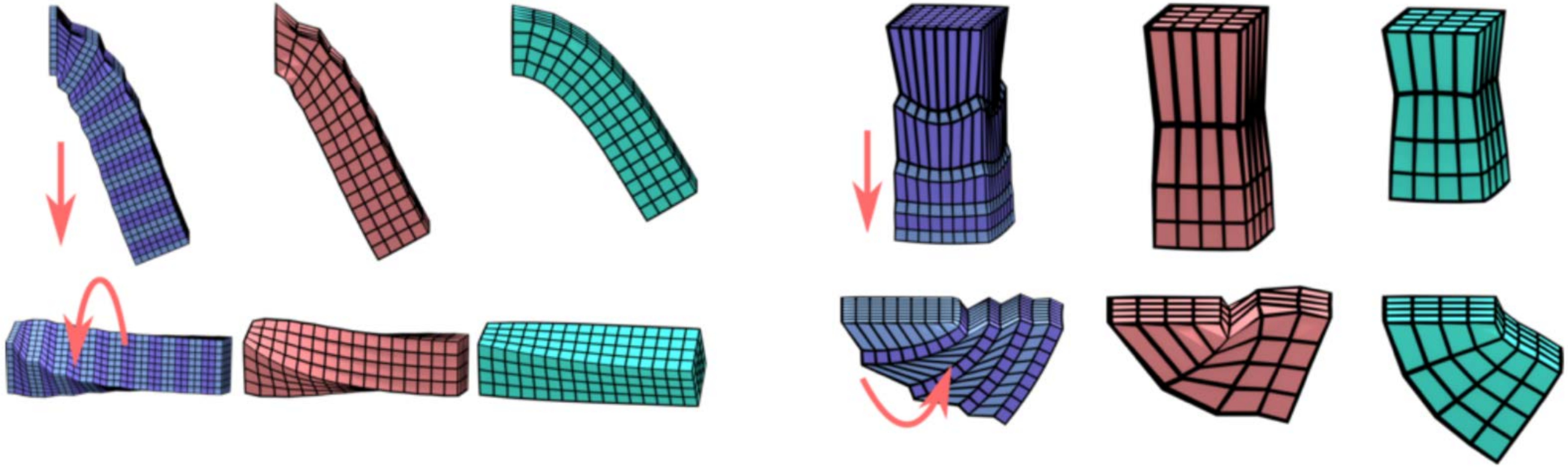




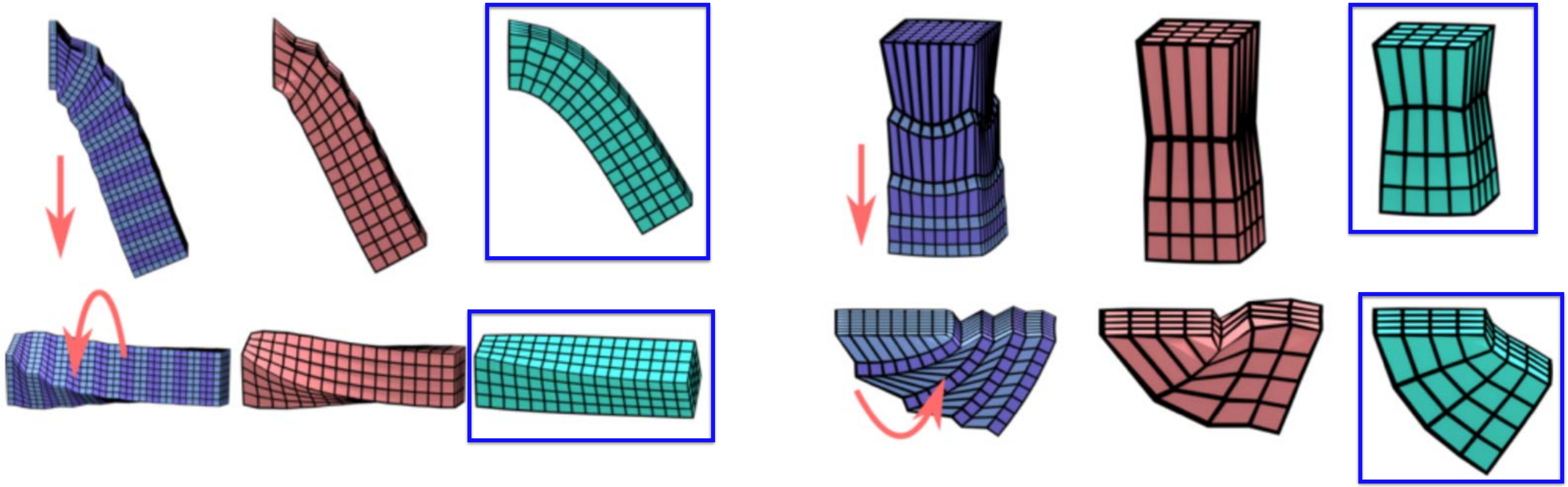




Relation to [Nesme 2009]



Relation to [Nesme 2009]



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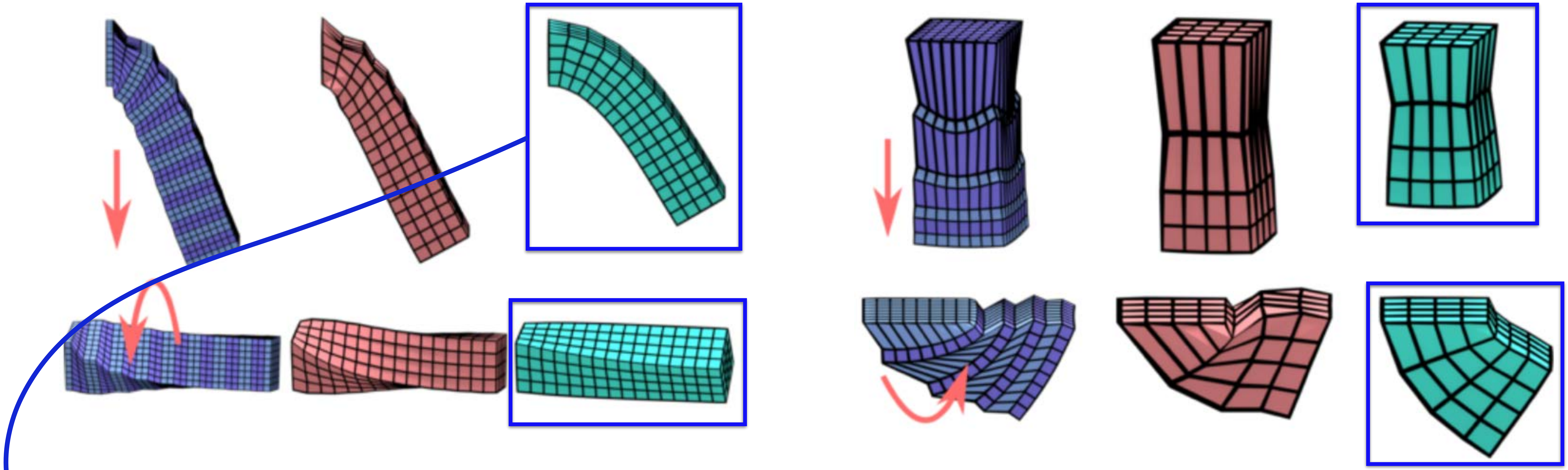
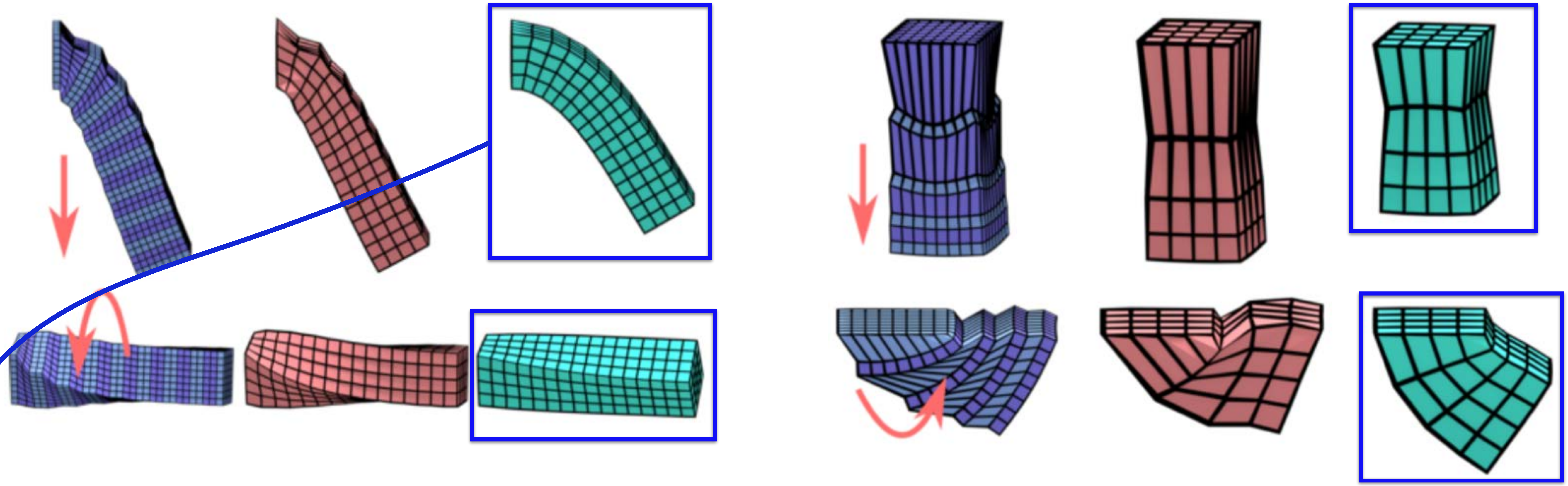


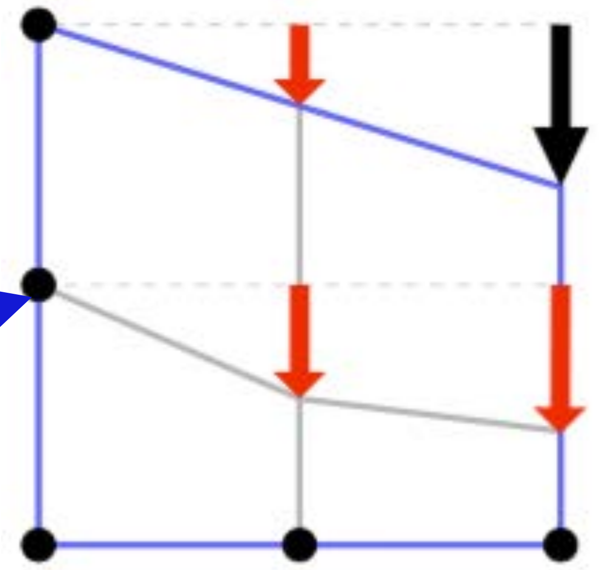
Diagonal basis
Translation invariance
Rotation invariance

Far boundary vanishing
Node interpolation
Psi-harmonic

Relation to [Nesme 2009]

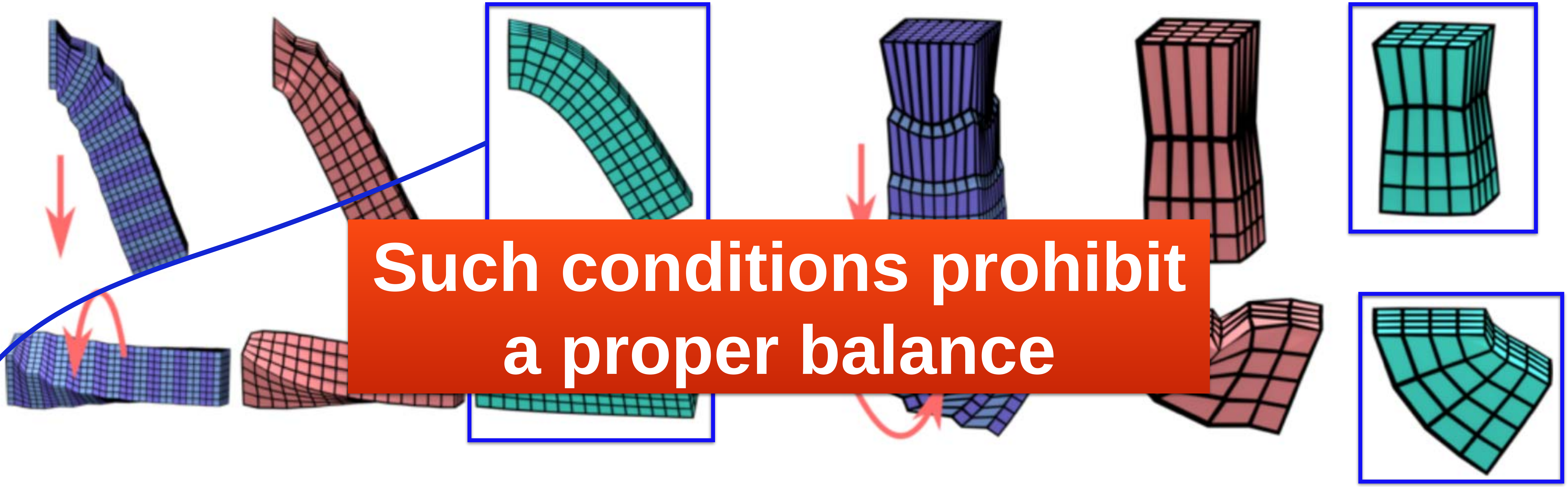


Diagonal basis	Far boundary vanishing
Translation invariance	Node interpolation
Rotation invariance	Psi-harmonic



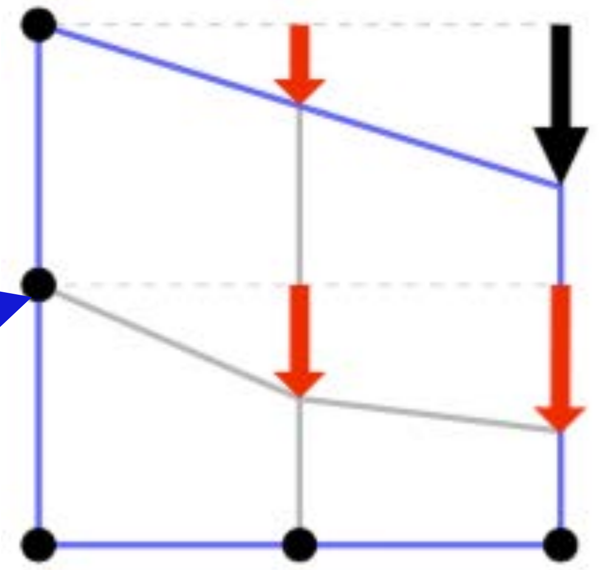
[Nesme 2009]

Relation to [Nesme 2009]



Such conditions prohibit a proper balance

Diagonal basis	Far boundary vanishing
Translation invariance	Node interpolation
Rotation invariance	Psi-harmonic



[Nesme 2009]

Relation to DDFEM

Results of DDFEM will be likely impacted by …

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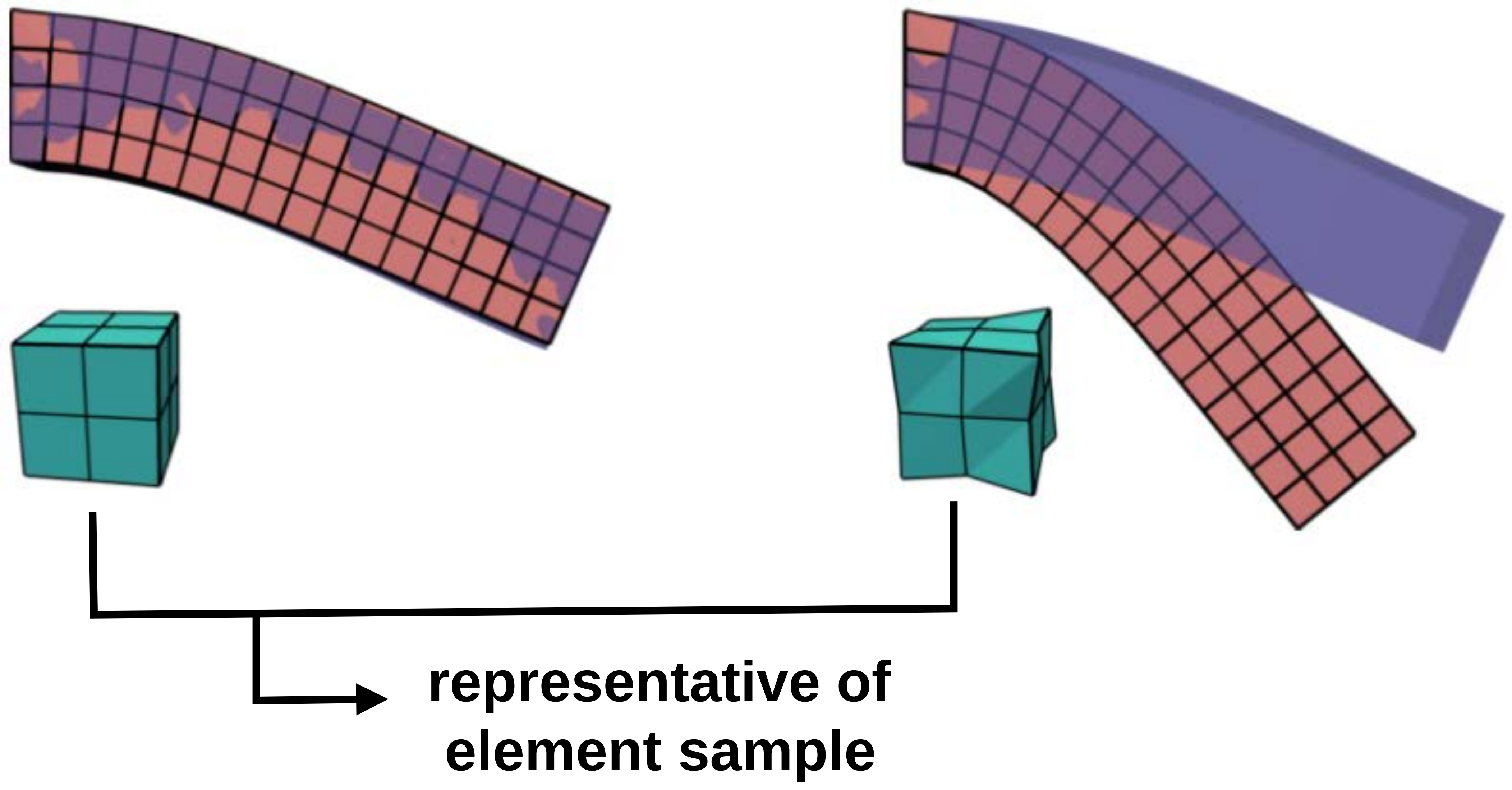
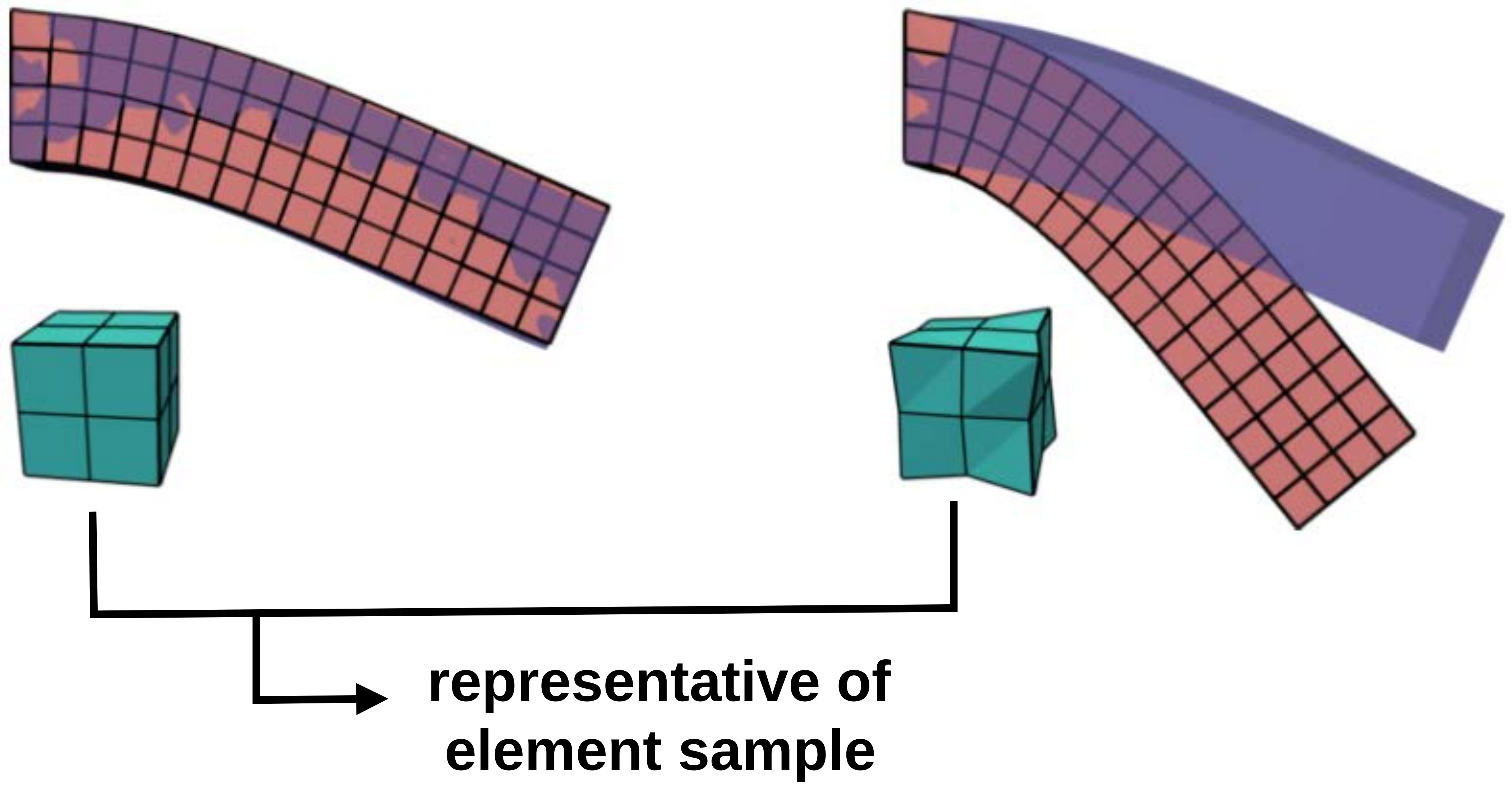


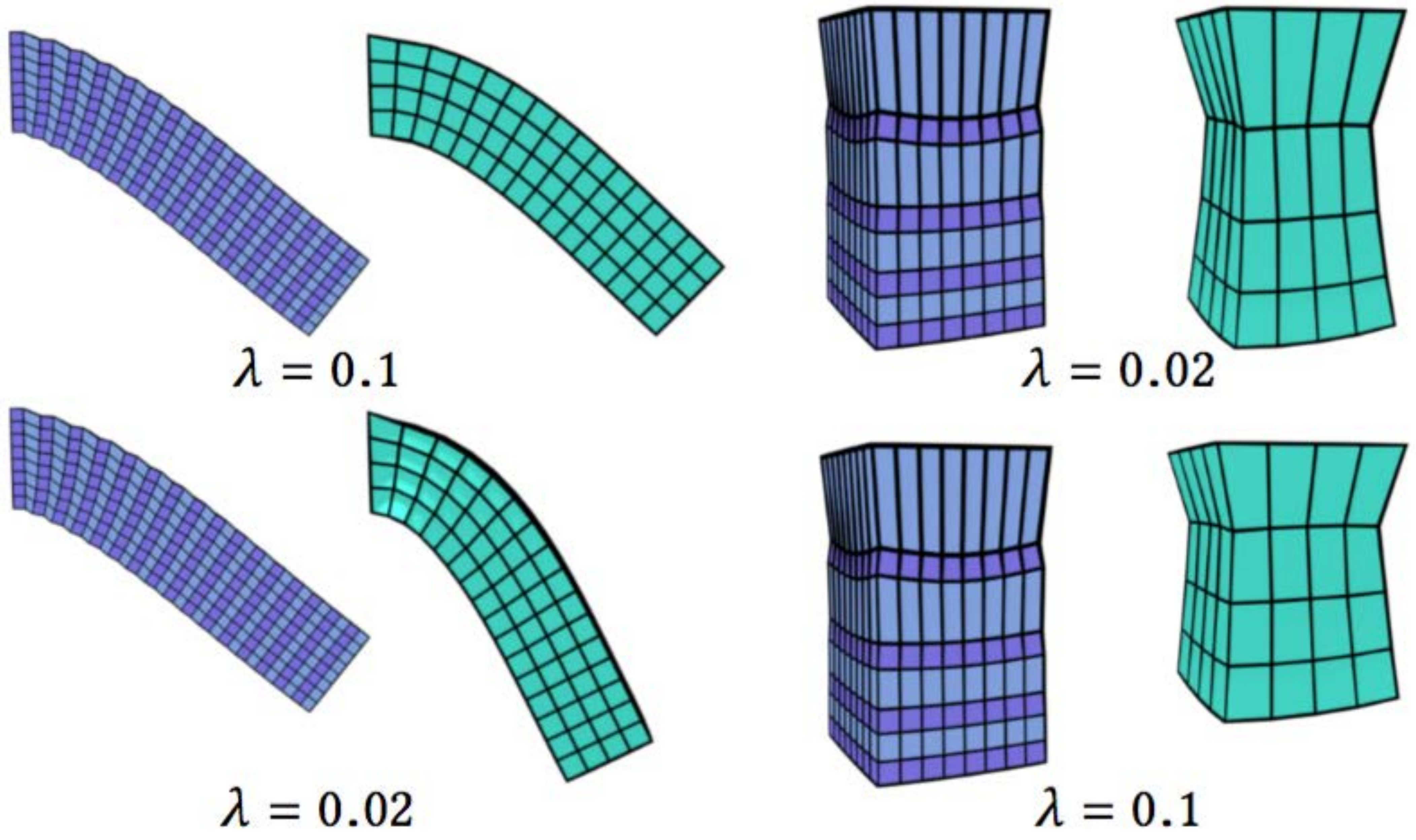
Training set

Relation to DDFEM

Results of DDFEM will be likely impacted by …



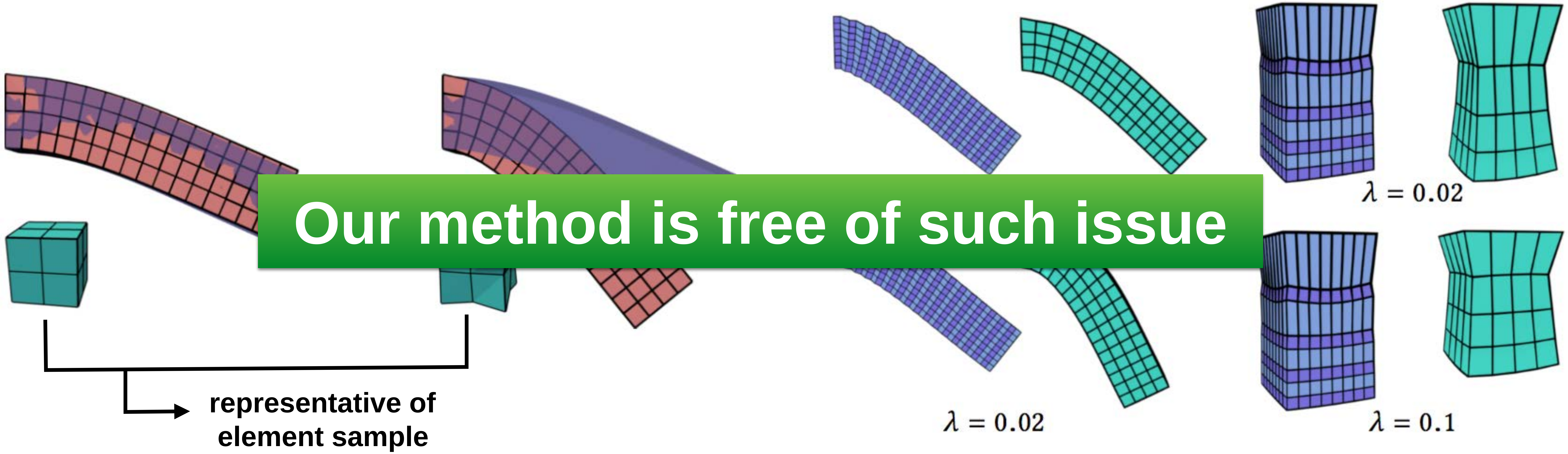
Training set



Parameters for regression

Relation to DDFEM

Results of DDFEM will be likely impacted by …

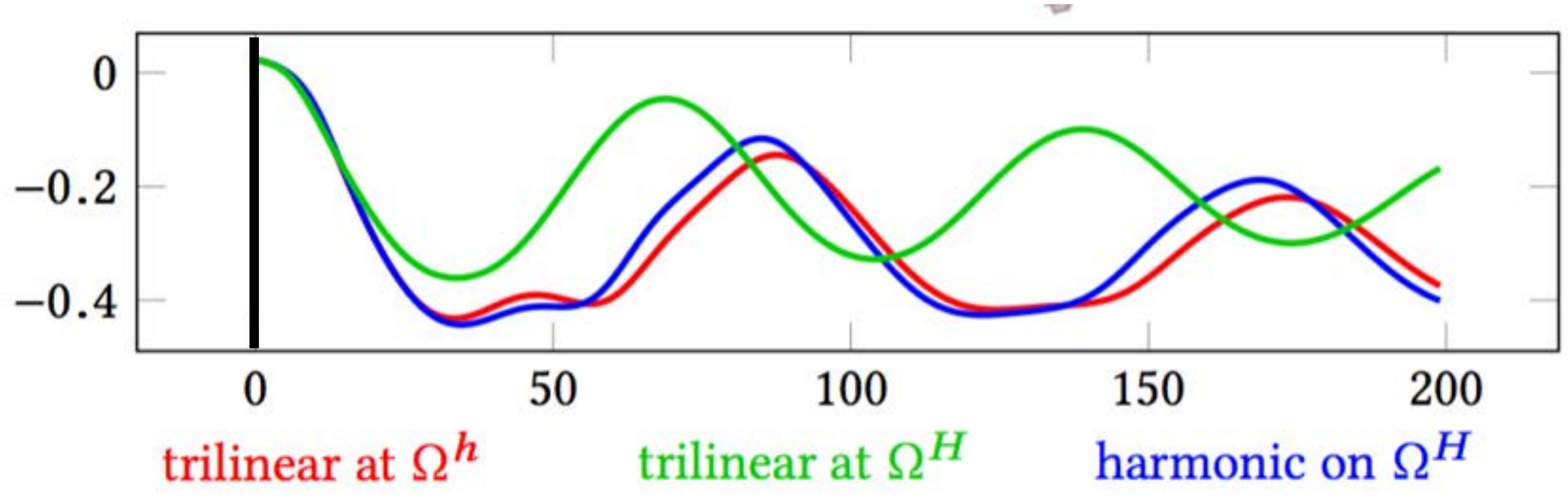


Our method is free of such issue

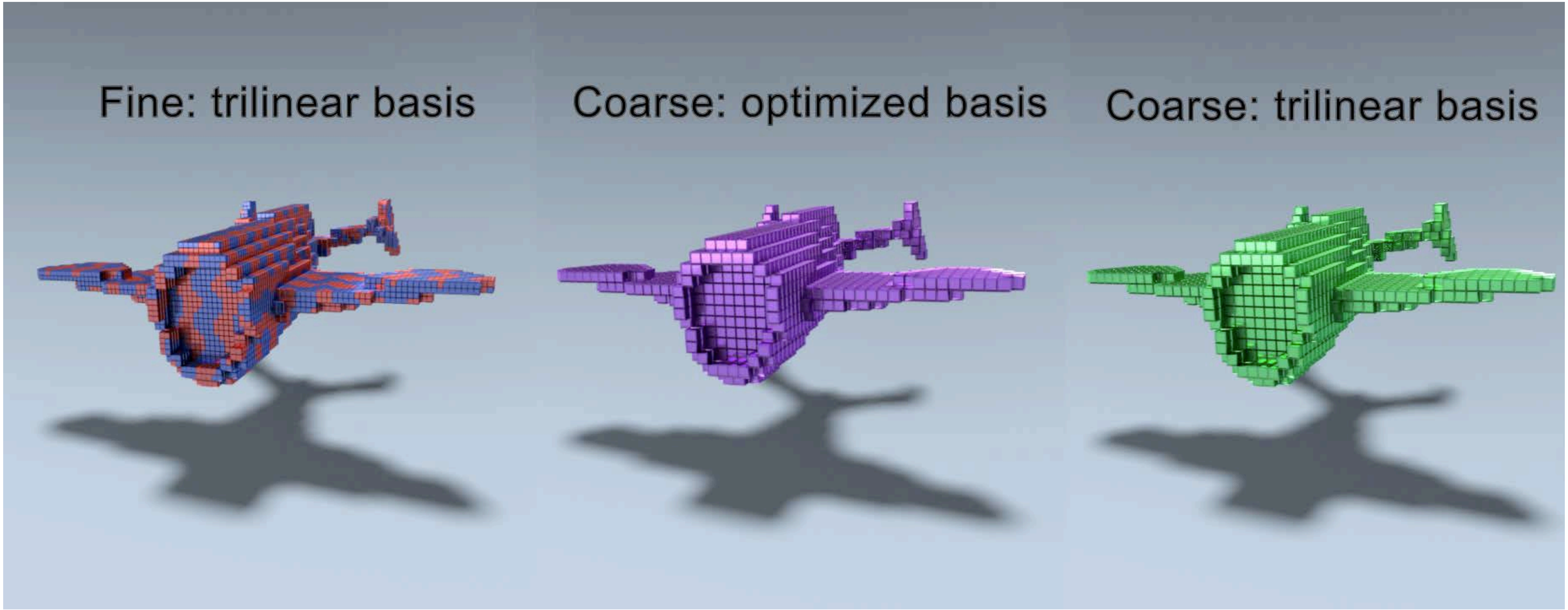
Training set

Parameters for regression

Dynamics

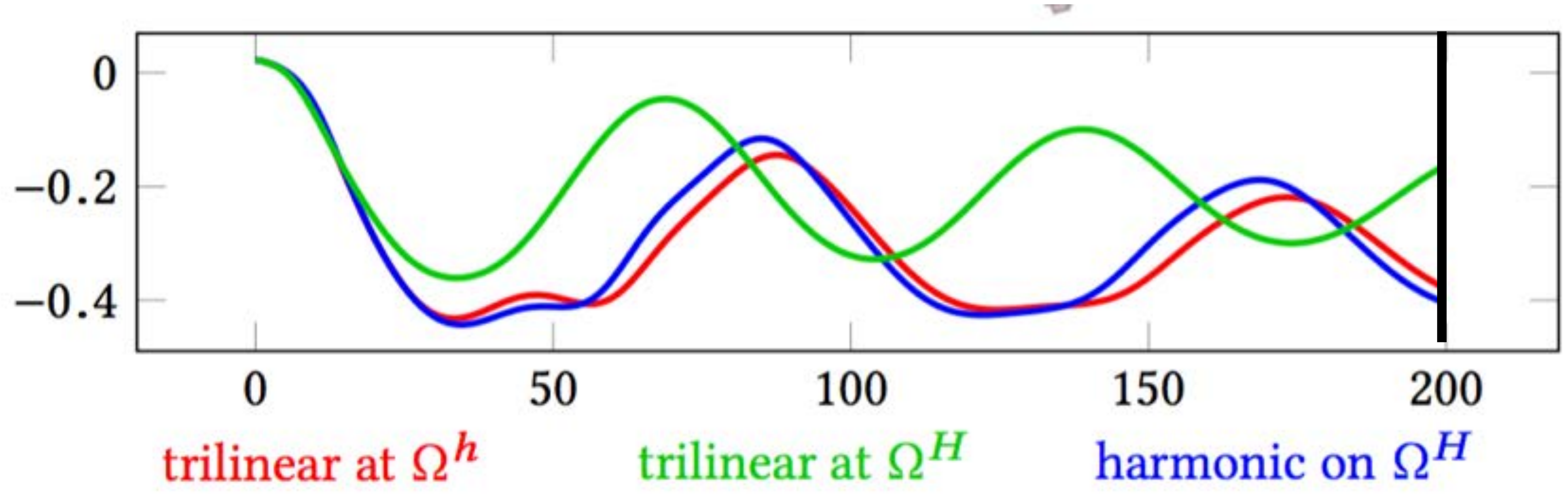


0
−0.2
−0.4
0
50
100
150
200
trilinear at Ω^h
trilinear at Ω^H
harmonic on Ω^H



Fine: trilinear basis
Coarse: optimized basis
Coarse: trilinear basis

Dynamics



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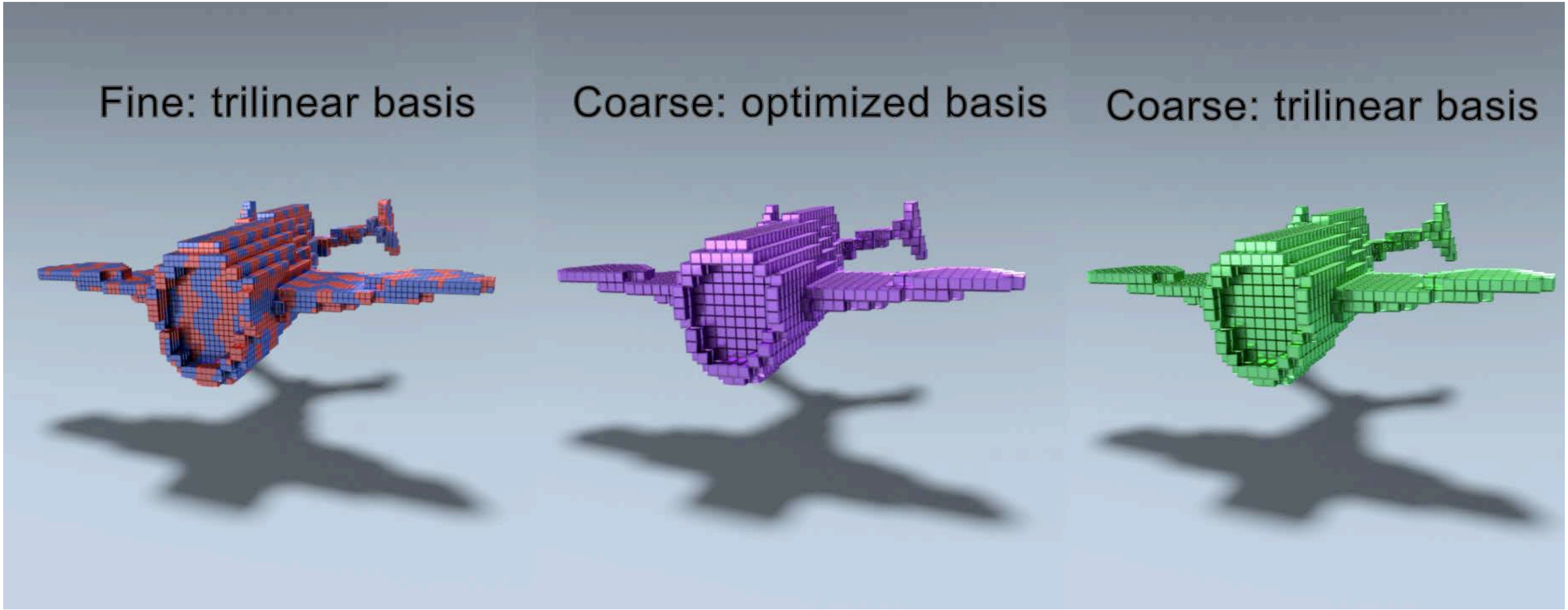

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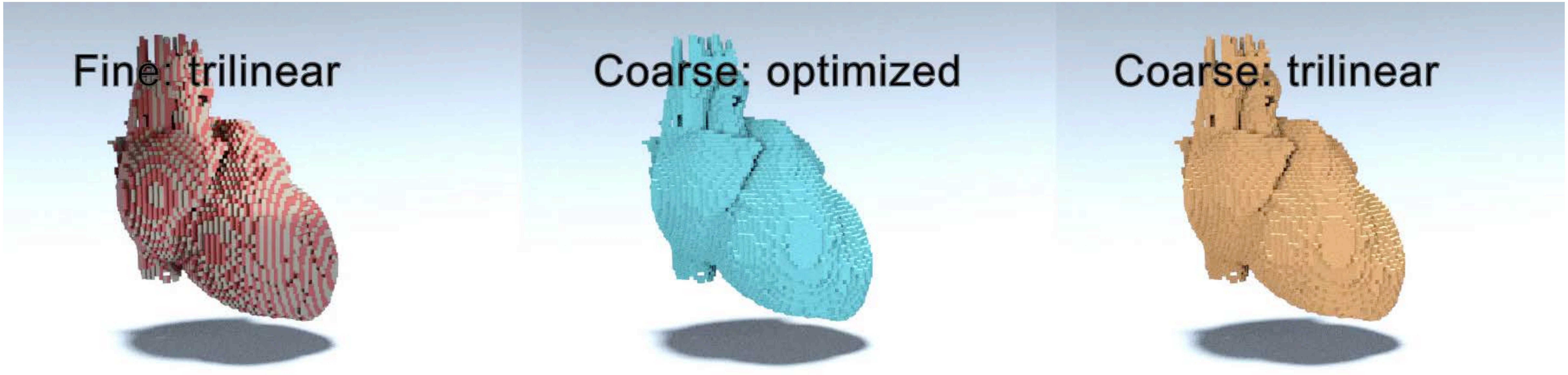
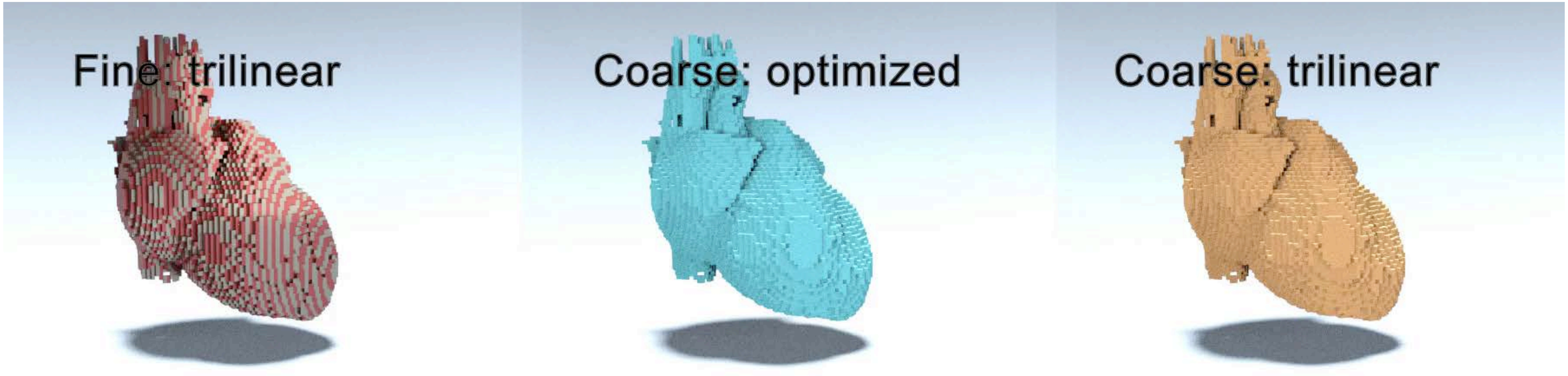
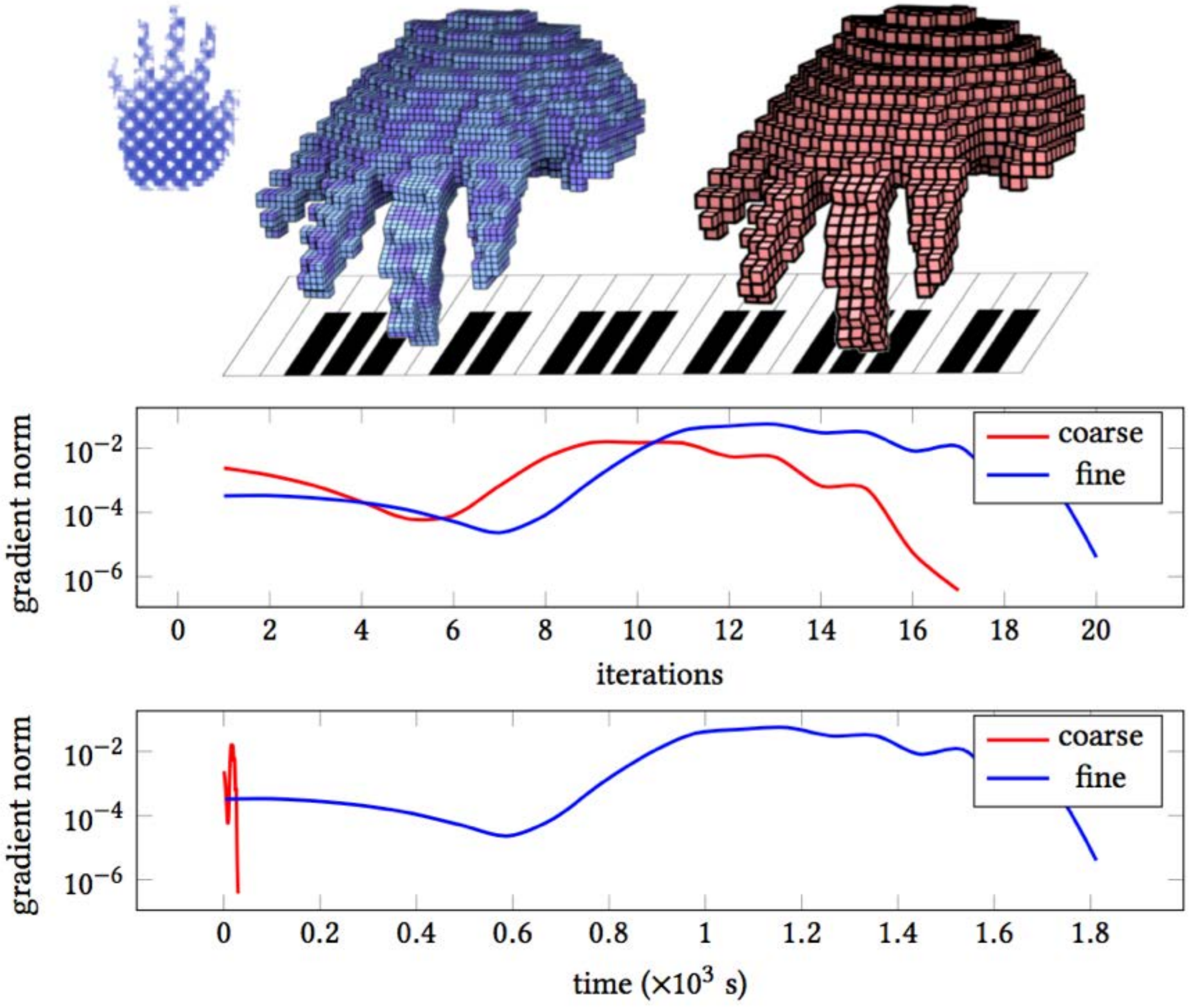


Dynamics



Acceleration



Fine mesh:
 # vert: 31337
 # elem: 26176

Coarse mesh:
 # vert: 4627
 # elem: 3272

Future work

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- Varying shape functions for very large deformation.

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- Varying shape functions for very large deformation.
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- Applied to other problems like acoustics.
- ***Problem-aware basis construction.***

Thanks!
Q&A