IGA Spline Modeling with Local Refinement and Engineering Applications

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Outline

1. Introduction
2. Literature Review
3. Truncated Hierarchical Catmull-Clark Subdivision (THCCS)
4. Extended Truncated Hierarchical Catmull-Clark Subdivision (eTHCCS)
5. Truncated T-splines
6. Truncated Hierarchical Splines on Unstructured Hex Meshes
7. Blended B-Spline Construction on Unstructured Quad/Hex Meshes
8. Conclusion

1. Introduction

Importance

- ❑ Local refinement has been an important issue since isogeometric analysis (IGA) was proposed
  - Local editing without superfluous control points
  - Adaptive analysis with higher efficiency
- ❑ Representation of complex geometries with extraordinary points
  - Geometric smoothness (G^1)
  - Optimal convergence rate

Objectives

- ❑ Develop highly localized refinement scheme
  - Support complex geometries
  - Suitable for geometric design and analysis
- ❑ Develop a method that achieves optimal convergence rates with 3D extraordinary points

1. Introduction

Local Refinement on Complex Geometries

Catmull-Clark Subdivision

T-splines

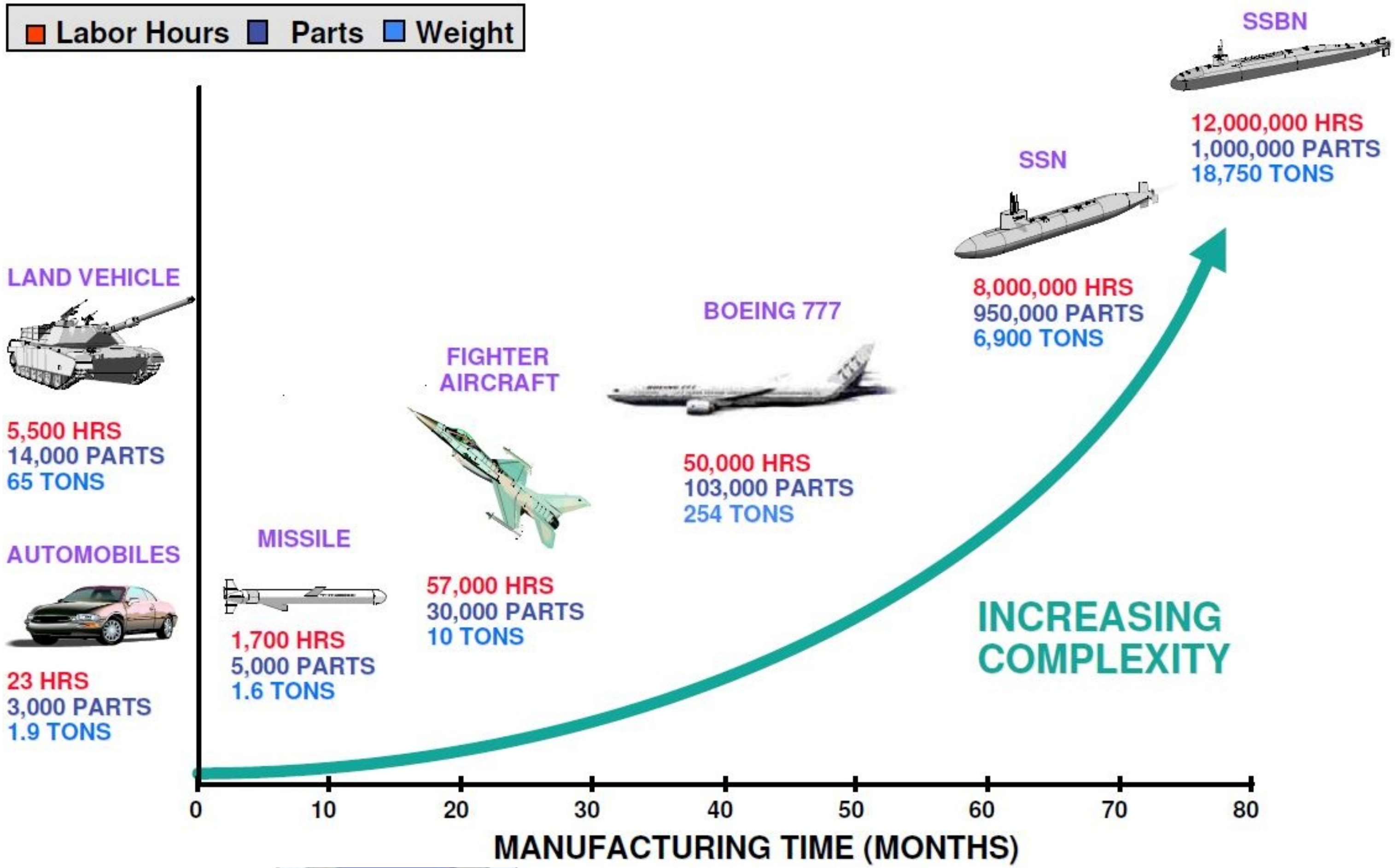
Optimal Convergence Rate

Truncation Mechanism

Adding Extra Functions

2.1. Isogeometric Analysis (IGA)

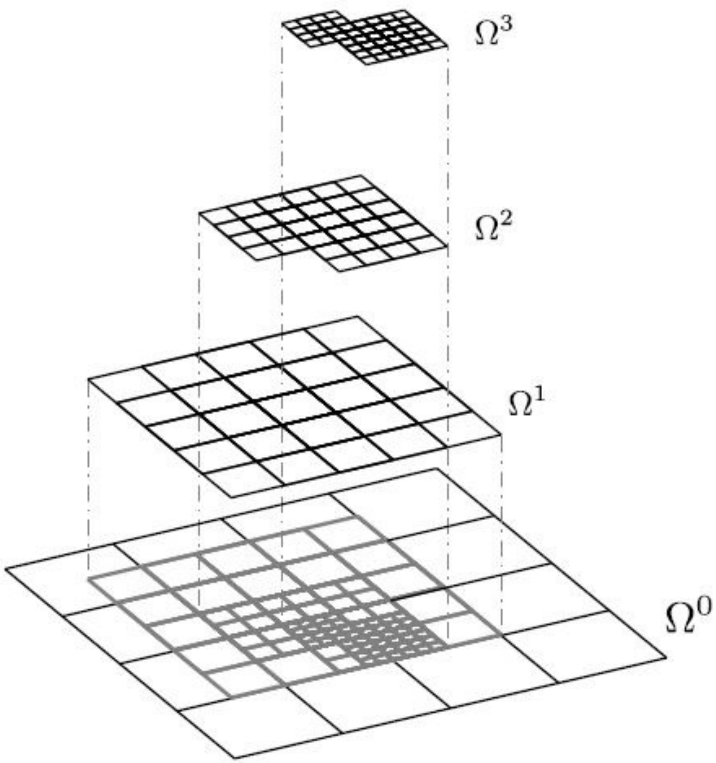
- IGA employs the same basis for geometric representation in analysis, to integrate CAD and FEM [*Hughes et al. 2005*]



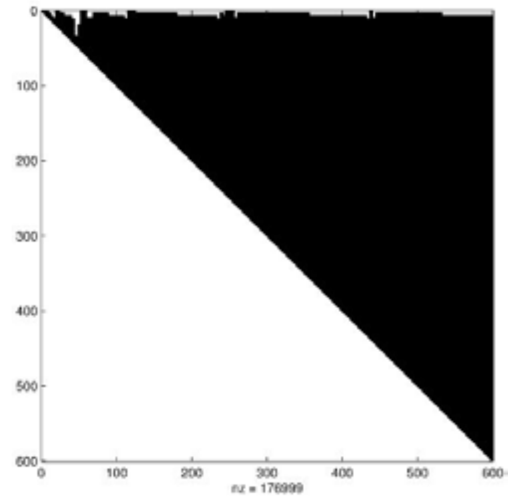
[Cottrell et al. 2009]

2. Review: Truncated Hierarchical B-splines (THB-splines)

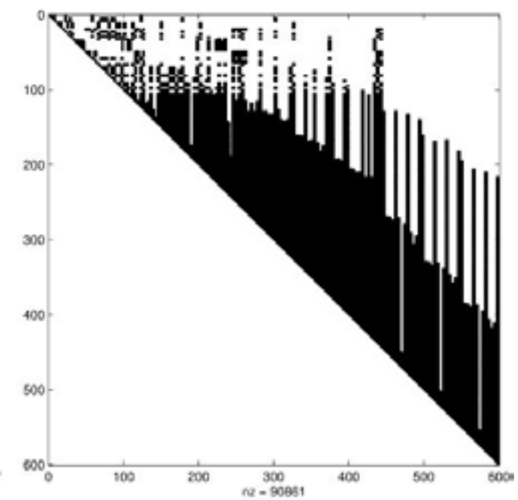
- THB-splines introduce truncation mechanism to improve HB-splines [*Giannelli et al. 2012*]



Hierarchical nested subdomains



Hierarchical B-splines

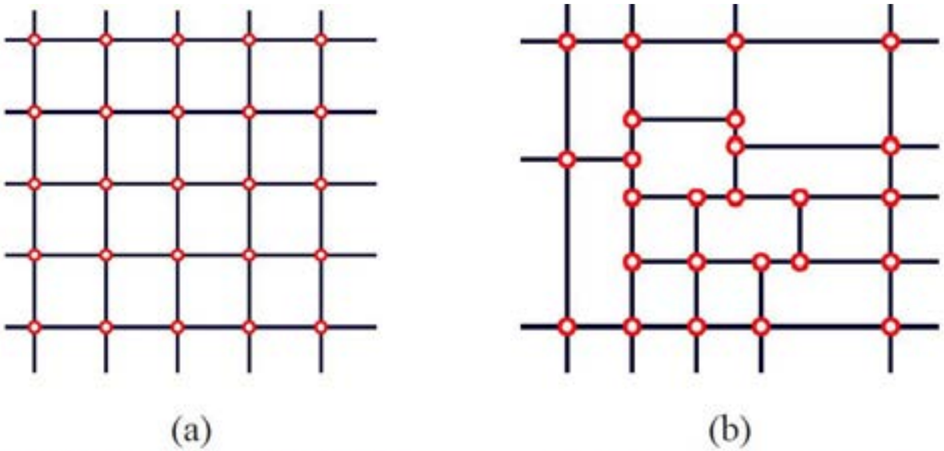


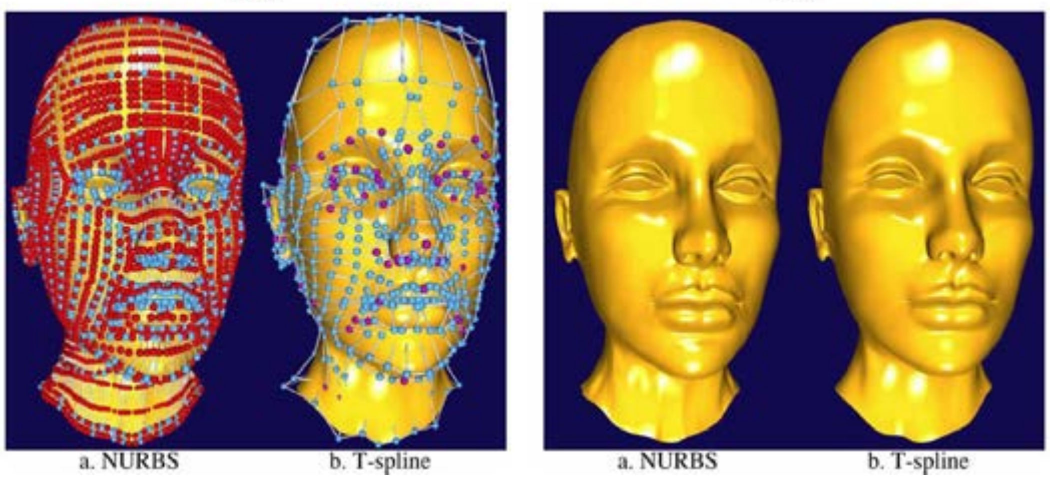
THB-splines

[Giannelli et al. 2012]

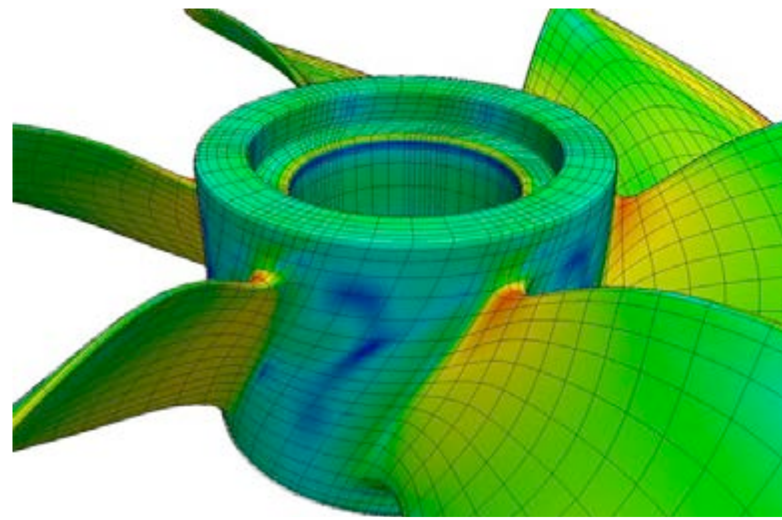
2. Review: T-splines

- T-splines break down the global tensor product of NURBS into local ones, and allow T-junctions [*Sederberg et al. 2003*]





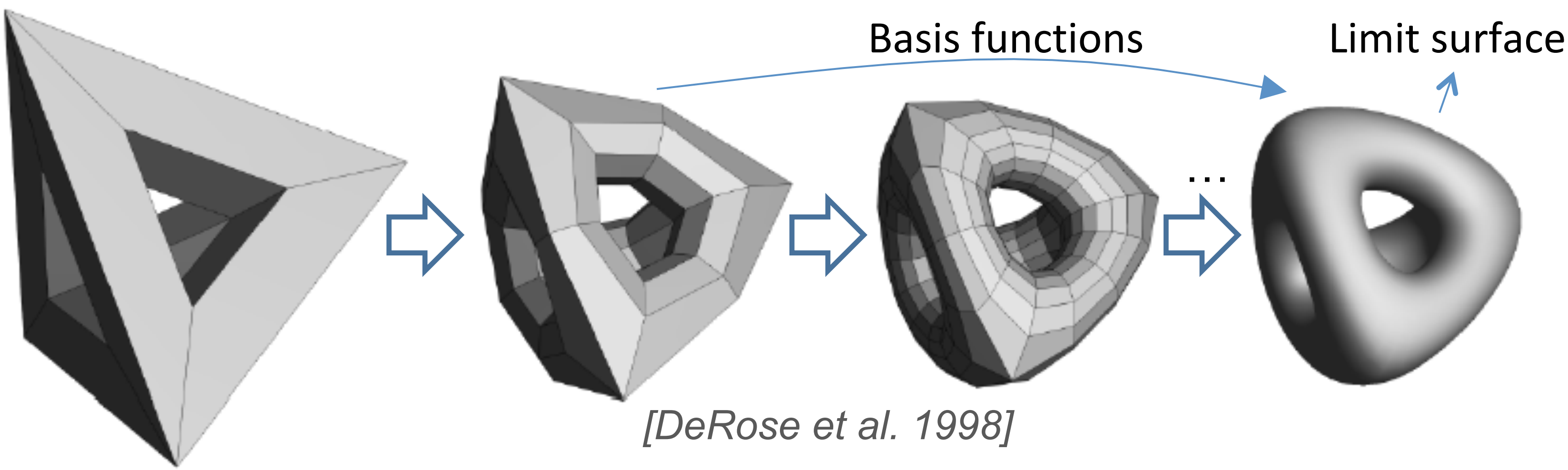
[Sederberg et al. 2003]



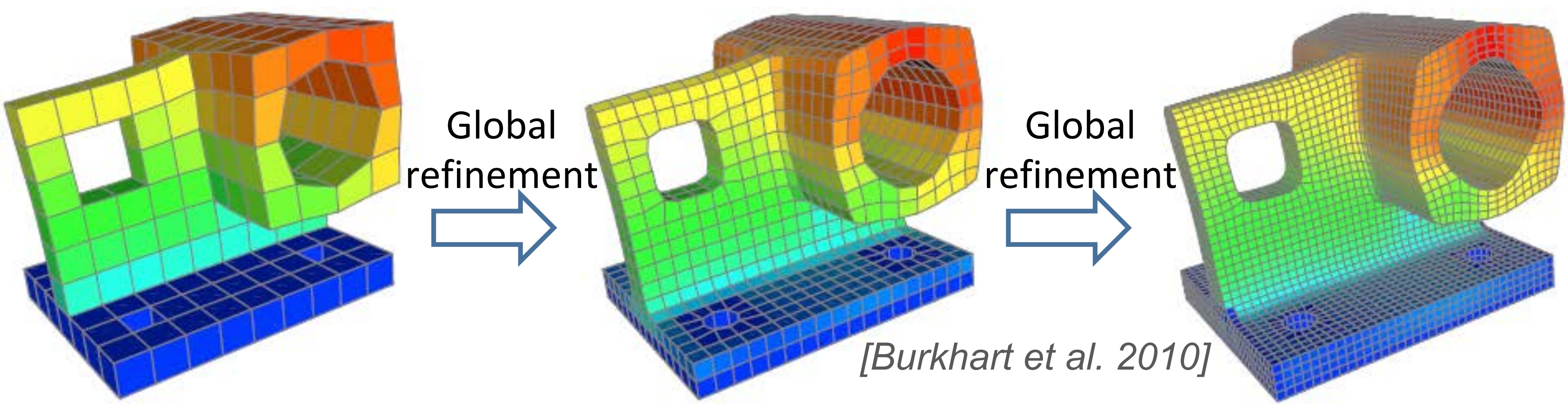
[Scott et al. 2013]

2. Review: Catmull-Clark Subdivision

❑Catmull-Clark subdivision generalizes bicubic B-splines to control meshes of arbitrary topologies [*Catmull et al. 1977*]



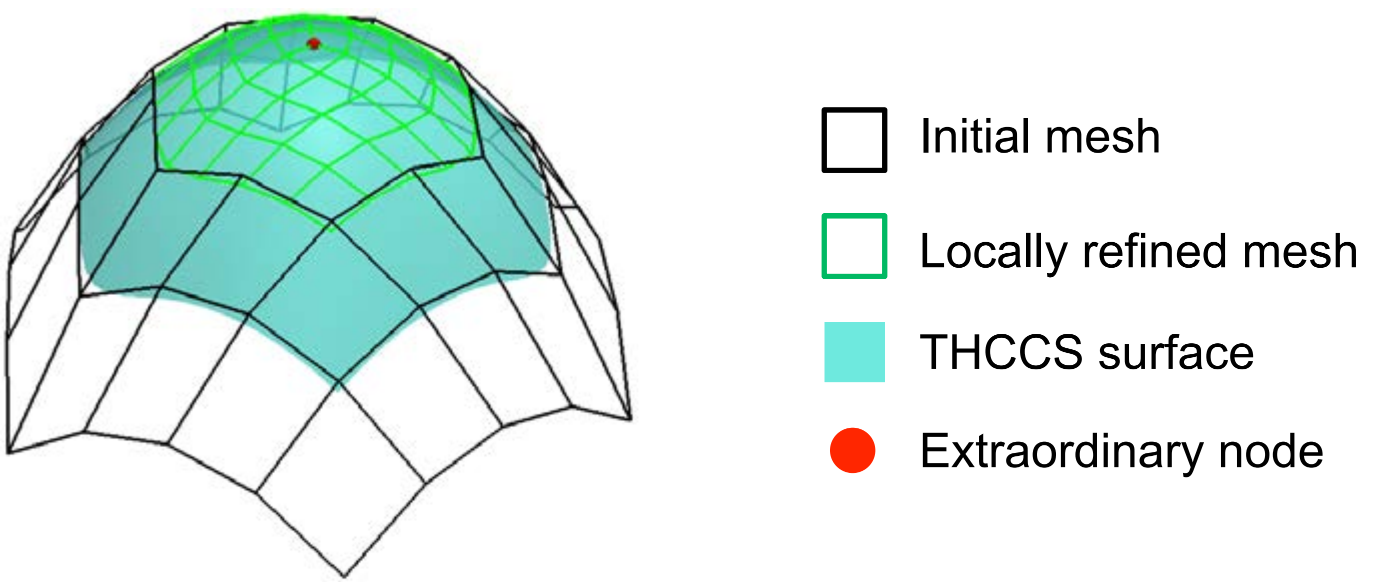
[DeRose et al. 1998]



[Burkhart et al. 2010]

3. Truncated Hierarchical Catmull-Clark Subdivision (THCCS)

- Motivation
  - THB-splines do not support general 2-manifold domains
  - Hierarchical Catmull-Clark is not suitable for analysis
  - CHARMS is not suitable for geometric design
- Objective: Develop a technique suitable for both design and analysis on general domains

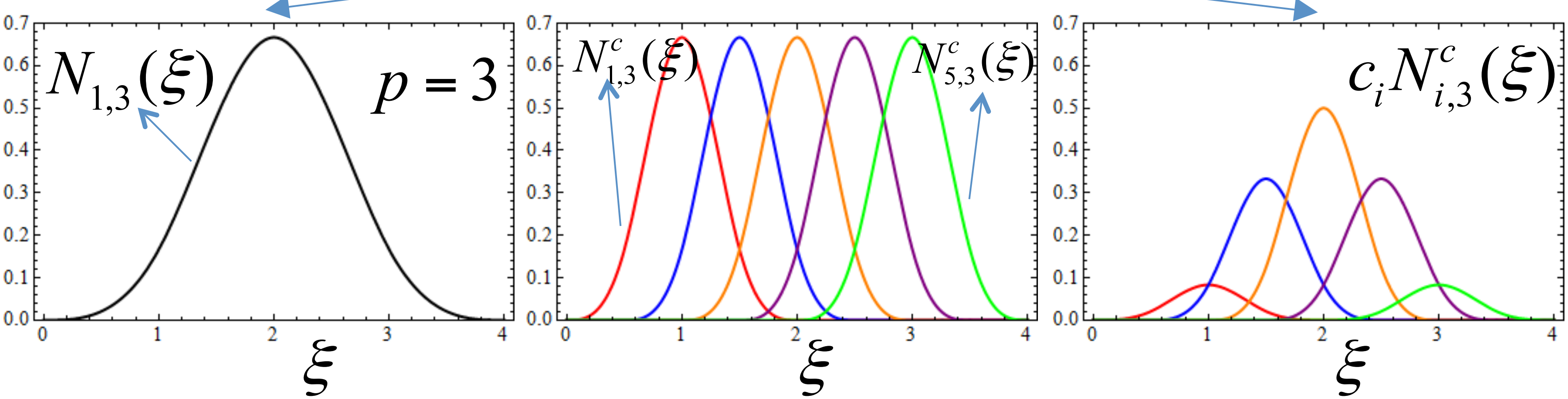


3.1. B-spline Refinability

- A B-spline basis function can be represented by a linear combination of refined basis functions (children)

$$N_{1,3}(\xi) = \sum_{i=1}^{5} c_i N_{i,3}^c(\xi)$$

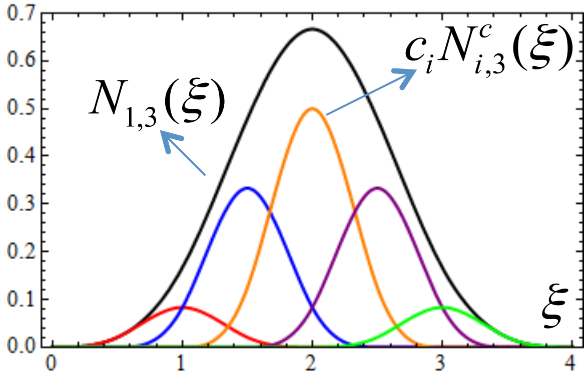
Children



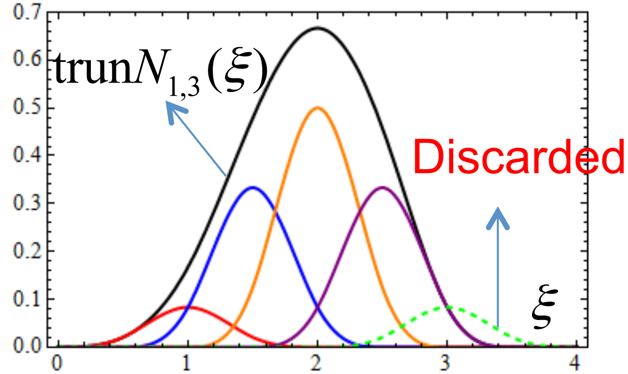
$\Xi = \{0, 1, 2, 3, 4\}$ $\quad$ $\Xi^c = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\}$ $\quad$ $c_i = \frac{1}{8}\binom{4}{i-1}$

3.2. Truncation

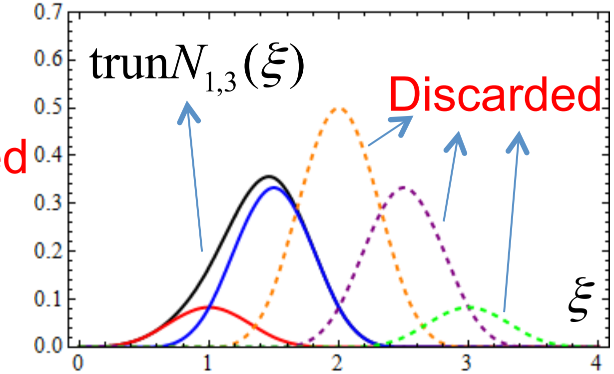
- Truncation: Discard identified children from refinability relationship



$$N_{1,3}(\xi) = \sum_{i=1}^{\boxed{5}} c_i N^c_{i,3}(\xi)$$



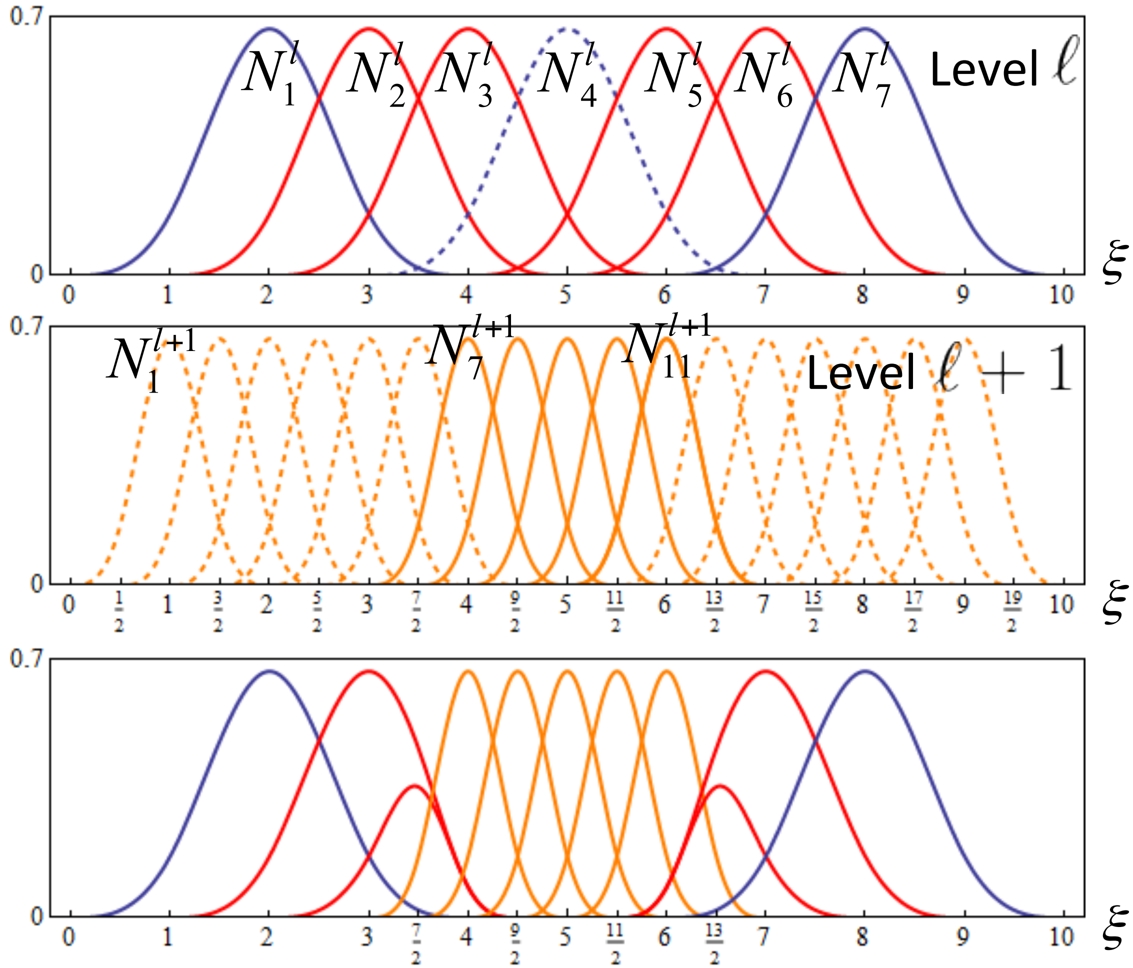
$$\mathrm{trun}N_{1,3}(\xi) = \sum_{i=1}^{\boxed{4}} c_i N^c_{i,3}(\xi)$$



$$\mathrm{trun}N_{1,3}(\xi) = \sum_{i=1}^{\boxed{2}} c_i N^c_{i,3}(\xi)$$

3.3. THB-splines

- Truncation mechanism: Discard children shared with to-be-refined basis functions



Identification:
To-be-refined basis functions at low level ℓ

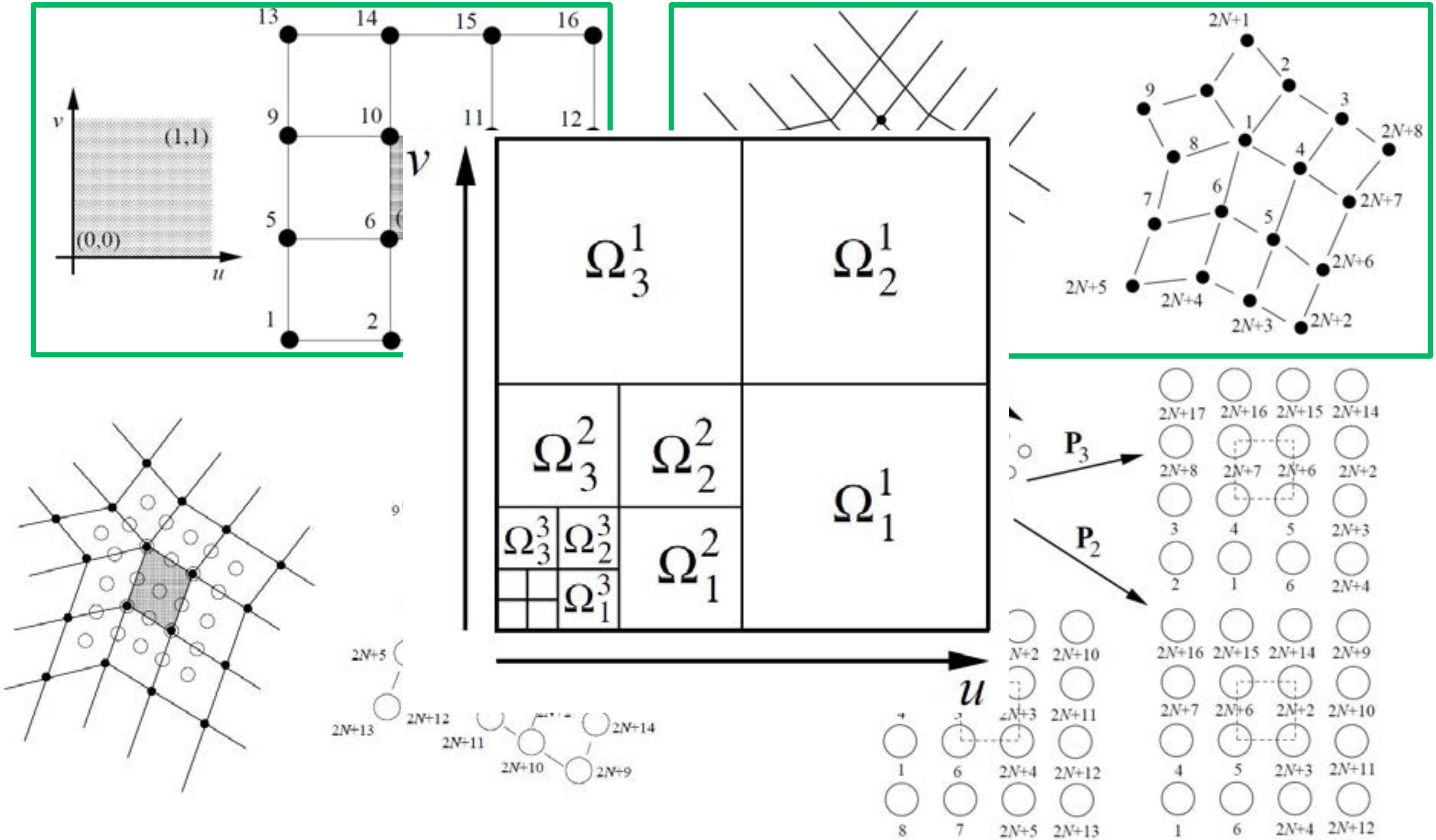
Refinement:
Add active splines at high level $\ell + 1$

Truncation and Collection:
Form basis of THB-splines

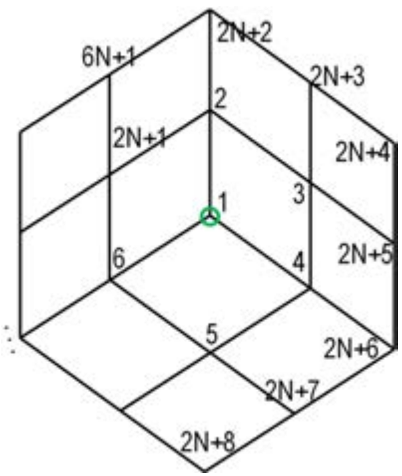
$$\mathrm{trun}N_i^l = \sum_{\text{passive children}} c_{ij} N_j^{l+1}$$

3.4. Catmull-Clark Basis Functions

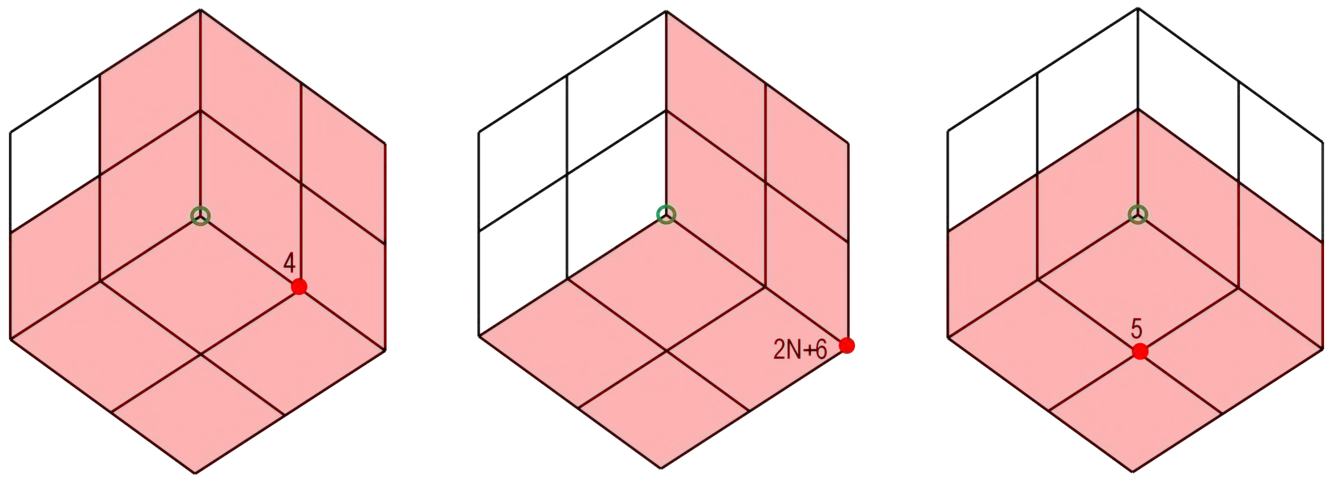
- ❑ Stam's basis functions [*Stam, 1998*] $\mathbf{B}^{\ell}(\xi,\eta) = (\mathbf{V}^{-1})^{T}\mathbf{\Lambda}^{n-\ell-1}(\mathbf{P}_k\bar{\mathbf{A}}\mathbf{V})^{T}\mathbf{b}(\xi,\eta)$



3.5. Example of truncation

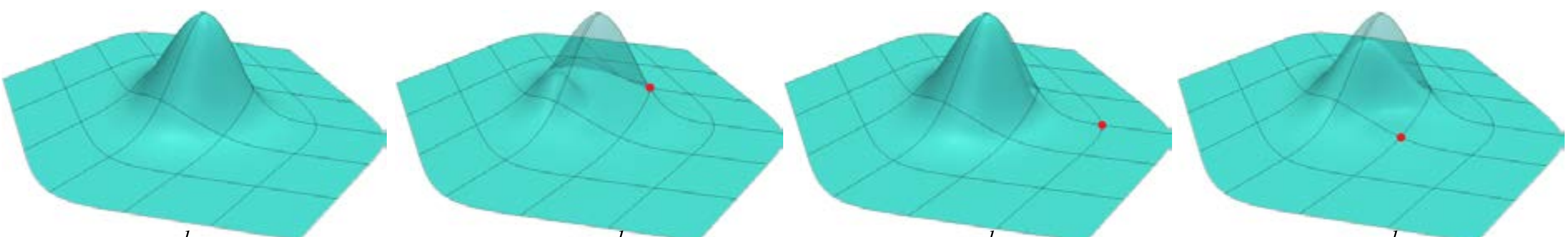


Example 1 | Example 2 | Example 3



$$\mathrm{trun}B_i^l = \sum_{\text{Passive children}} \boxed{c_{ij}} B_j^{l+1}$$

c_{ij} → subdivision rule



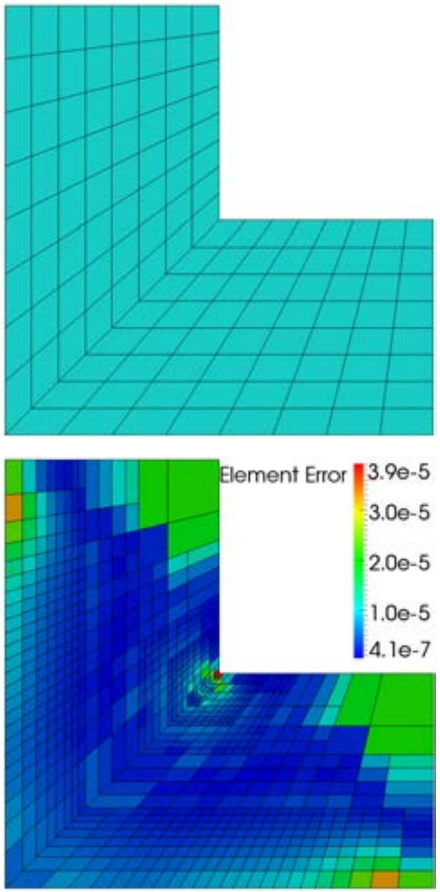
B_1^l | $\mathrm{trun}B_1^l$ | $\mathrm{trun}B_1^l$ | $\mathrm{trun}B_1^l$

Before truncation

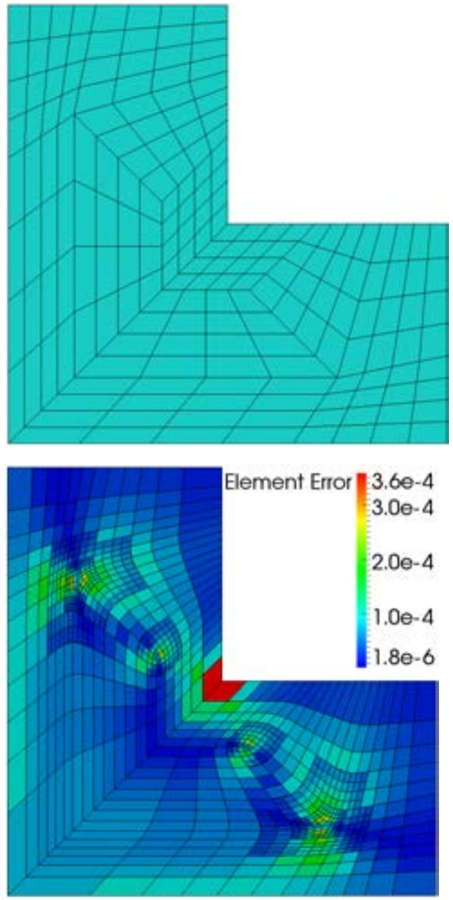
Comparison before and after truncation

3.6. Benchmark problem: *L*-shaped domain

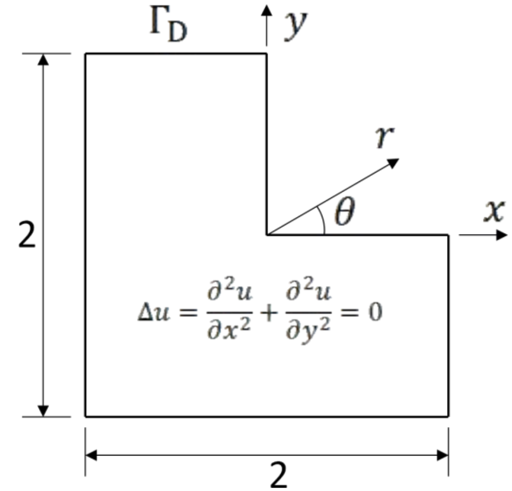
- ❑ Solving Laplace equation over *L*-shaped domain

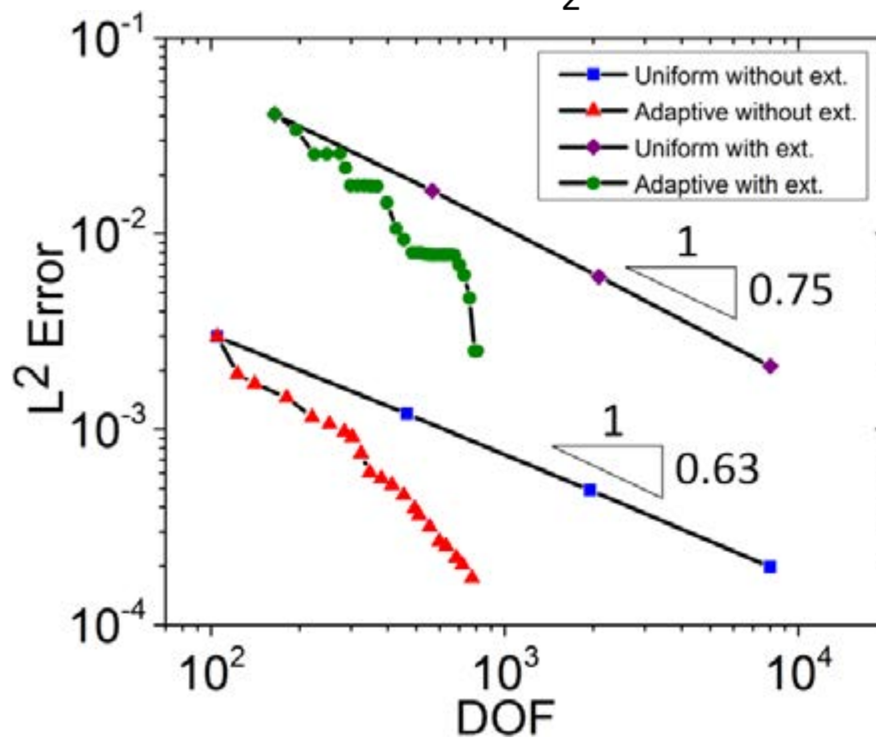


Regular input quad mesh



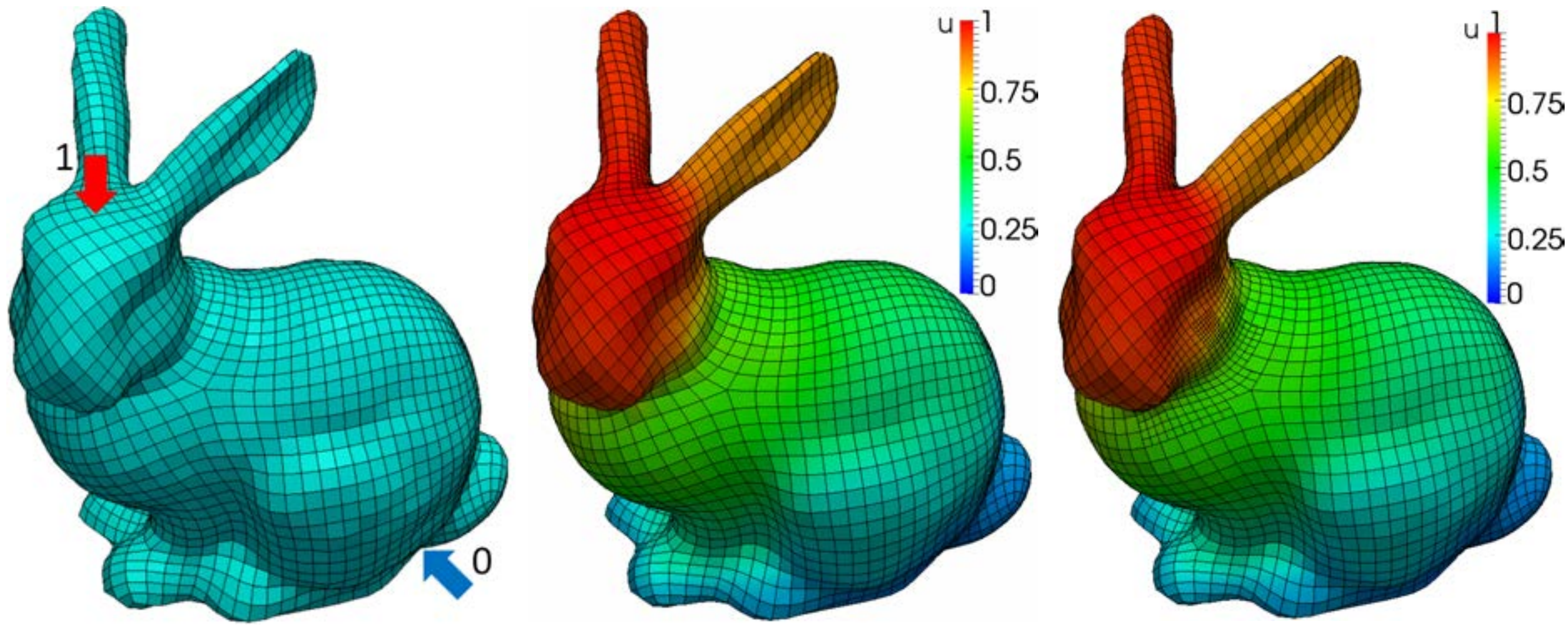
Irregular input quad mesh





3.6. Complex geometries

- Solving Laplace-Beltrami equation with local refinement
  - Preprocessing: Subdivide invalid elements to separate extraordinary nodes



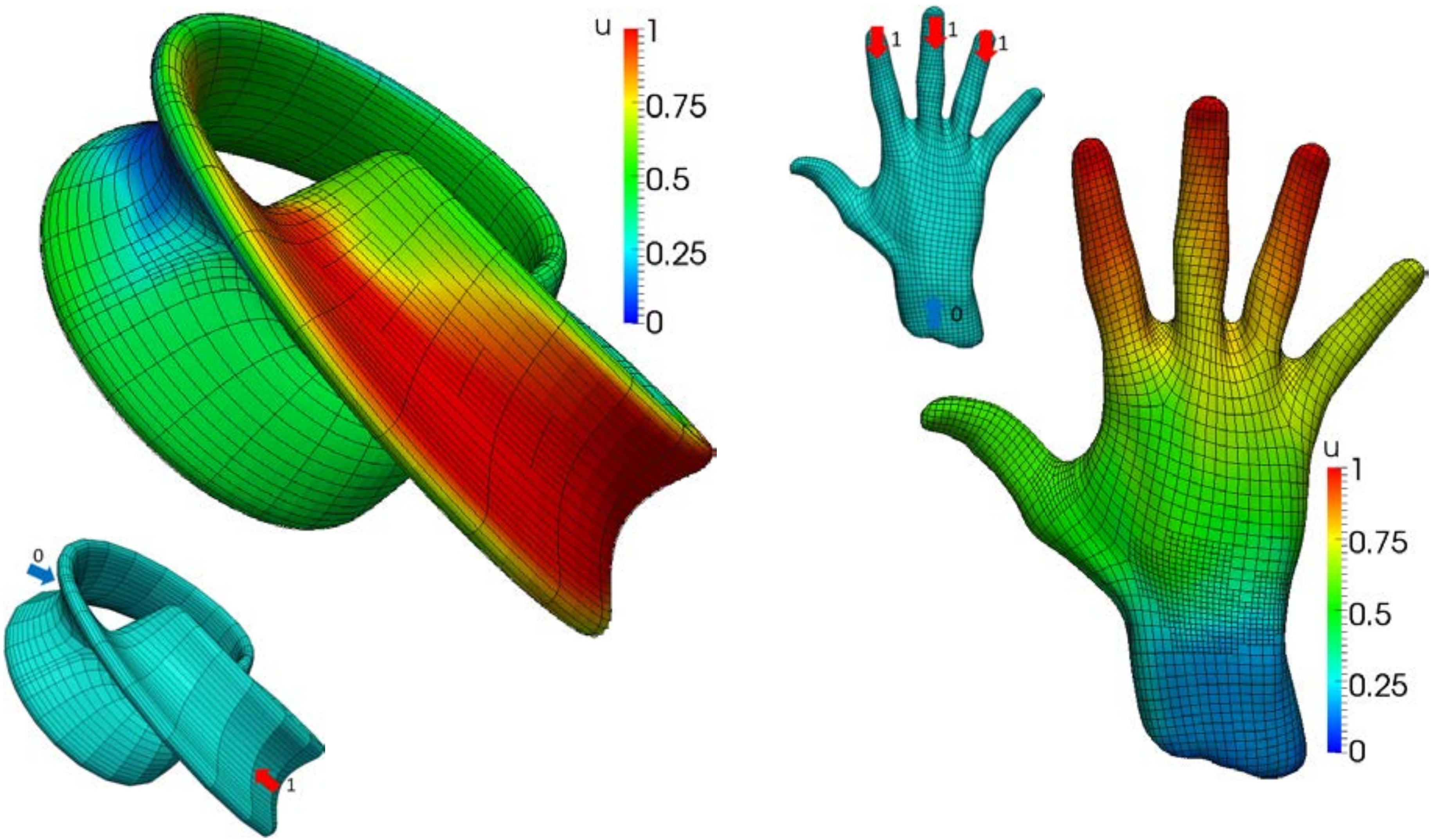
Input & boundary condition

Solution after preprocessing

Solution after 20 steps

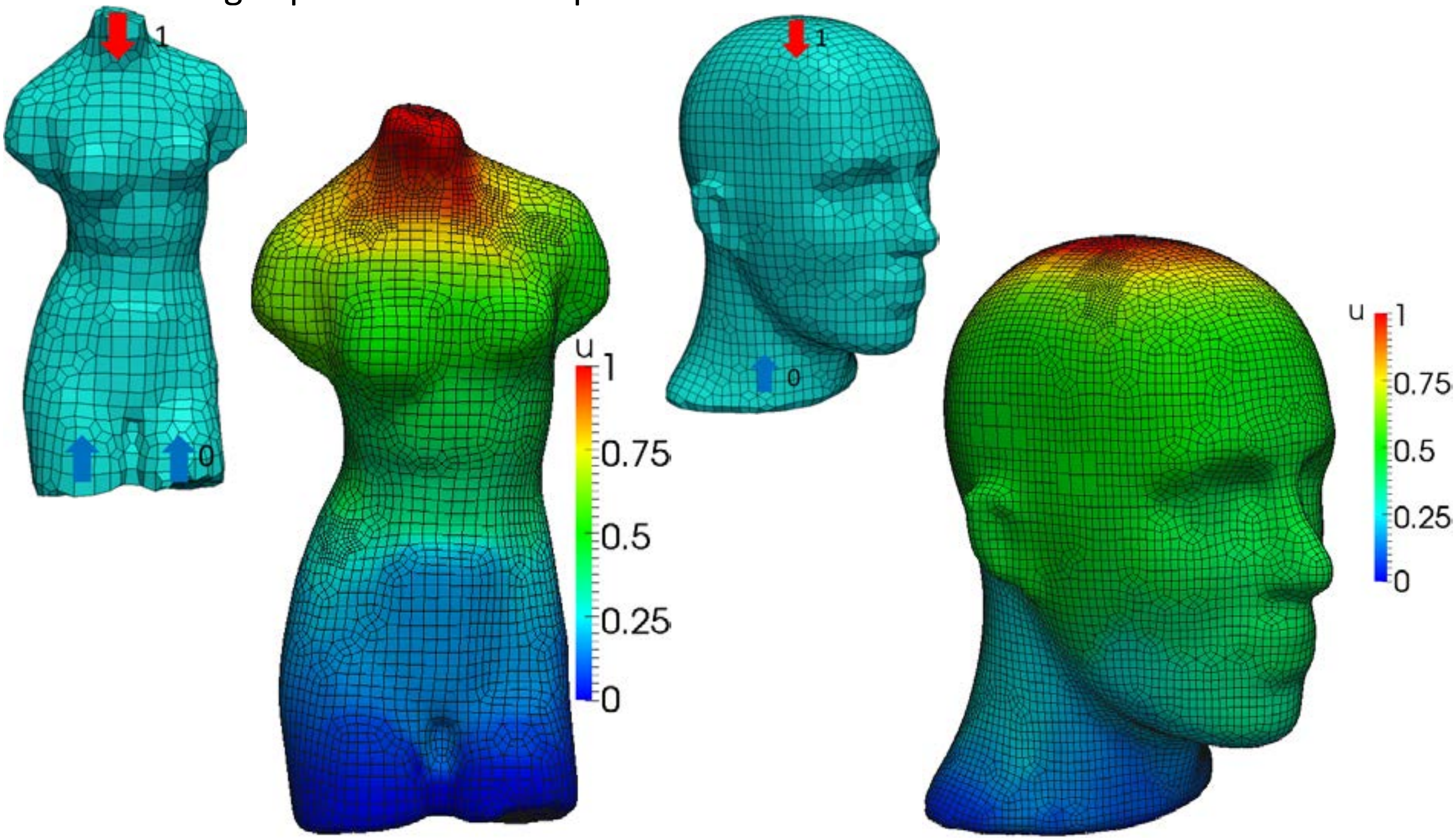
3.6. Complex geometries

- Solving Laplace-Beltrami equation with local refinement



3.6. Complex geometries

- Solving Laplace-Beltrami equation with local refinement



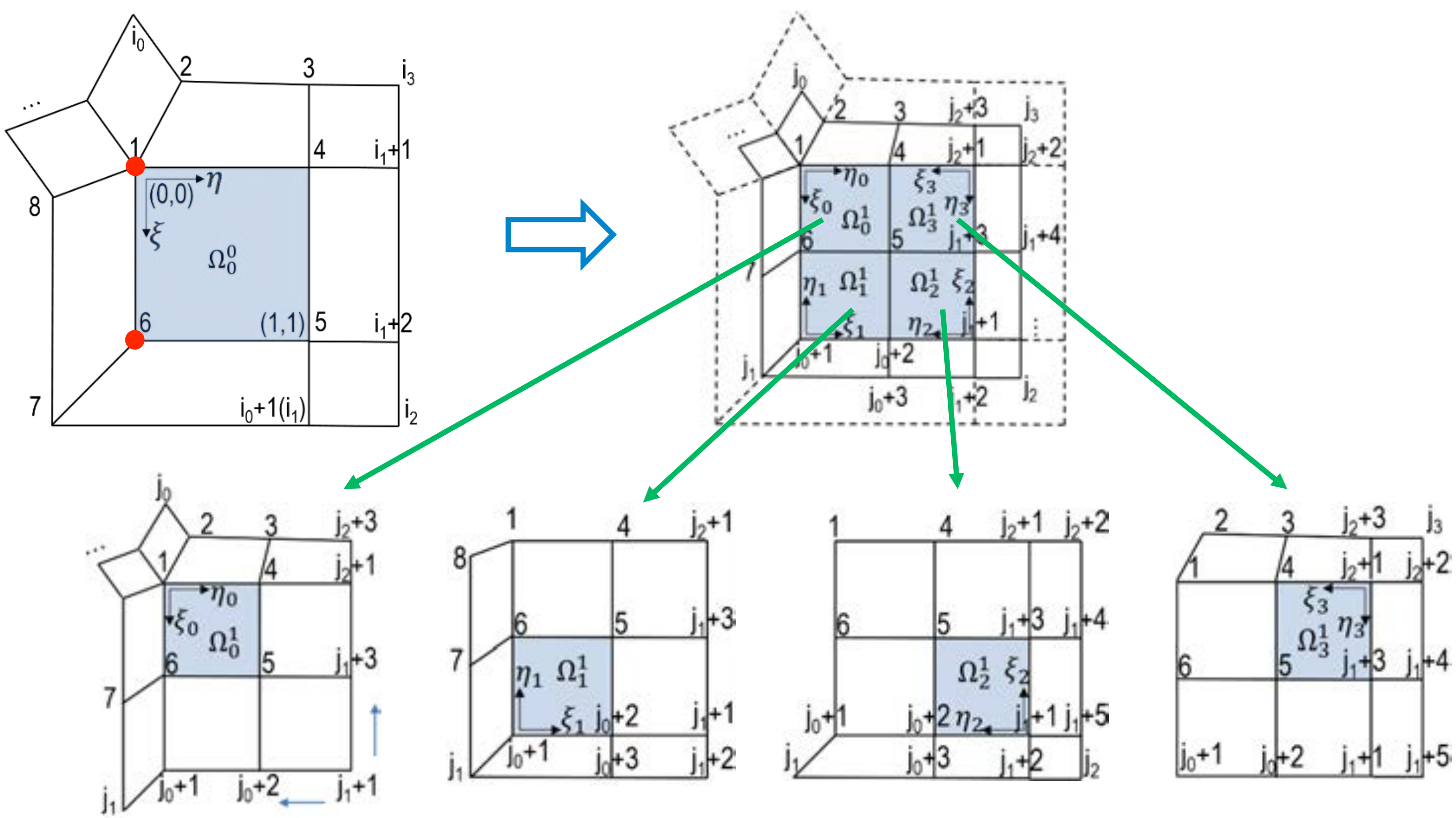
4. Extended THCCS (eTHCCS)

- Motivation
  - Local refinement of THCCS has a strong constraint, especially for quad meshes with excessive invalid elements
- Objective: Improve the refinement locality of THCCS

	THCCS	eTHCCS
Minimum to-be-refined region at each refinement step	Two-ring neighborhood	One-ring neighborhood
Invalid element (More than one extraordinary node)	Preprocessing	No preprocessing

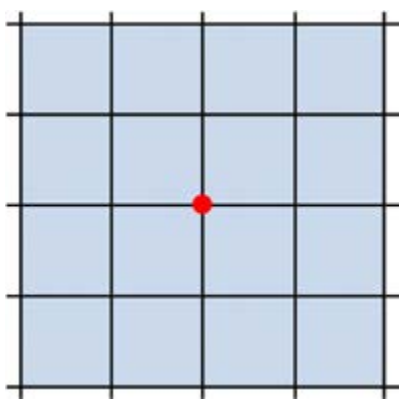
4.1. Catmull-Clark basis functions on invalid elements

- ❑ Build basis on arbitrary quad meshes for analysis
- ❑ Applied in the context of THCCS

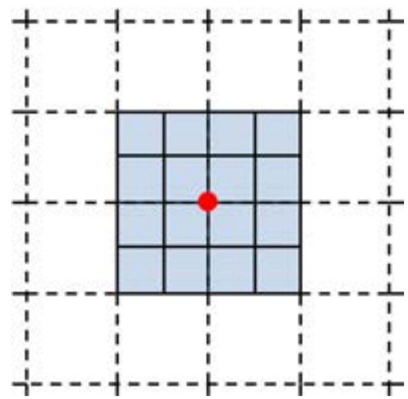


4.2. Minimum to-be-refined subdomain

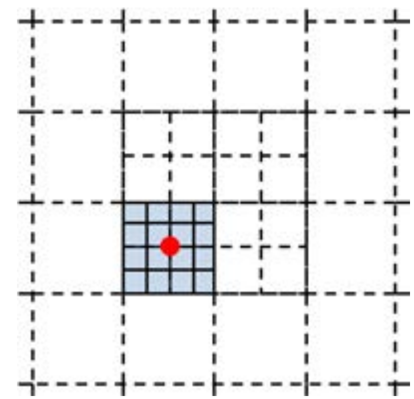
- ❑ Union of support of high-level basis functions
  - THCCS: Union of support of low-level basis functions
- ❑ At least insert one high-level basis function



THCCS



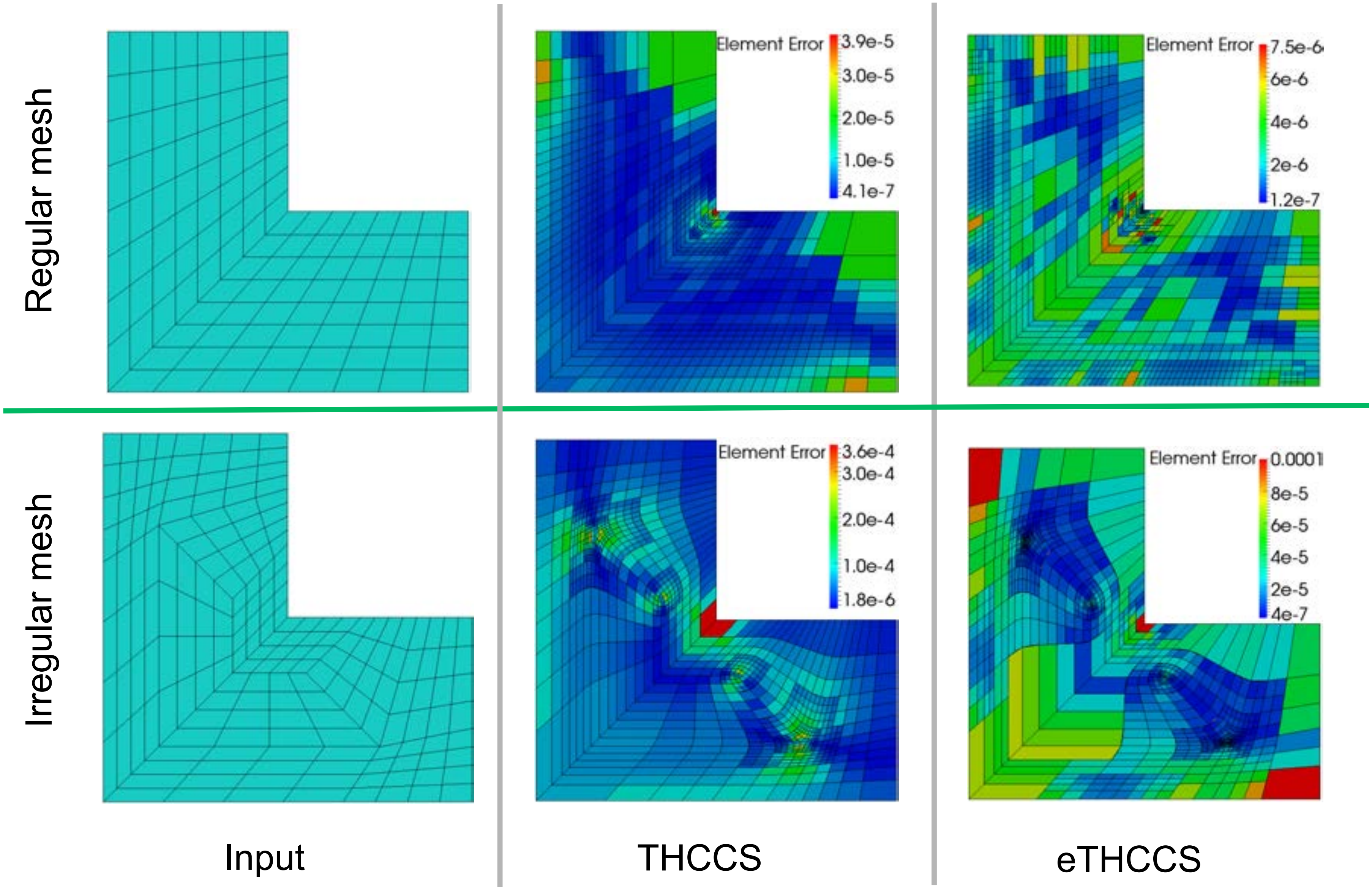
eTHCCS



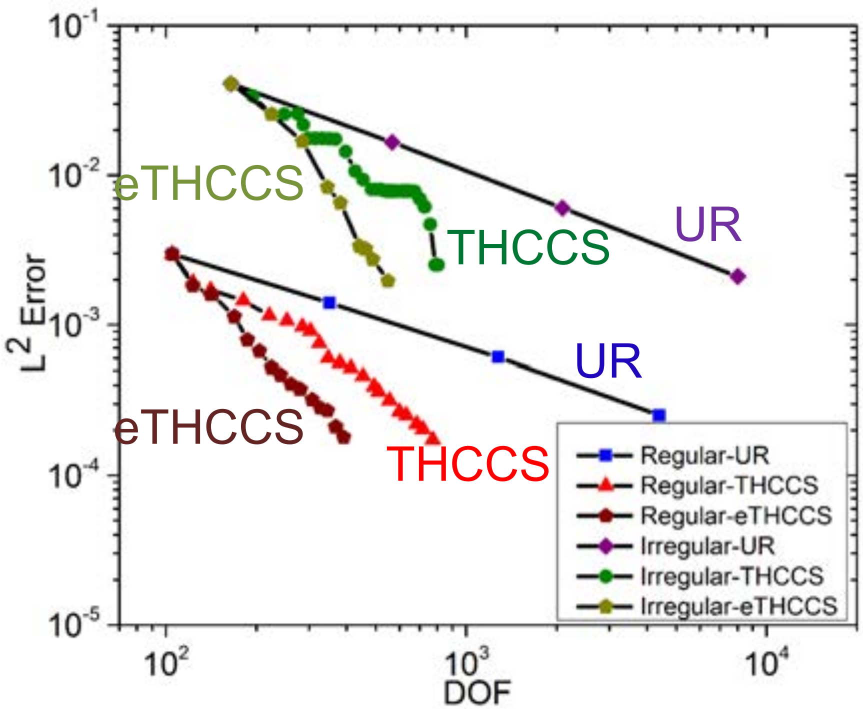
eTHCCS

One-ring refinement

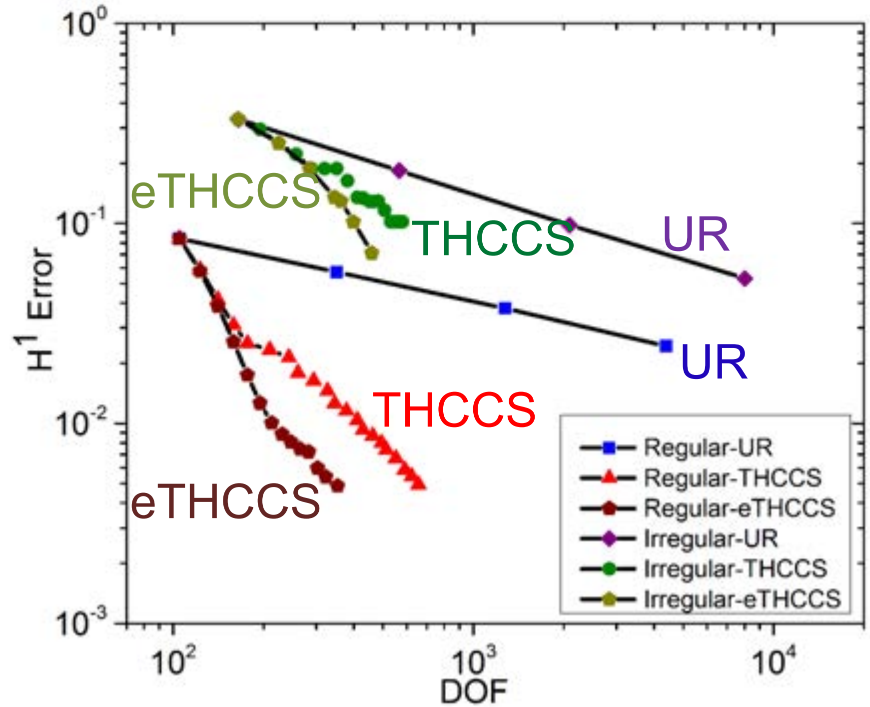
4.3. Benchmark problem – *L*-shaped domain



4.3. Benchmark problem – *L*-shaped domain



L^2 norm

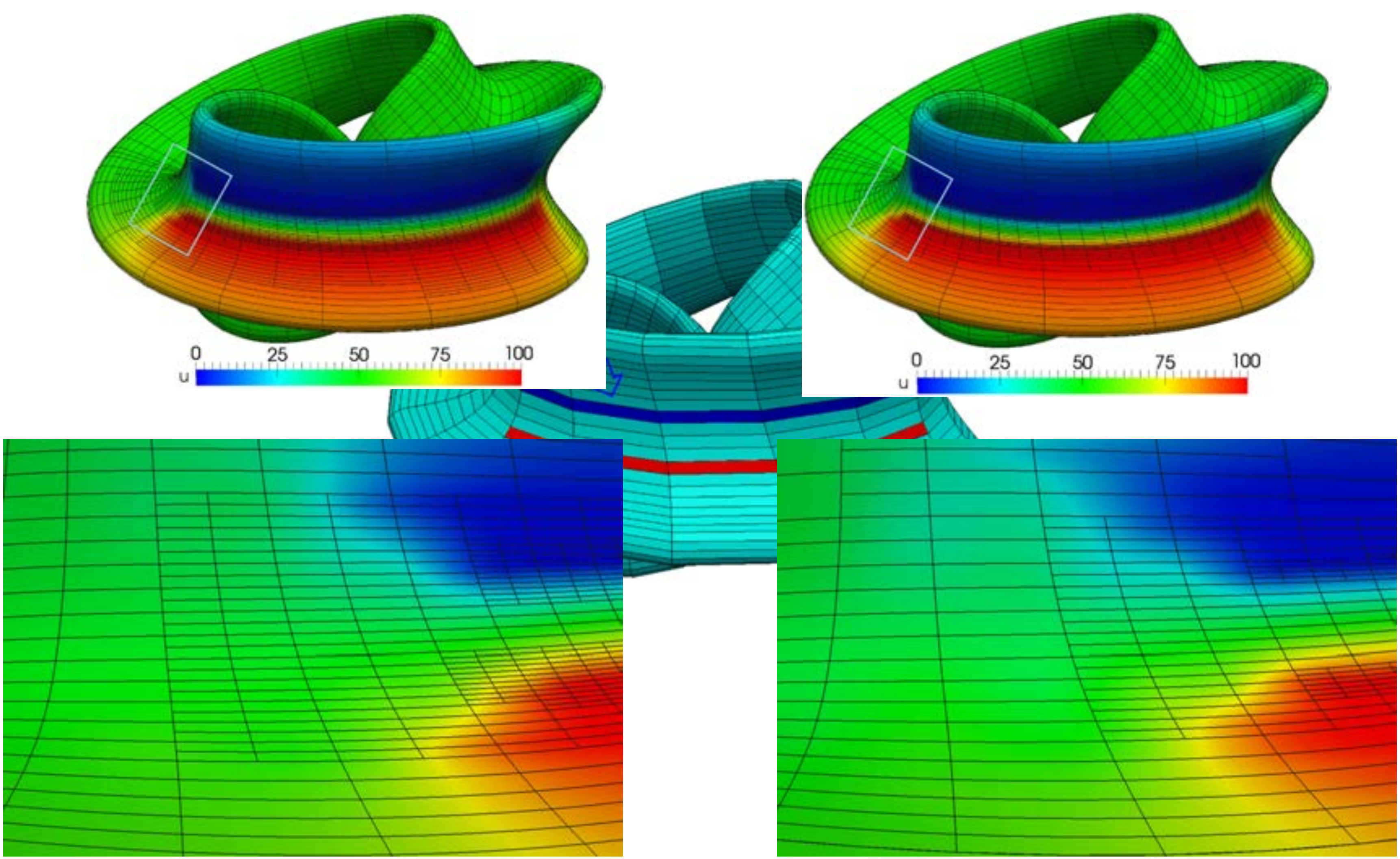


H^1 semi-norm

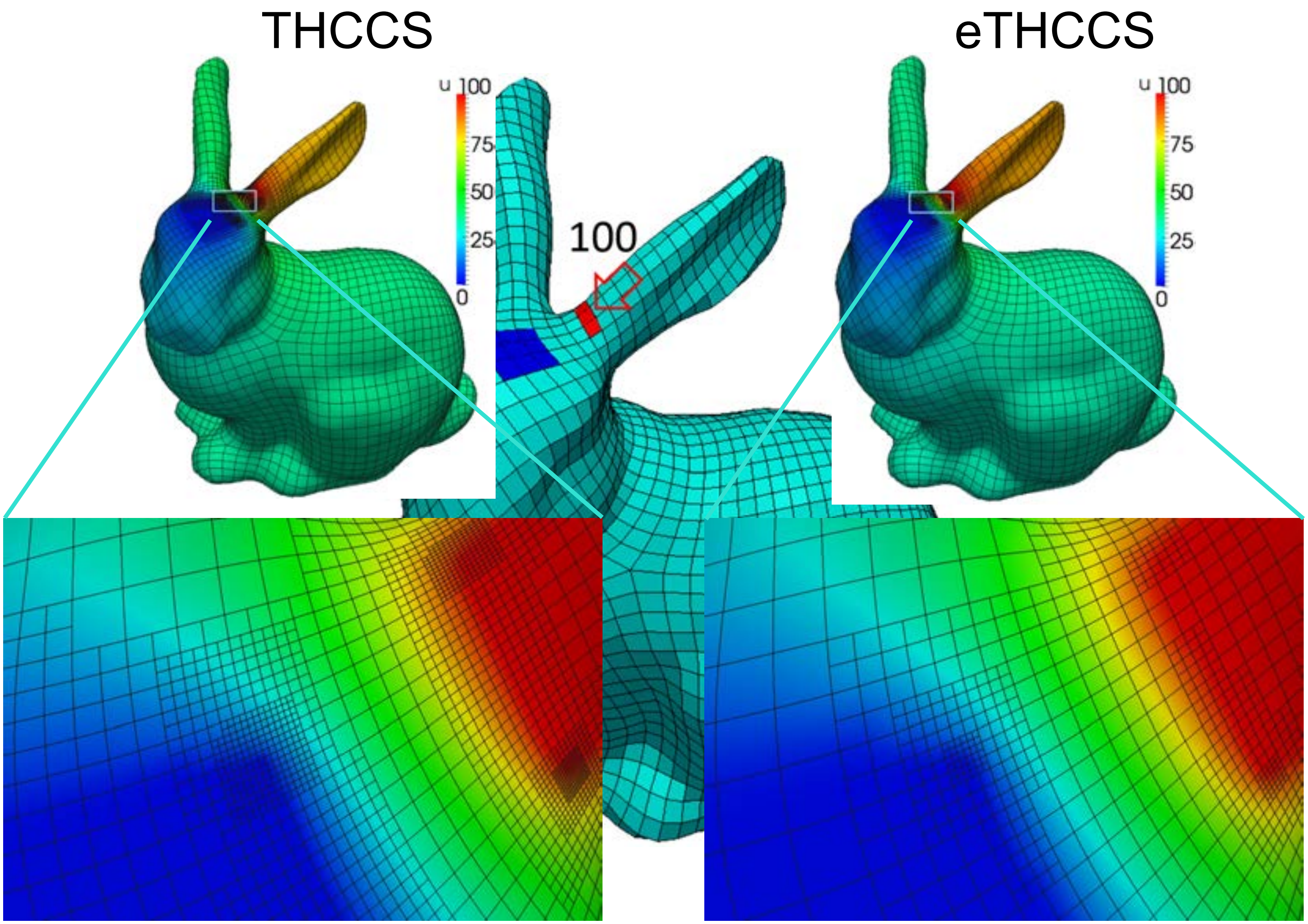
eTHCCS is more efficient than THCCS, which utilizes fewer DOF given the same error.

4.3. Complex surface models

THCCS | eTHCCS

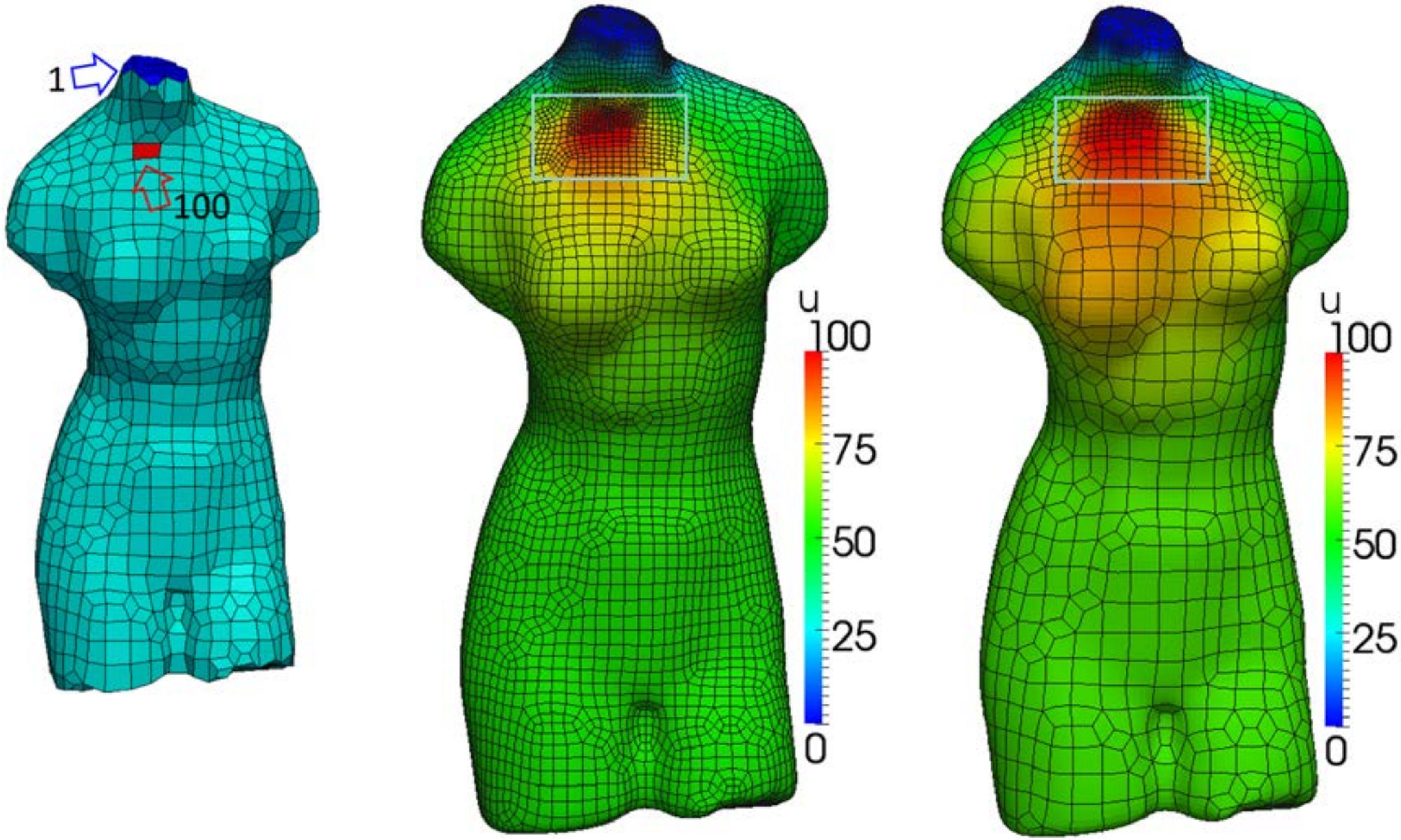


4.3. Complex surface models



4.3. Complex surface models

THCCS eTHCCS



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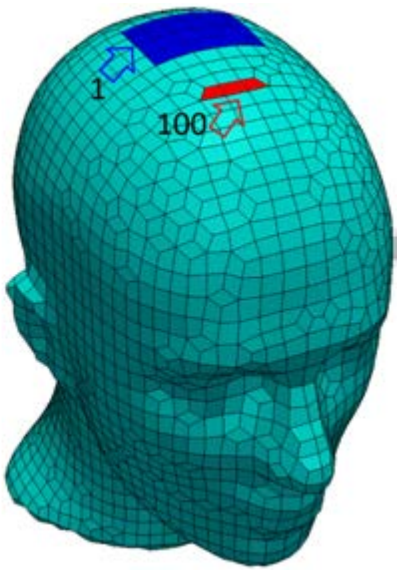
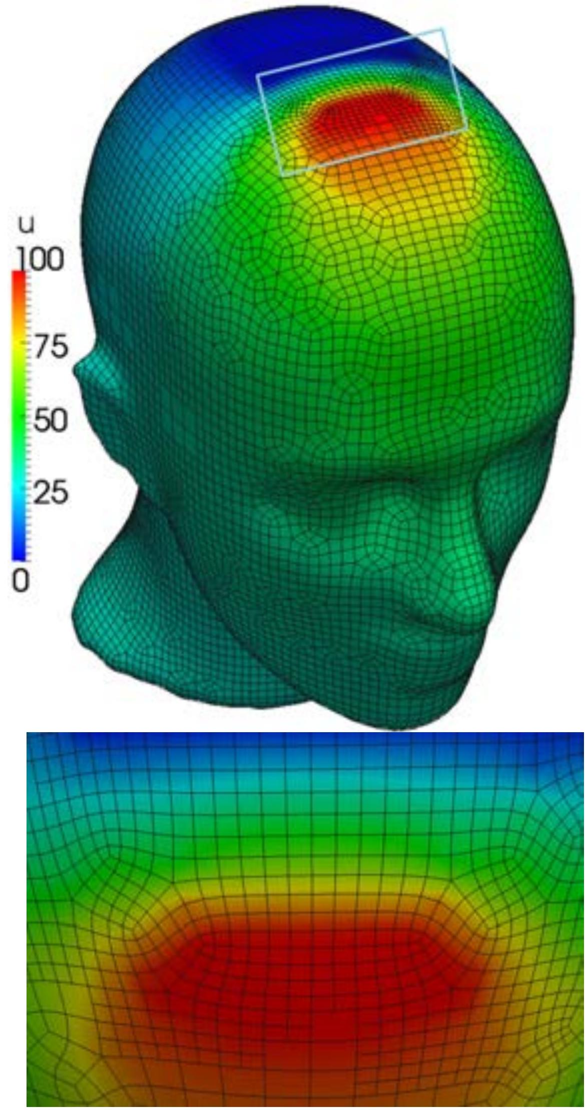
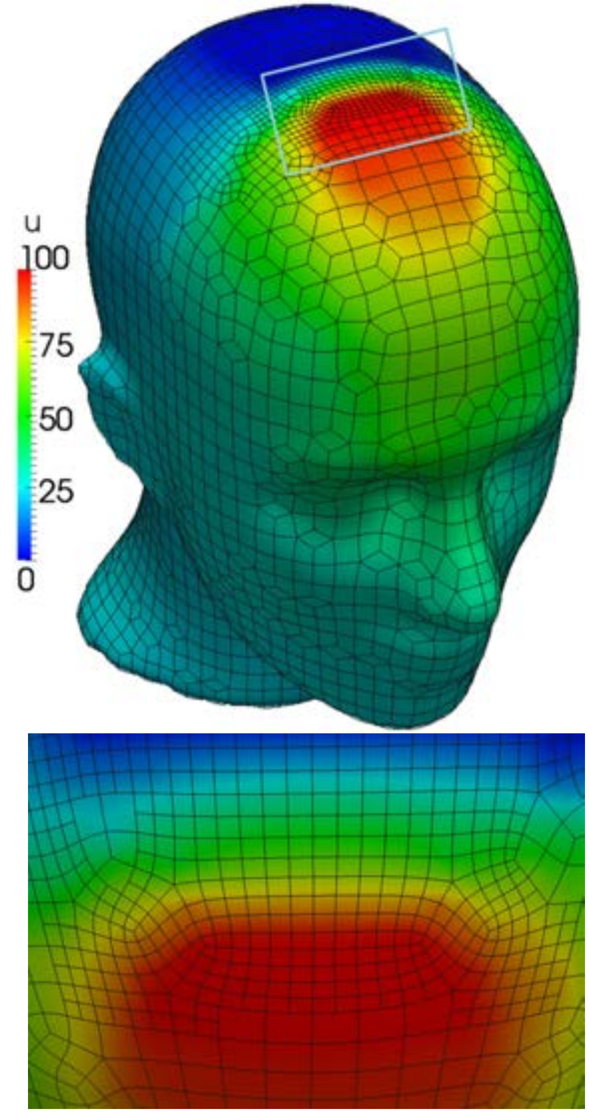


THCCS

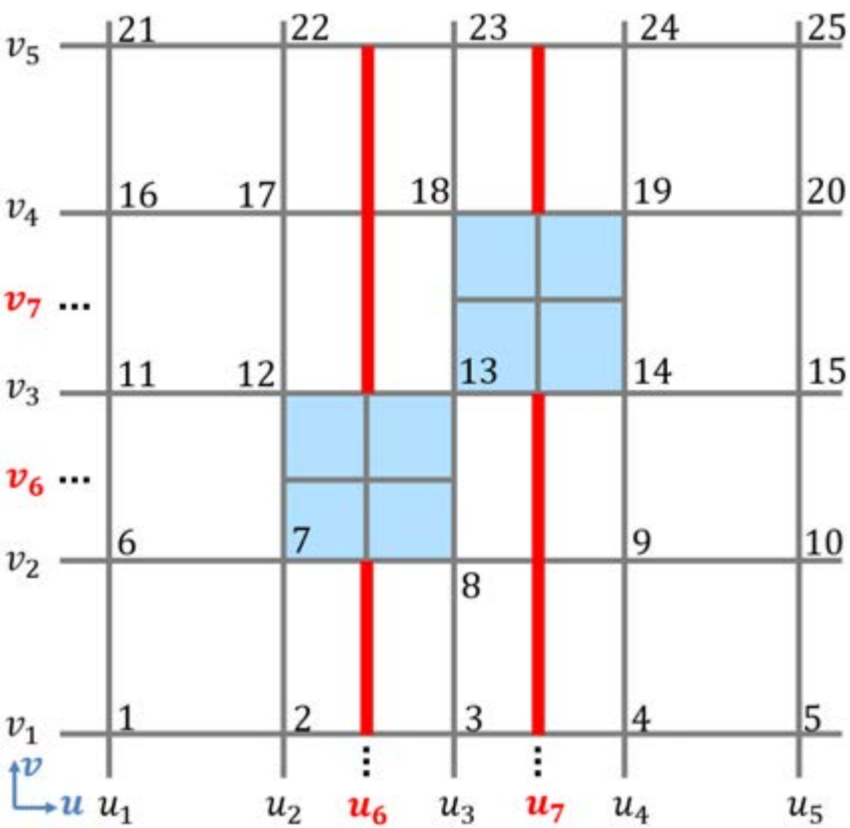


eTHCCS

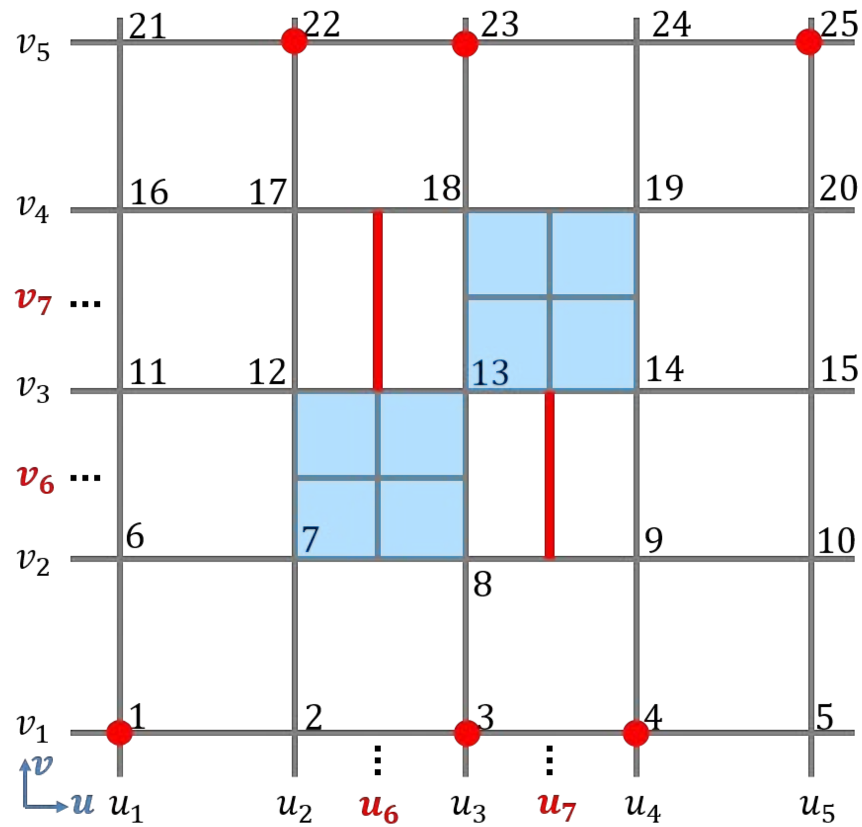


5. Truncated T-splines

- ❑ Motivation: Local refinement of analysis-suitable T-splines is not as local as expected
- ❑ Objective: Improve the refinement locality of analysis-suitable T-splines



Analysis-suitable T-splines



Truncated T-splines

5.1. Analysis-suitable T-splines vs Truncated T-splines

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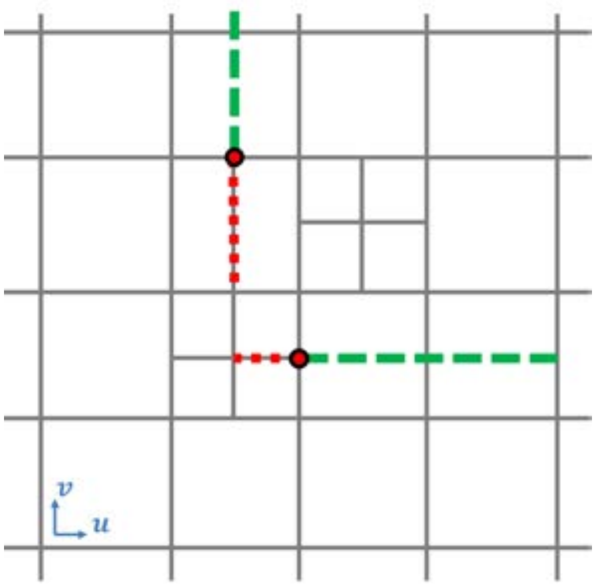
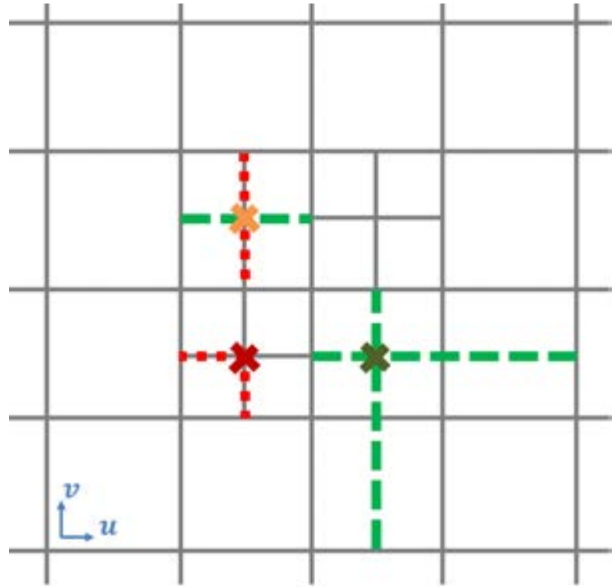
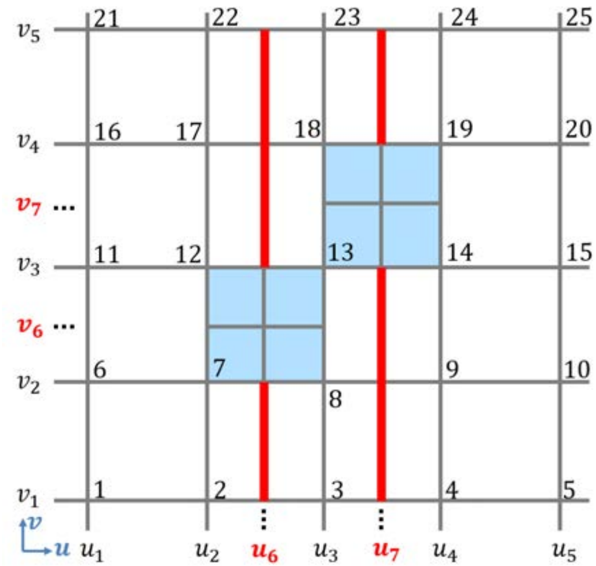
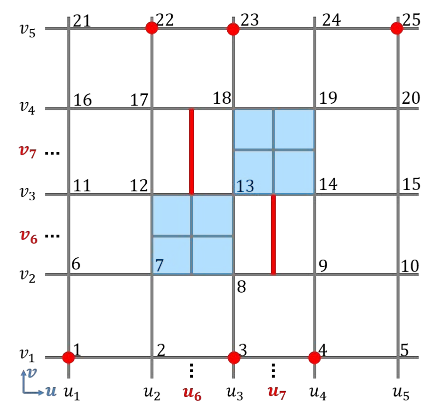
- ❑ Topological constraints: No intersection of T-junction extensions
- ❑ T-junction extensions: Face extension and edge extension







- ❑ T-splines with truncated basis functions
- ❑ Topological constraint: No face-face intersection
  - To ensure nestedness of spline spaces

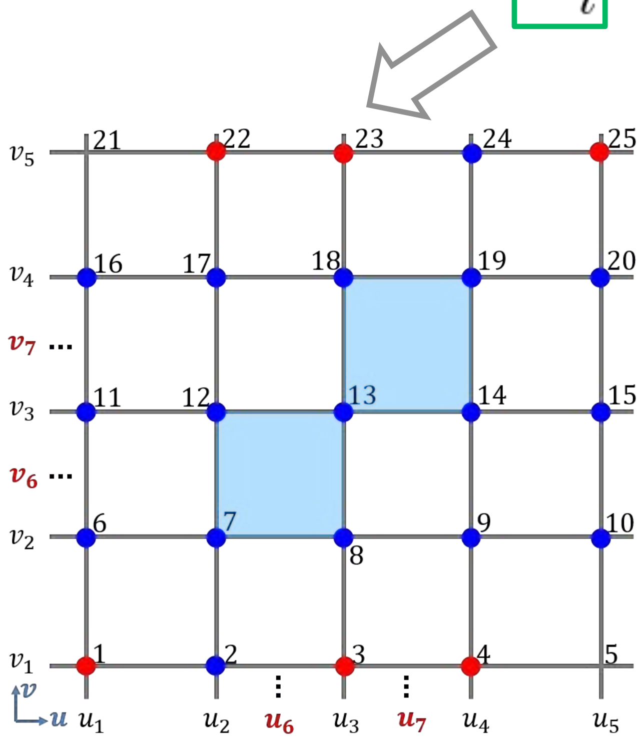


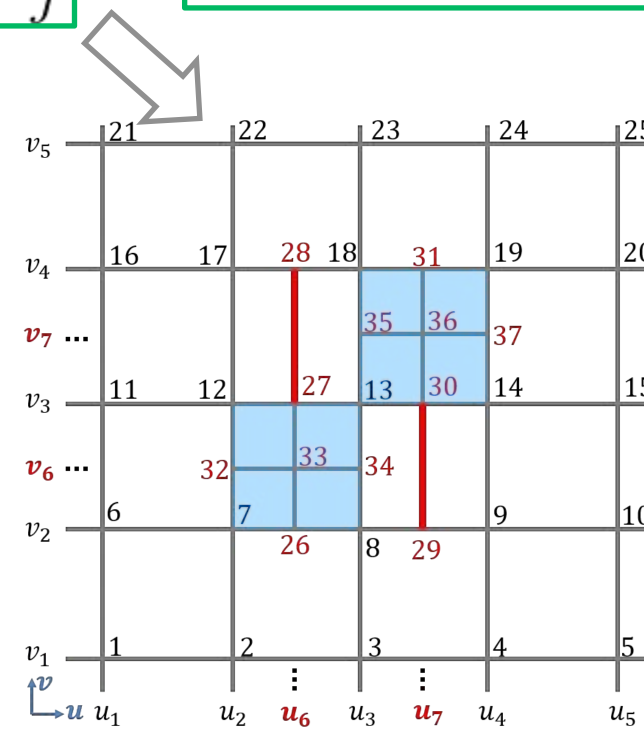
5.2. Two Types of Basis Functions

- ❑ Fully refined and partially refined basis functions

$$B_i \stackrel{?}{=} \sum c_{ij} B'_j$$

By local knot insertion





Insert u_6 and v_6

$$B_2 = B'_2 + \frac{5}{36} B'_7 + \frac{1}{12} B'_{26}$$

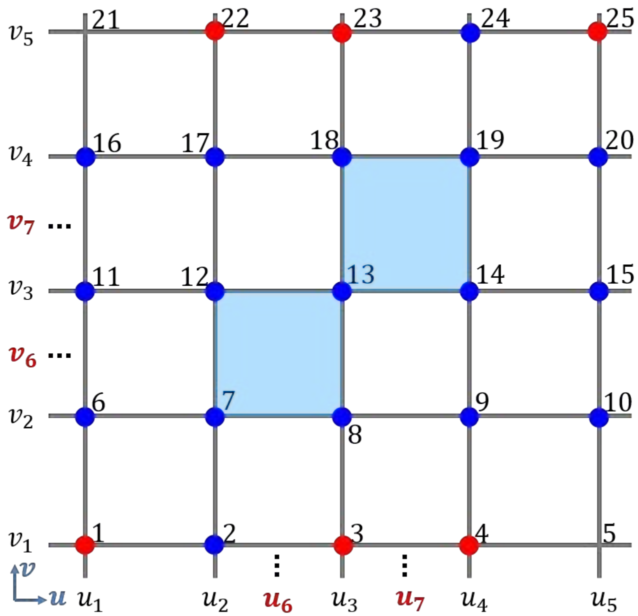
$$B_1 = \frac{1}{36} B'_7 + \cdots$$

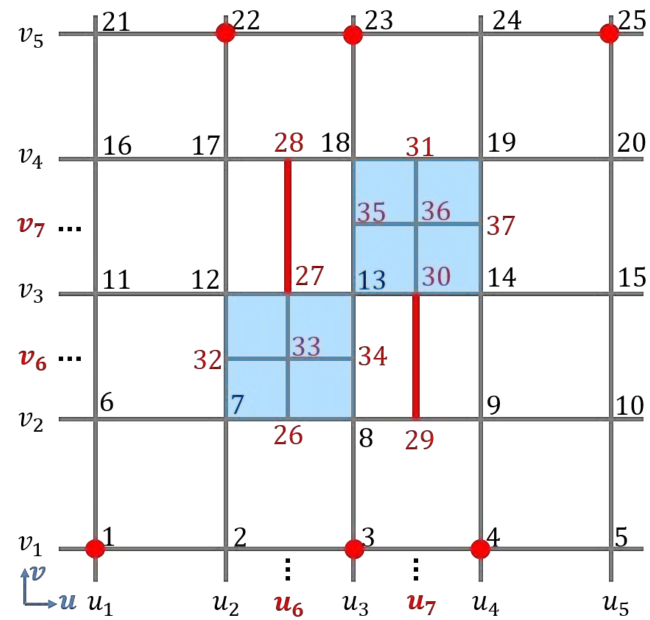
Not defined on

5.3. Truncated T-spline Basis Functions

- Partially refined basis functions need truncation by discarding children defined on the refined T-mesh

$$trunB_i = \boxed{B_i} - \sum_{j \neq i} c_{ij} B'_j$$





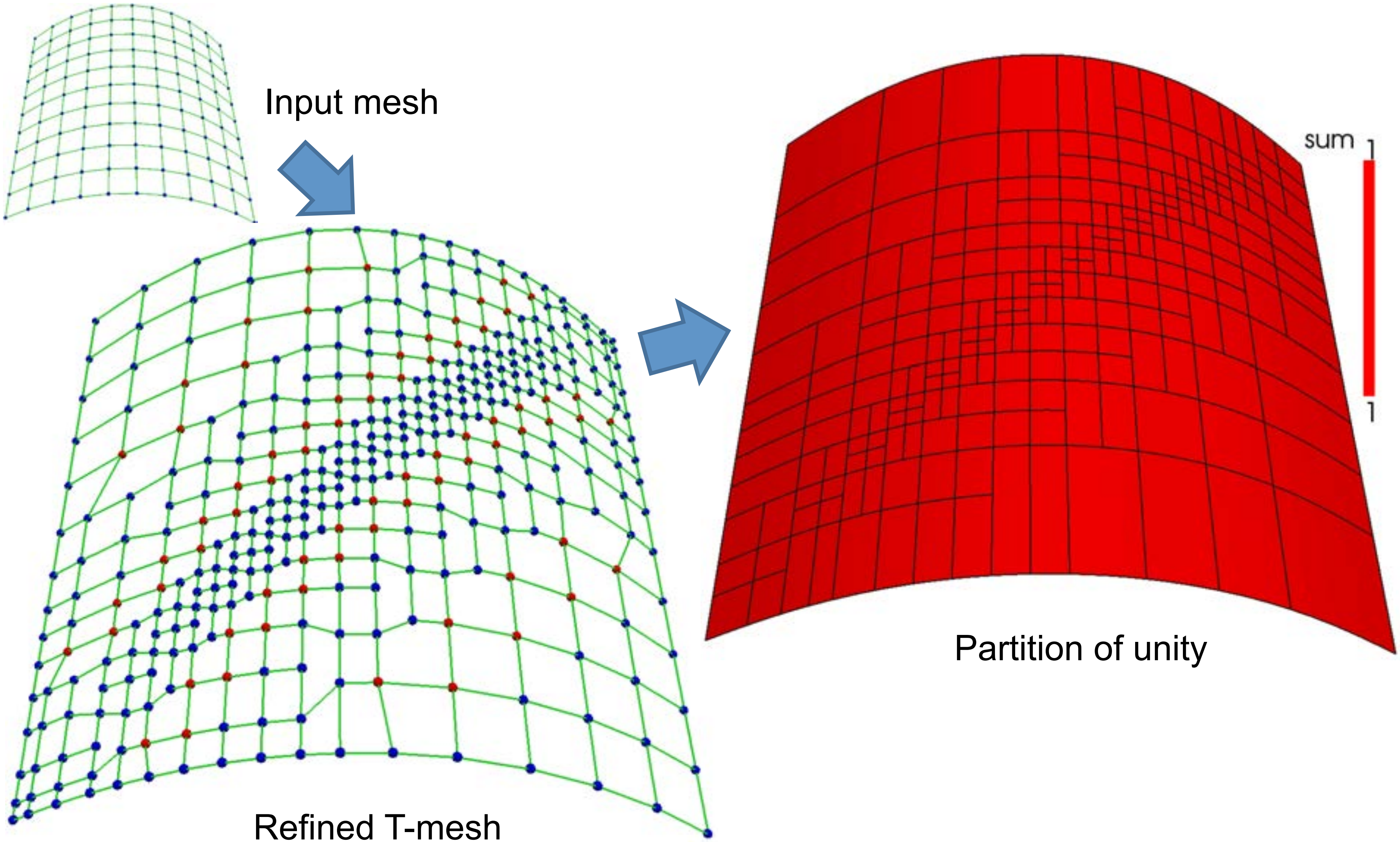
$$\boxed{B_1} = \frac{1}{36}B'_7 + \cdots$$

$$\boxed{B_3} = B'_3 + \frac{5}{36}B'_8 + \frac{1}{12}B'_{26} + \cdots$$

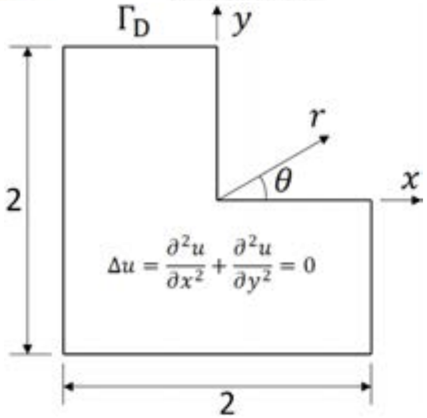
$$\boxed{trunB_1} = B_1 - \frac{1}{36}B'_7$$

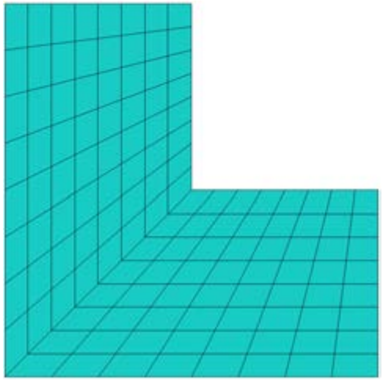
$$\boxed{trunB_3} = B_3 - \frac{5}{36}B'_8 - \frac{1}{12}B'_{26}$$

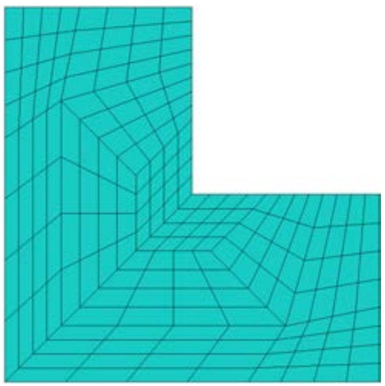
5.4. Refinement along diagonal direction

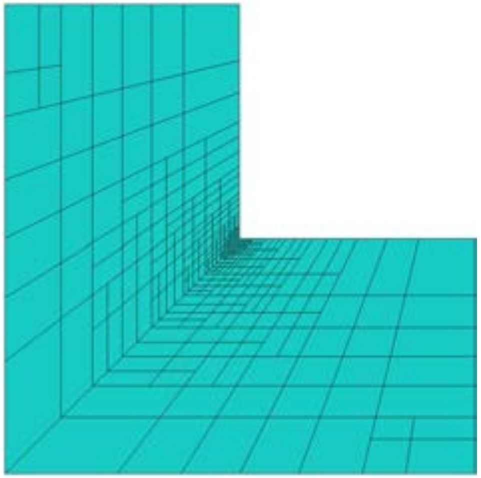


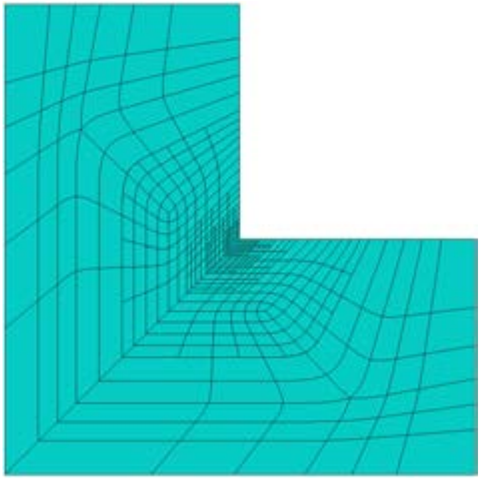
5.5. Benchmark problem: *L*-shaped domain

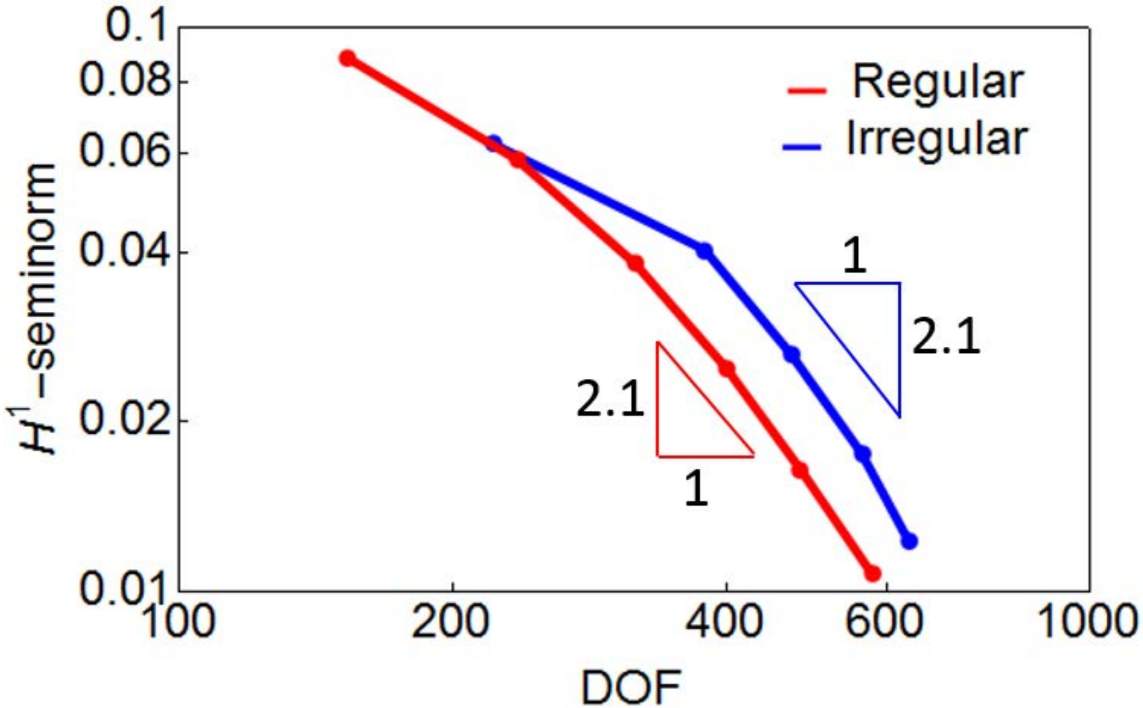




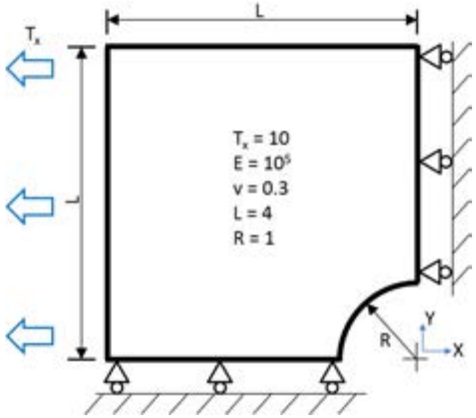


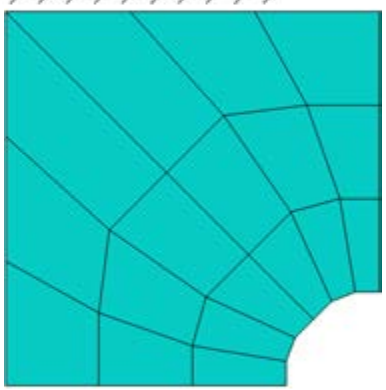


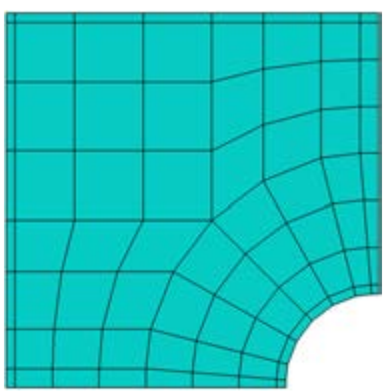


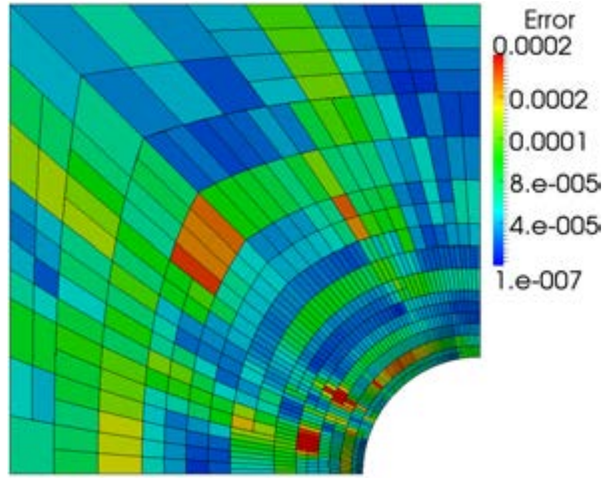


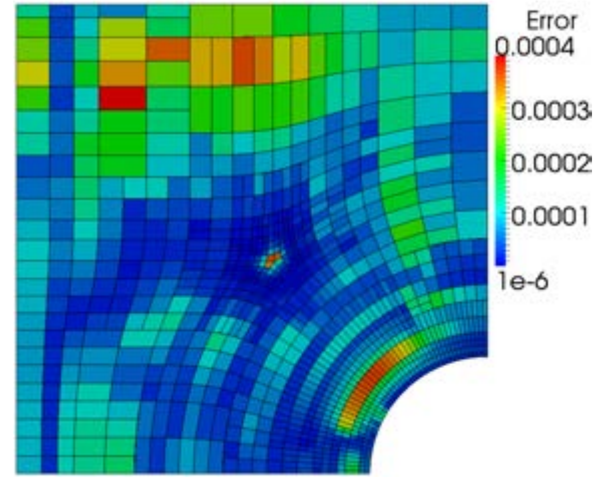
5.5. Benchmark problem: Plate with a hole

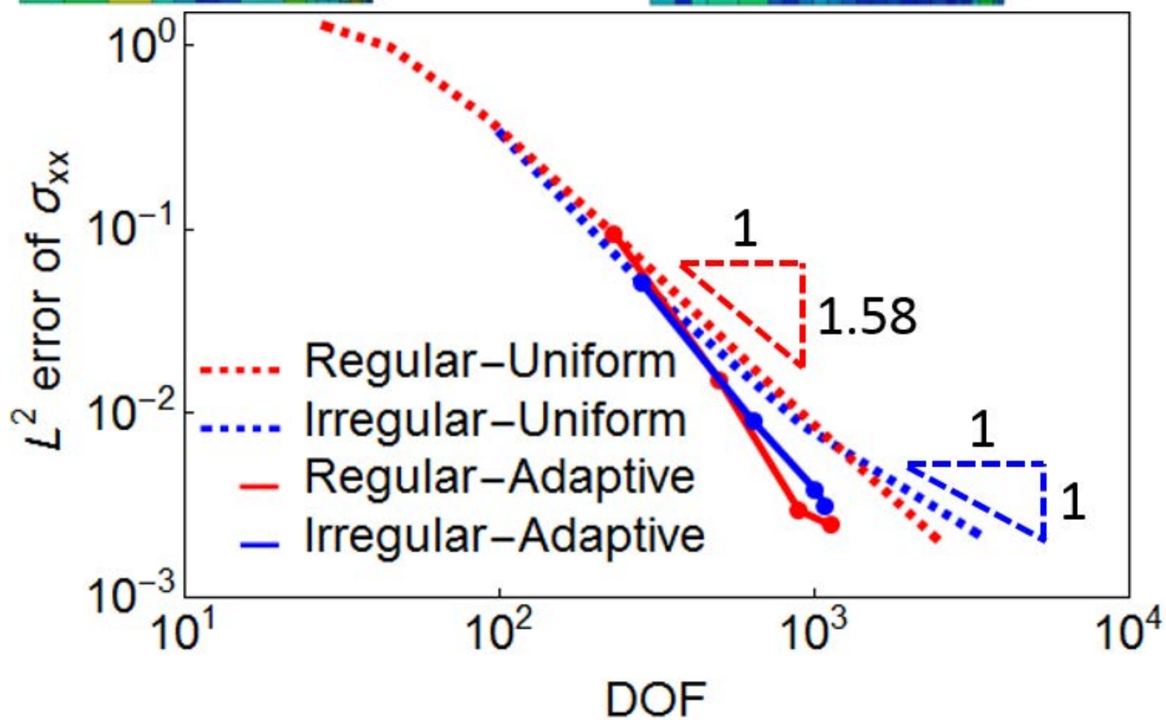












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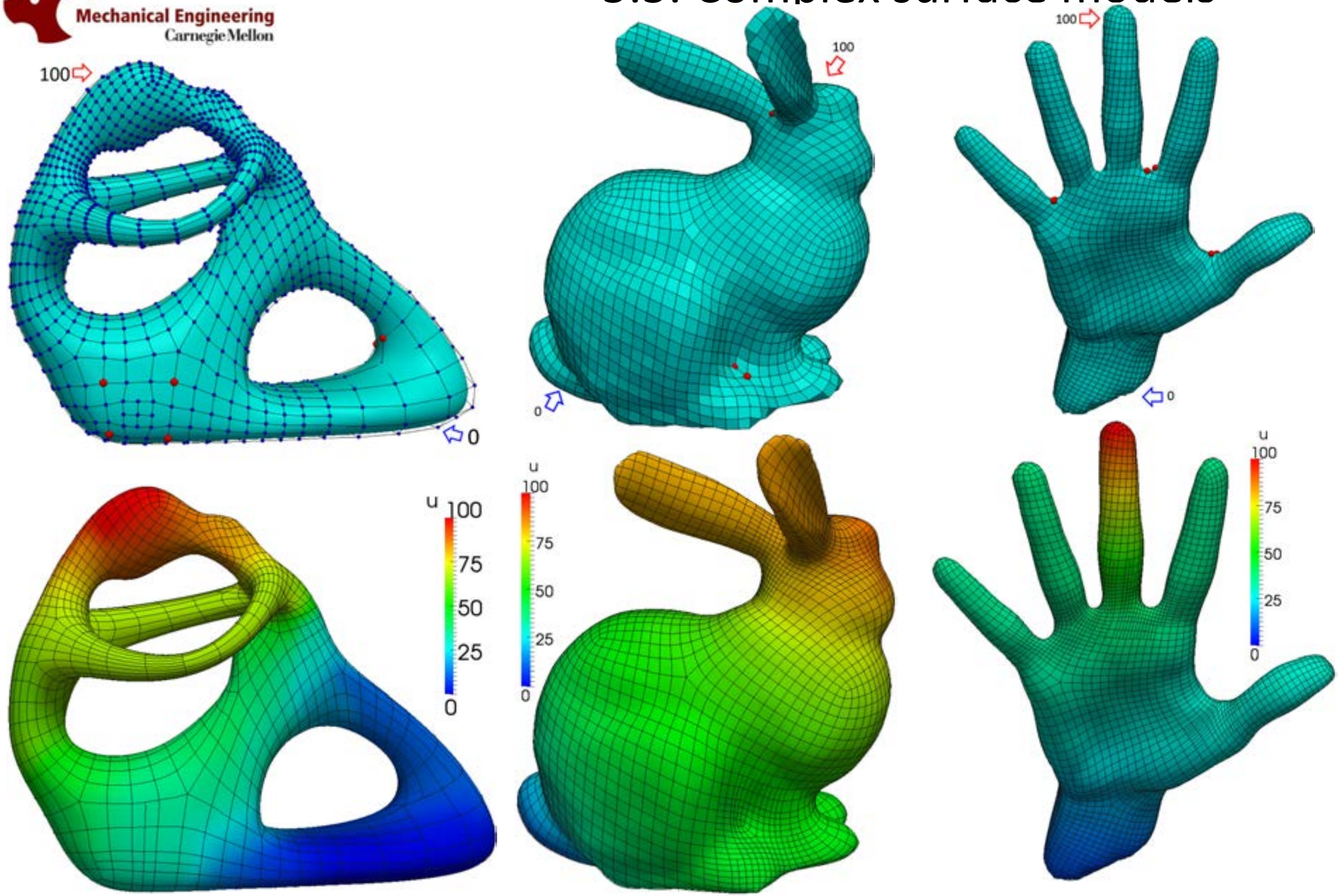
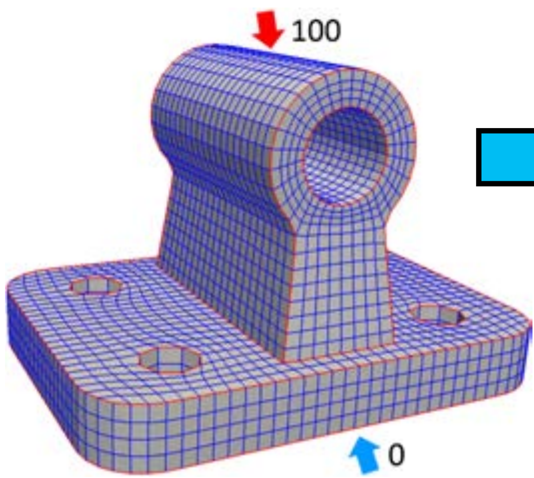
5.5. Complex surface models

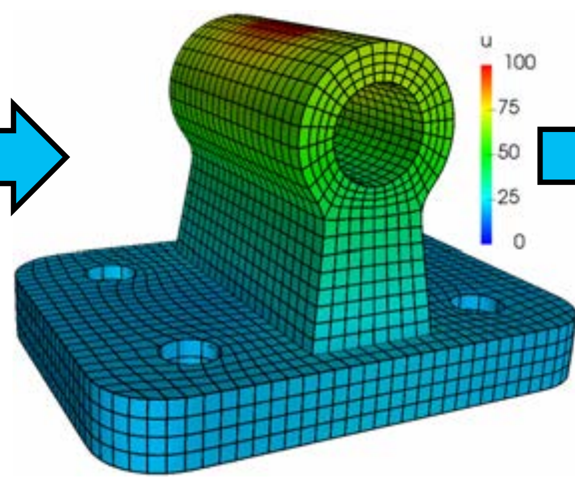


6. Truncated hierarchical splines in 3D (TH-spline3D)

- Motivation: Analysis-driven local refinement on unstructured hex meshes has not been studied in IGA
- Objective: Enable adaptive IGA on unstructured hex meshes

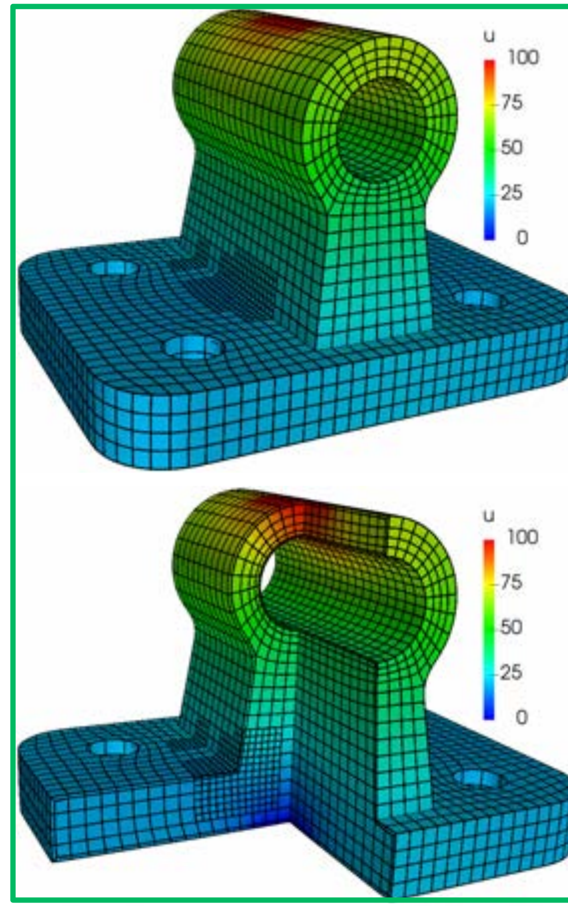


Input hex mesh



IGA on input mesh

1. Develop (tricubic) blending functions on unstructured hex meshes without changing mesh
2. Employ hierarchical structure and the truncation mechanism for local refinement



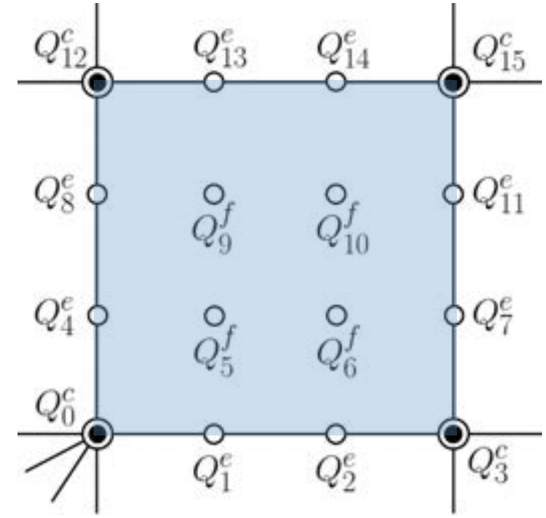
IGA on adaptive mesh

6.1. 2D blending functions

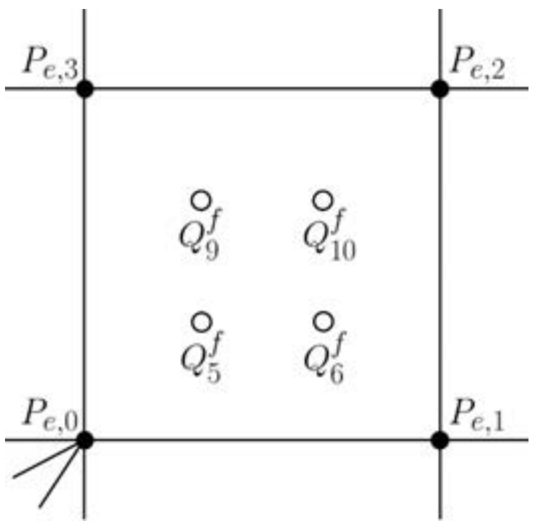
- Defined using Bernstein polynomials

$\mathbf{Q} = \mathbf{M}\mathbf{P}$	$\mathbf{B}$, $\mathbf{P}$ - spline
$\mathbf{B} = \mathbf{M}^T\mathbf{b}$	$\mathbf{Q}$, $\mathbf{b}$ - Bézier

$\mathbf{M}$: Bezier extraction matrix

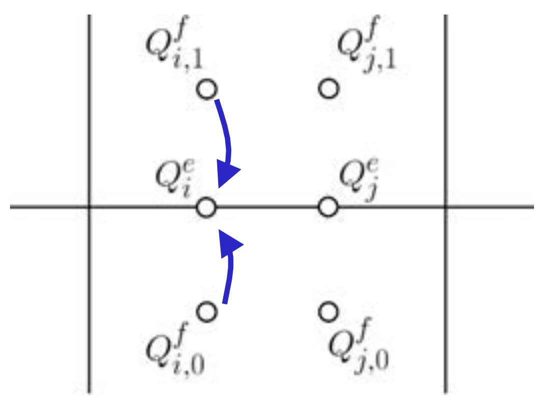


Bézier control points

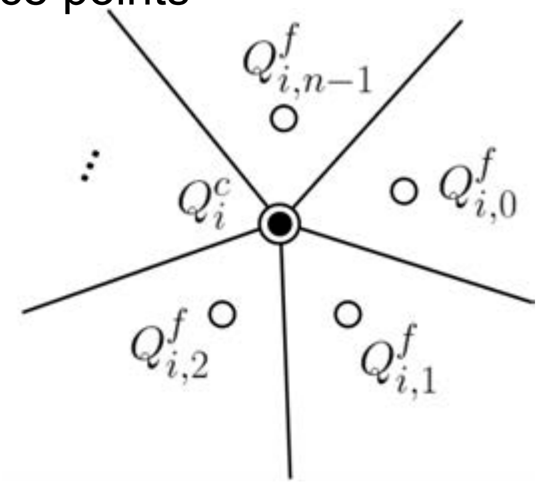


Face points

Nearest face points



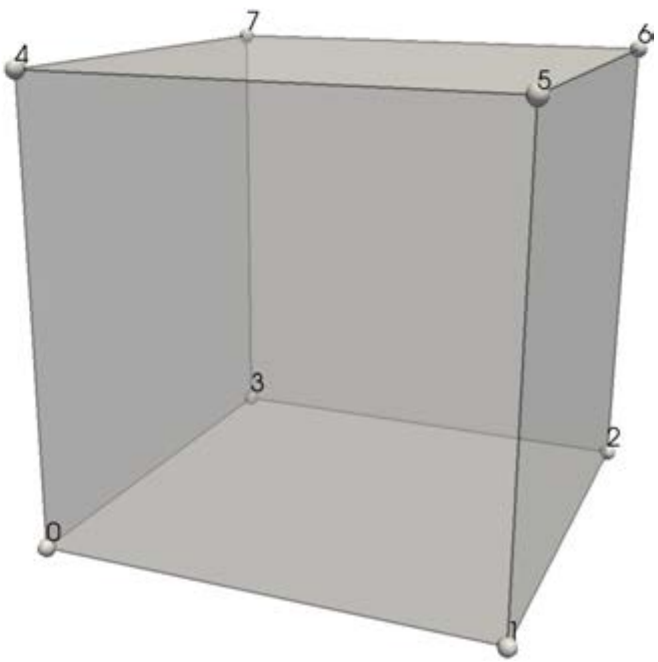
Edge points



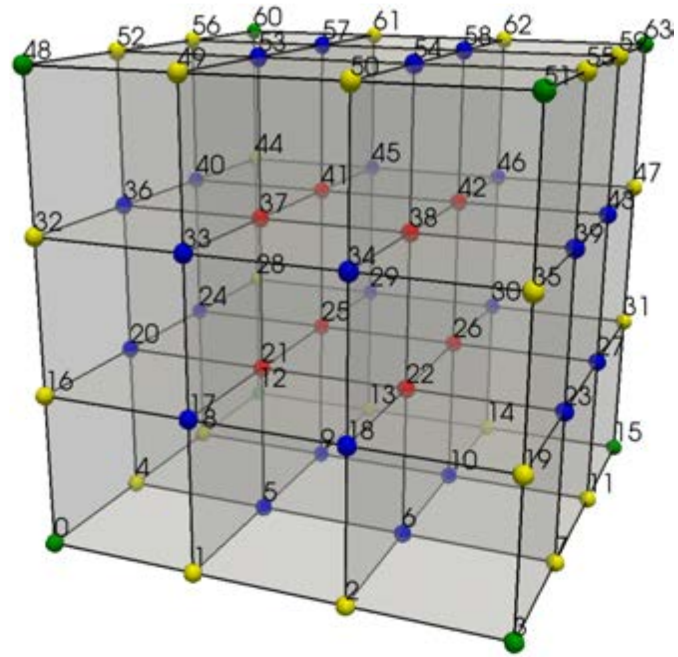
Corner points

6.2. 3D blending functions

- ❑ Generalization of 2D
- ❑ Different types of hex elements
  - Regular interior element (no extraordinary edge)
  - Boundary element
  - Irregular interior element (extraordinary edge)



A hex element

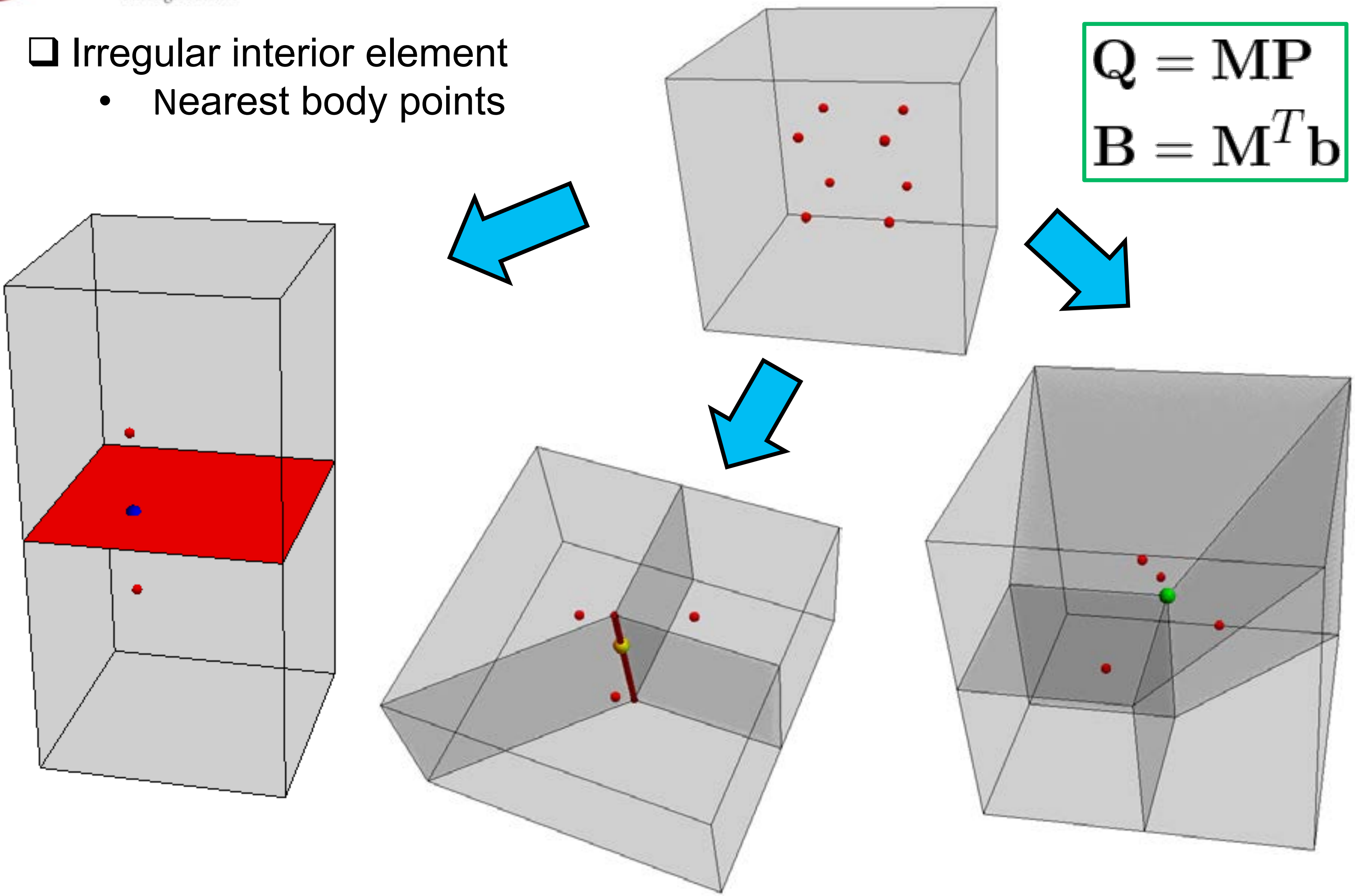


Bézier control points

6.2. Obtain 3D Bézier points

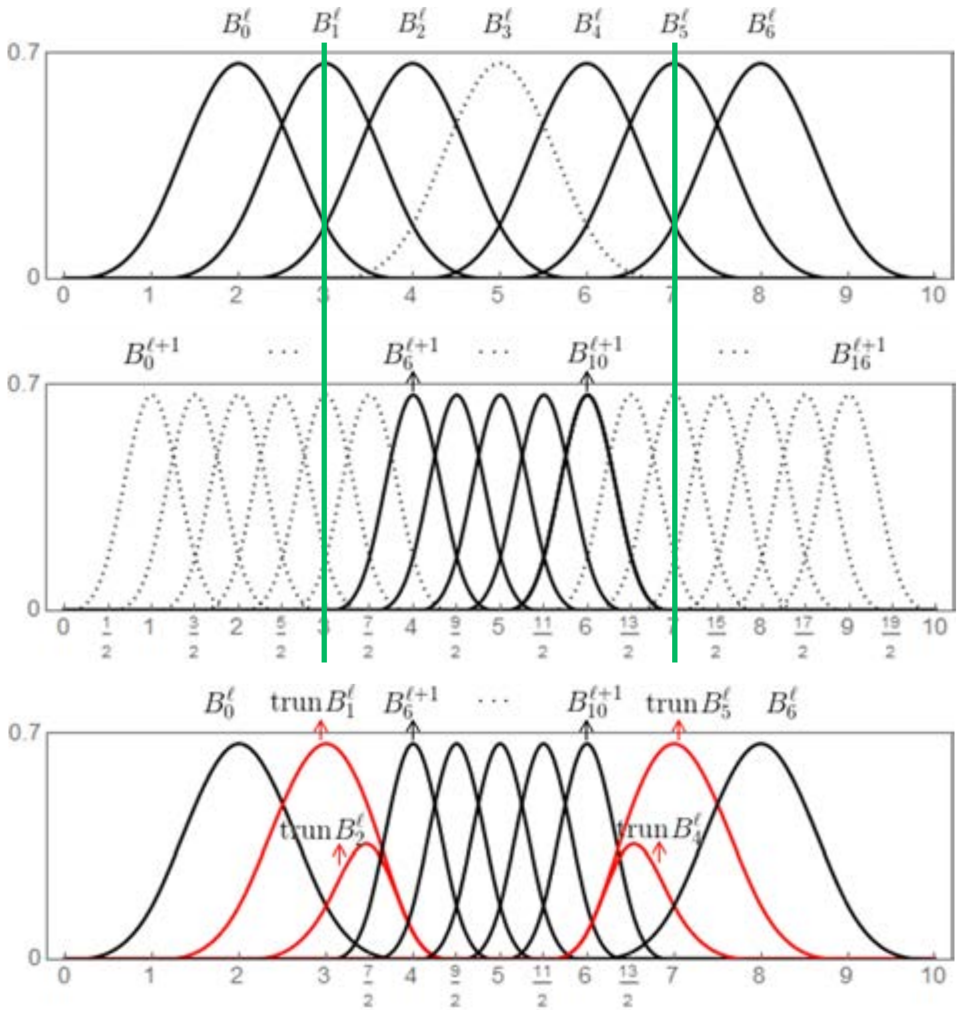
❑ Irregular interior element
- Nearest body points

$$\mathbf{Q} = \mathbf{M}\mathbf{P}$$
$$\mathbf{B} = \mathbf{M}^T\mathbf{b}$$



6.3. Three-step construction

- Similar to THB-splines
  - Special treatment before truncation due to lack of refinability



$$\Omega^{\ell+1} = [3,7] \subset \Omega^\ell = [0,10]$$

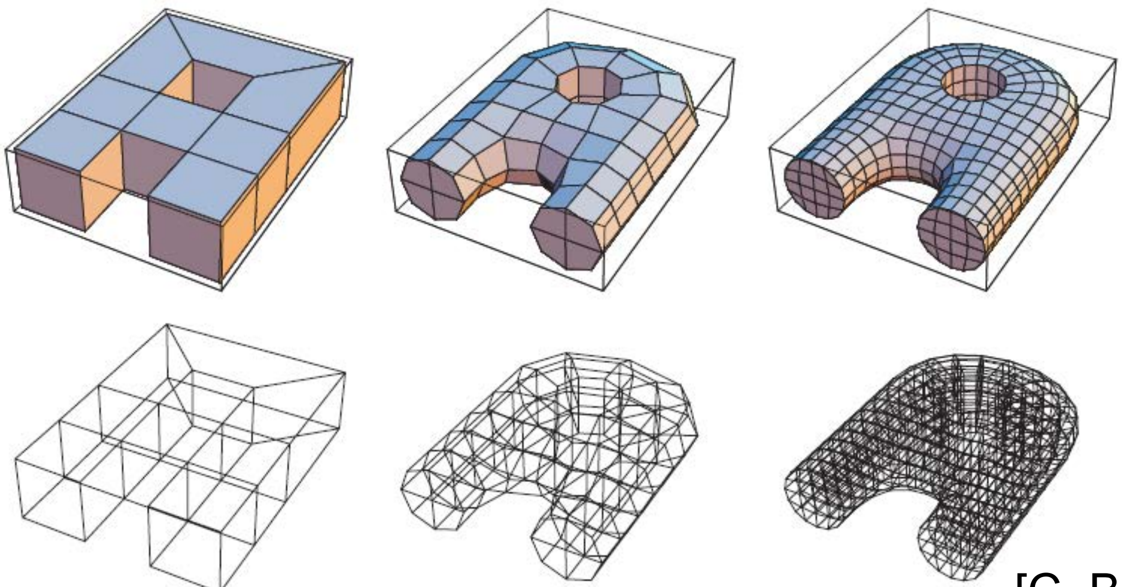
$$\mathcal{B}_a^\ell = \{B_i^\ell : \mathrm{supp} B_i^\ell \not\subseteq \Omega^{\ell+1}\}$$

$$\mathcal{B}_a^{\ell+1} = \{B_i^{\ell+1} : \mathrm{supp} B_i^{\ell+1} \subseteq \Omega^{\ell+1}\}$$

$$\mathrm{trun} B_i^\ell = \sum_{B_j^{\ell+1} \notin \mathcal{B}_a^{\ell+1}} c_{ij}^\ell B_j^{\ell+1}$$

6.3. Two-level relationship

- Two-level relationship via Catmull-Clark subdivision for solids



[C. Bajaj *et al*. 2002]

$$\mathbf{P}^{\ell+1} = \mathbf{S}^{\ell}\mathbf{P}^{\ell}$$

$$\tilde{\mathbf{B}}^{\ell} = \mathbf{S}^{\ell,T}\mathbf{B}^{\ell+1}$$

$$\mathbf{S}^{\ell} = [c_{ij}^{\ell}]$$

Subdivision matrix

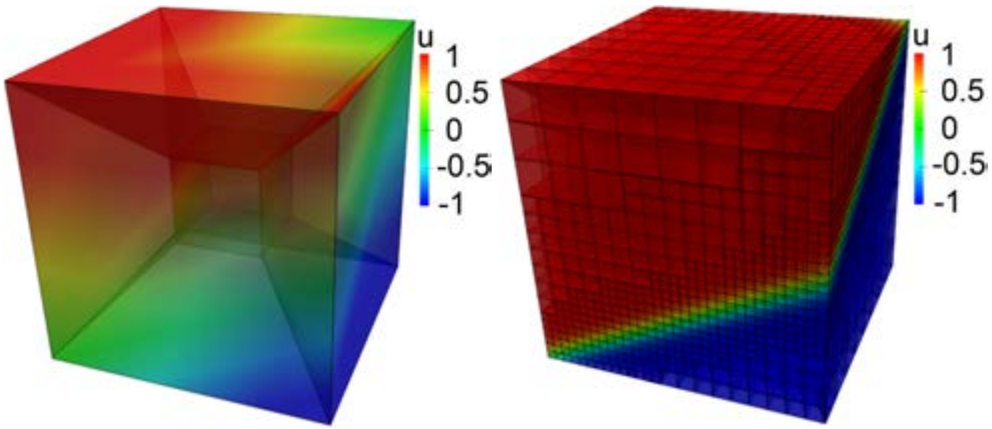
- Truncated blending functions

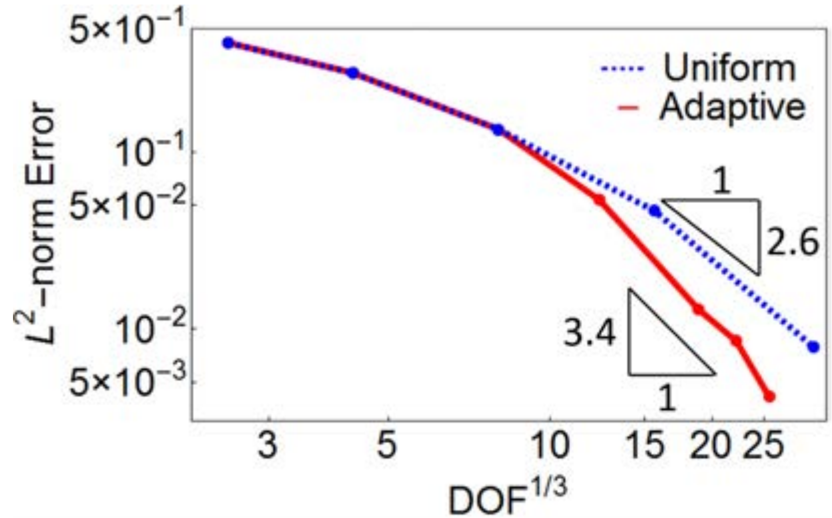
$$\mathrm{trun}B_i^{\ell} := \mathrm{trun}\tilde{B}_i^{\ell} = \sum_{B_j^{\ell+1} \notin \mathcal{B}_a^{\ell+1}} c_{ij}^{\ell} B_j^{\ell+1}$$

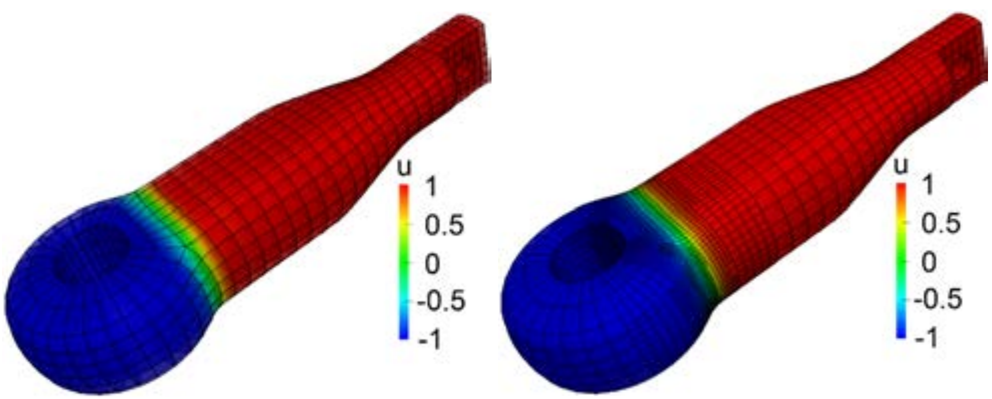
6.4. Results

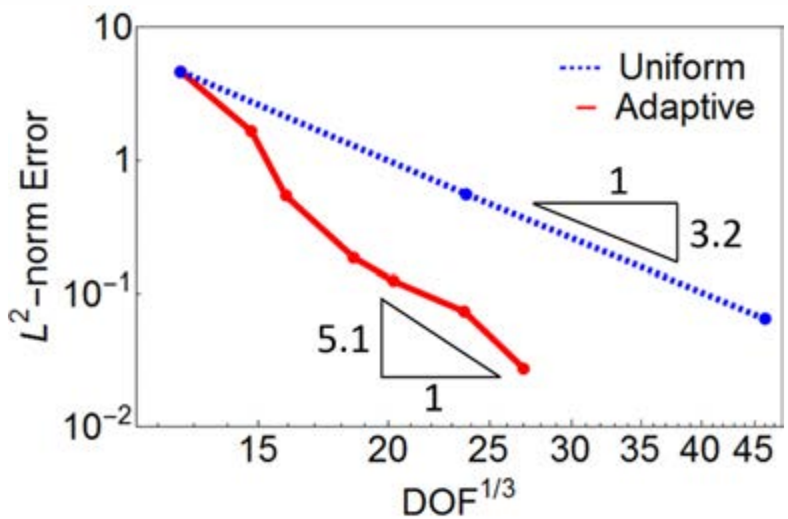
- Solve Poisson's equation with manufactured solution

$$u(\mathbf{x}) = \tanh(a(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n})$$





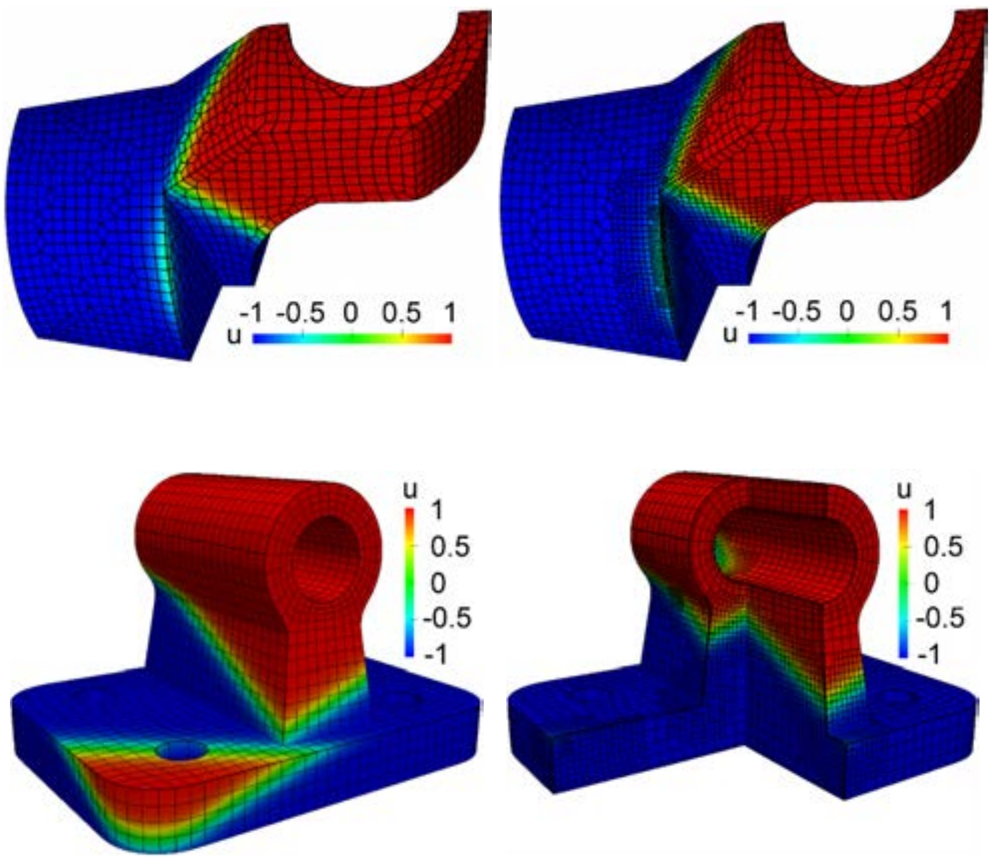


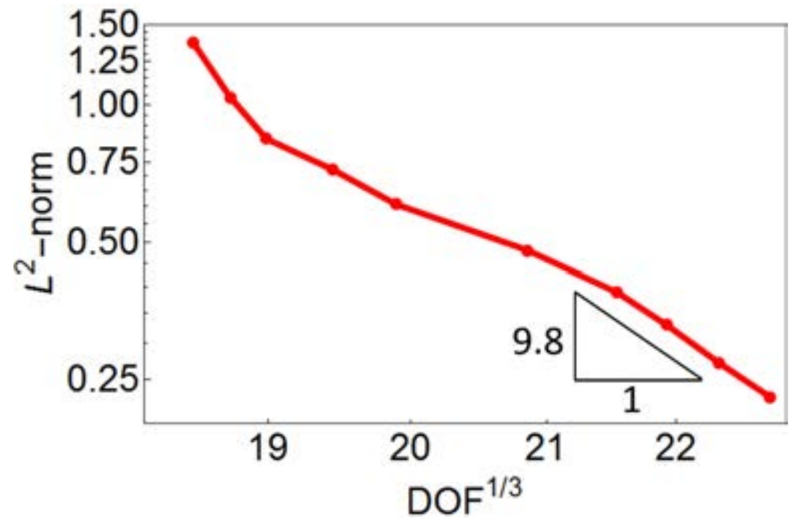


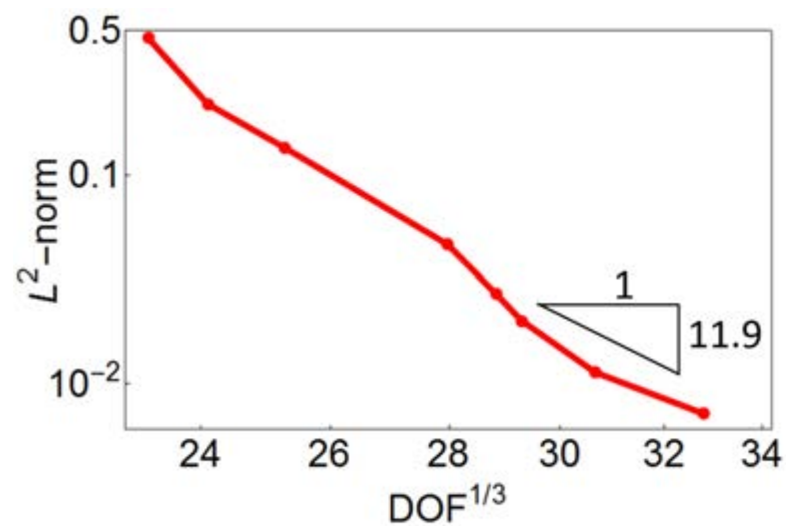
6.4. Results

- Solve Poisson's equation with manufactured solution

$$u(\mathbf{x}) = \tanh(a(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n})$$

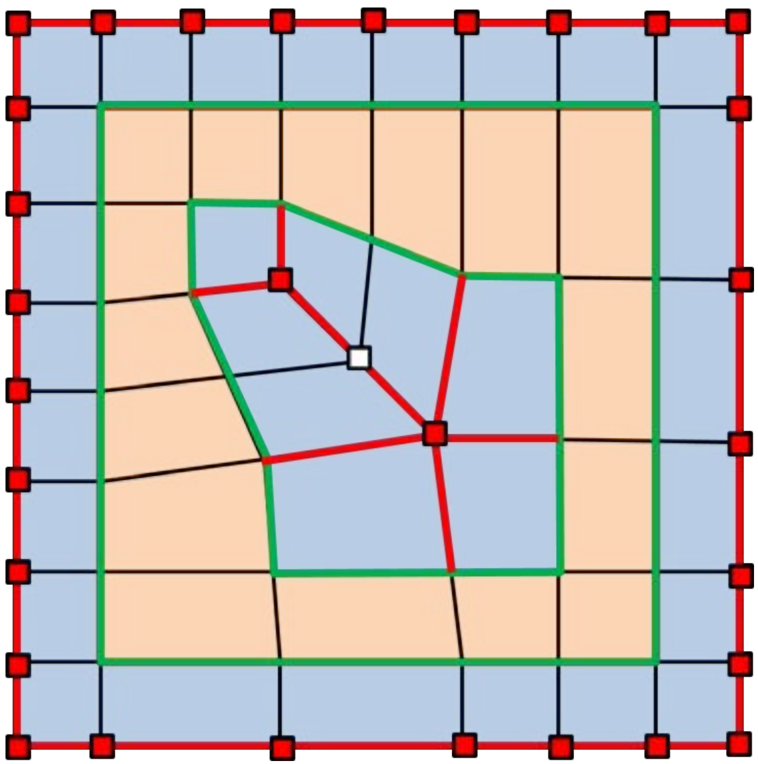




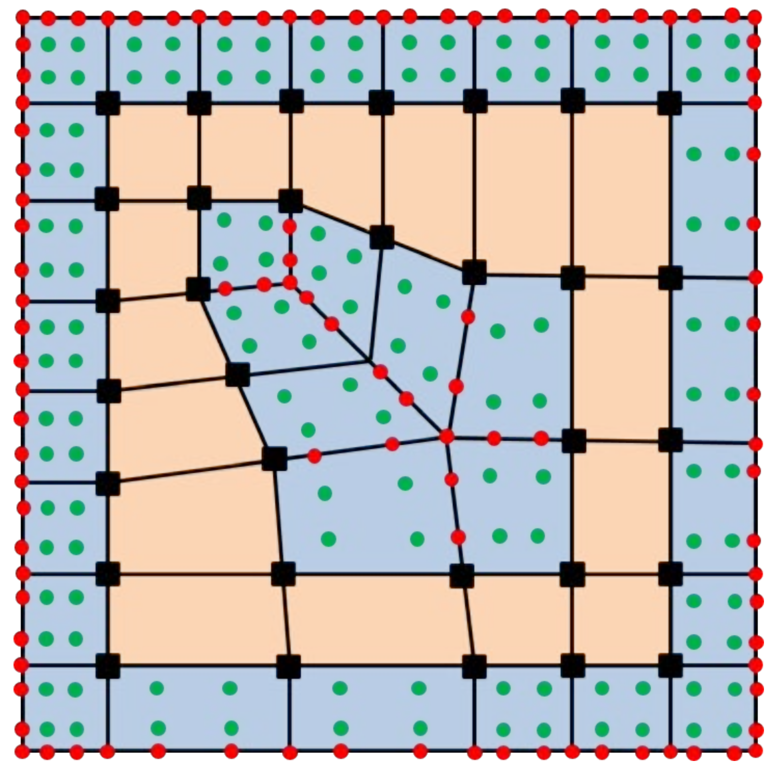


7. Blended B-spline construction

- ❑ Motivation: Achieving optimal convergence rates remains an open problem when using unstructured hex meshes in IGA
- ❑ Objective: Develop a method that can achieve optimal convergence rates
- ❑ Idea: Add minimal number of extra functions in the irregular region



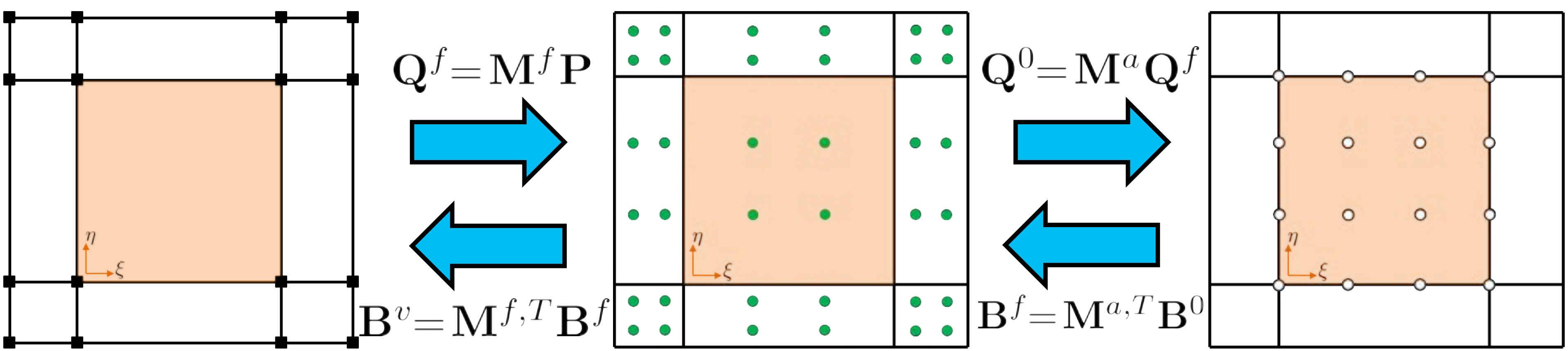
Input quad mesh



Add extra DOF

7.1. Three types of spline functions

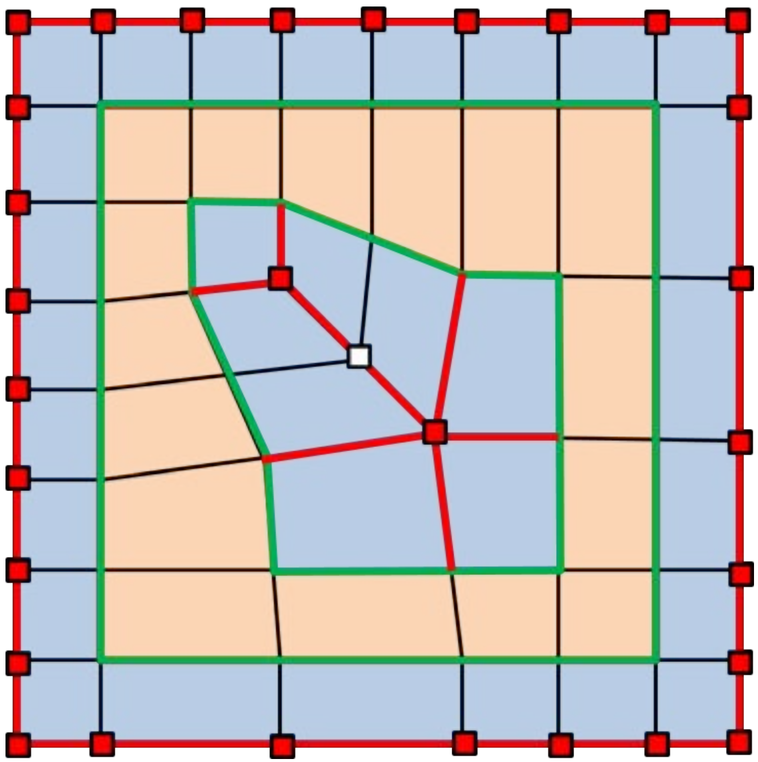
- Three types of spline functions through Bézier extraction
  - Vertex-associated functions $\mathbf{B}^v$ (uniform C^2 B-splines in regular region)
  - Face-point-associated functions $\mathbf{B}^f$ (C^1 B-splines in regular region)
  - Bezier functions $\mathbf{B}^0$ (C^0 B-splines, Bernstein polynomials)



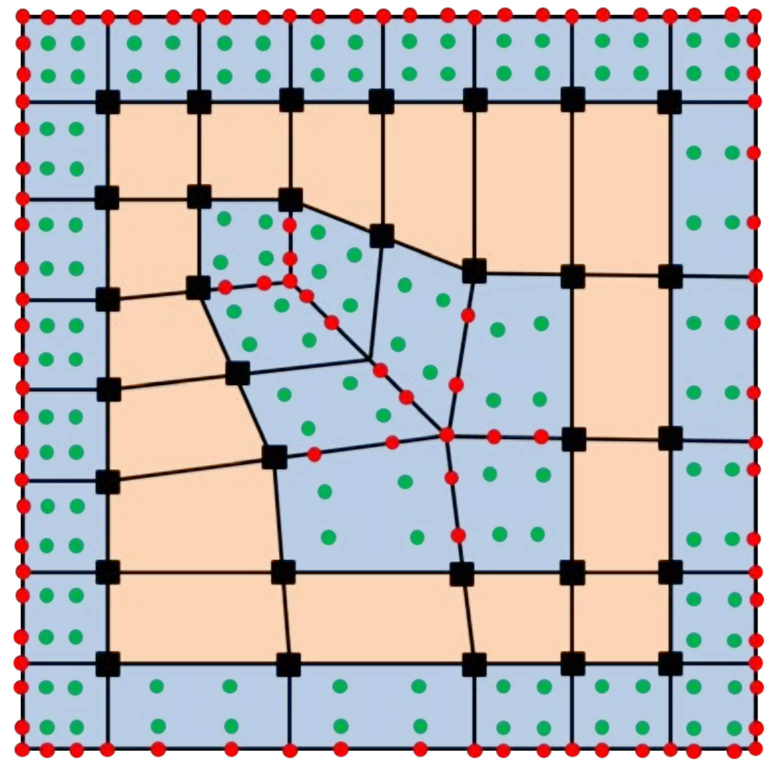
Vertex-based $\mathbf{P}^T \mathbf{B}^v$ Face-point-based $\mathbf{Q}^{f,T} \mathbf{B}^f$ Bézier $\mathbf{Q}^{0,T} \mathbf{B}^0$

7.2. Blended construction – Add extra DOF

- Regular elements (orange), irregular elements (blue), and interface (green)
- C^0 edges (red) and C^0 vertices (red squares)
- Add face points to irregular elements, and Bézier points to C^0 edges/vertices
- Active DOF: mesh vertices whose one-ring neighborhood are not all irregular, and added DOF



Input quad mesh



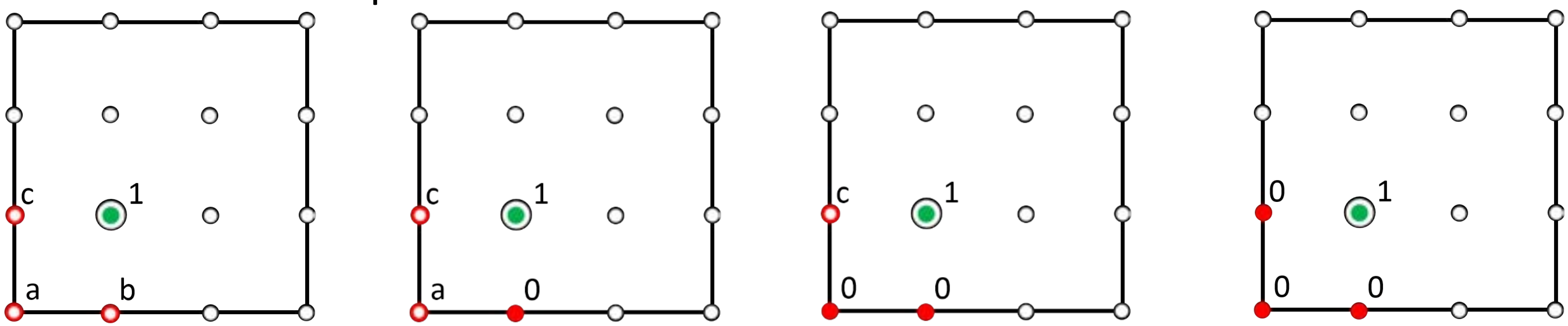
Add extra DOF

7.3. Blended construction – Truncation

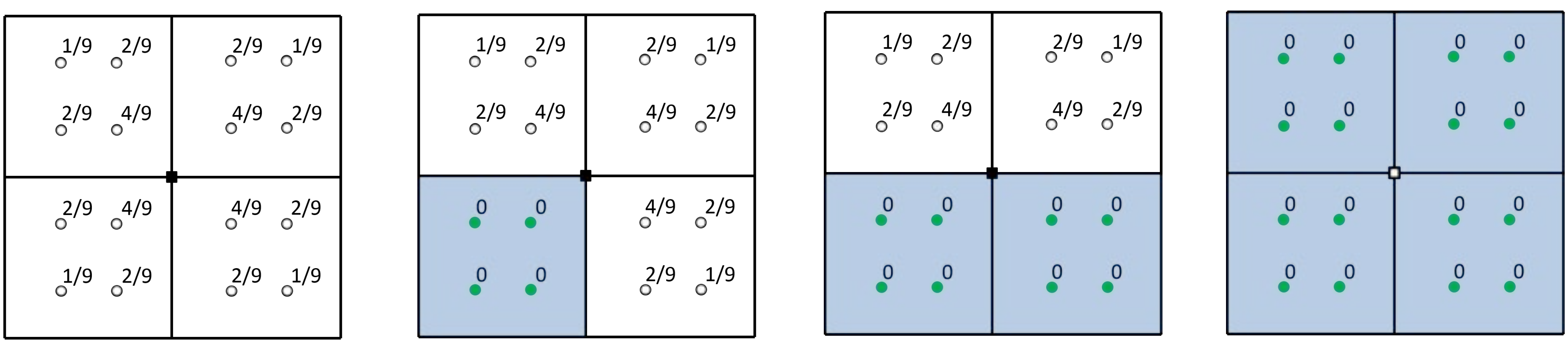
- Three parent-children relationships

$$\mathbf{B}^f = \mathbf{M}^{a,T}\mathbf{B}^0 \qquad \mathbf{B}^v = \mathbf{M}^{f,T}\mathbf{B}^f \qquad \mathbf{B}^v = \mathbf{M}^{f,T}\mathbf{M}^{a,T}\mathbf{B}^0 := \mathbf{M}\mathbf{B}^0$$

- Truncation: Setting the ordinates of active children to be zero
- Truncate face-point-associated functions $\mathbf{B}^f$ w.r.t. Bezier functions $\mathbf{B}^0$

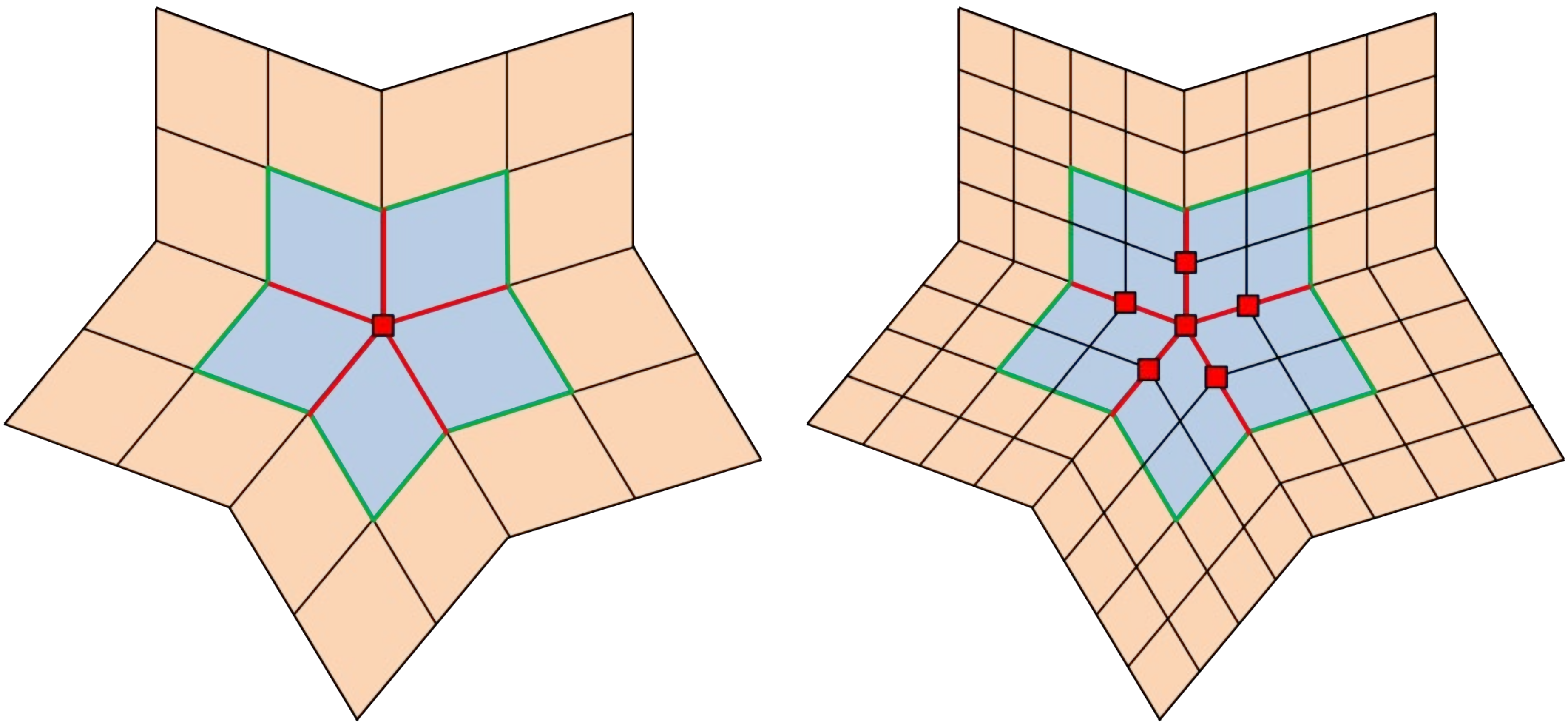


- Truncate vertex-associated functions $\mathbf{B}^v$ w.r.t. $\mathbf{B}^f$



7.3. Blended construction – Refinement

- Maintain the same continuity in irregular region
  - Properly pass “irregular tags” to refined mesh



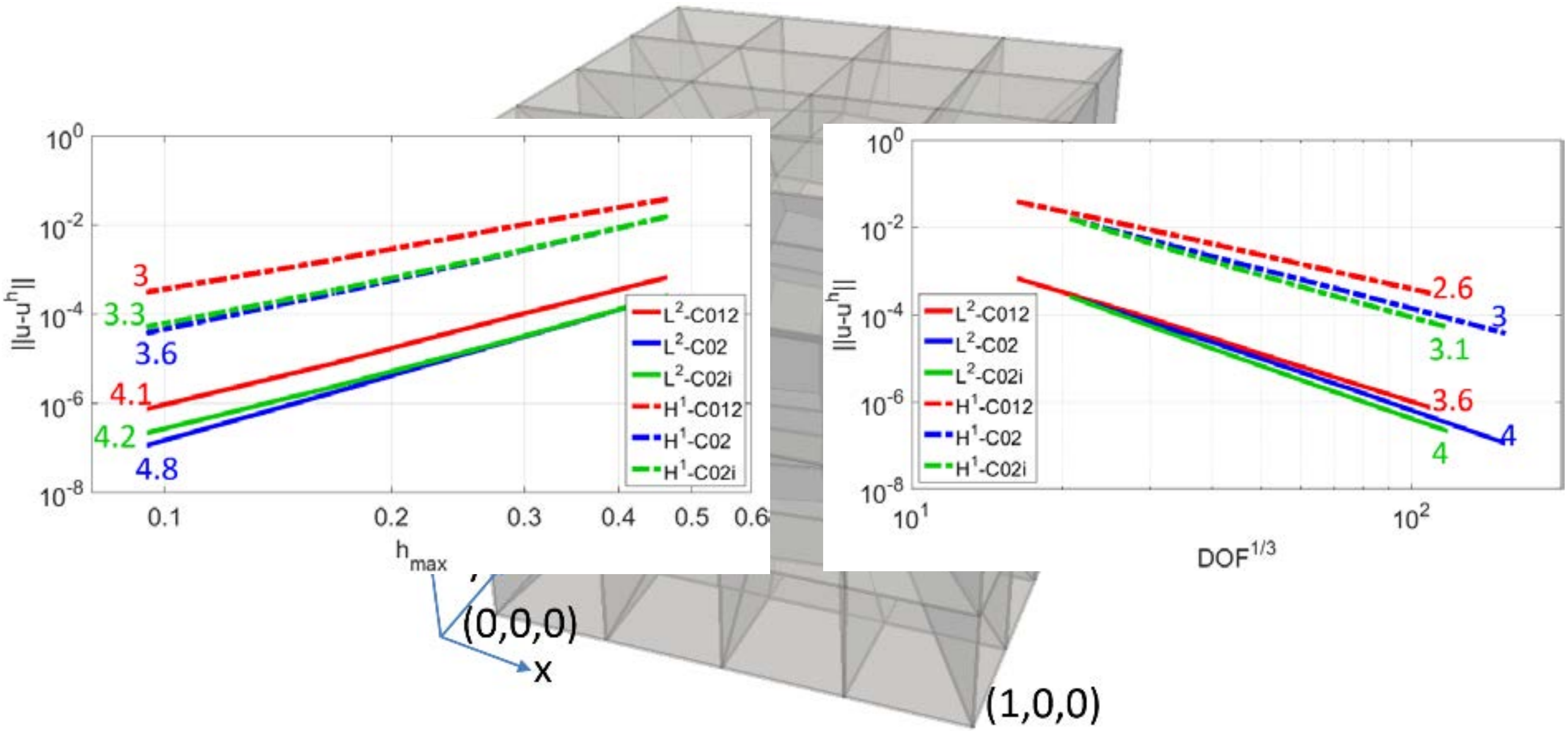
Given mesh

Refined mesh

7.4. Optimal convergence rates in 3D

- Solve Poisson's equation with manufactured solution

$$u(x,y,z) = \sin(\pi x)\sin(\pi y)\sin(\pi z)$$



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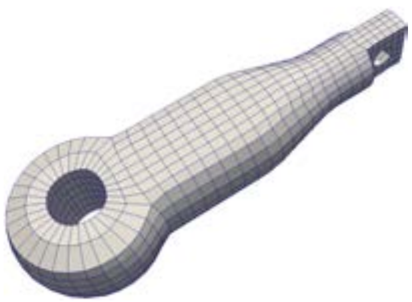
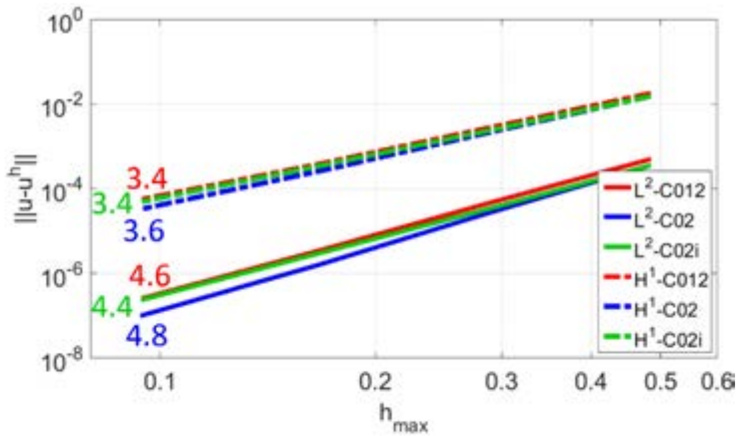
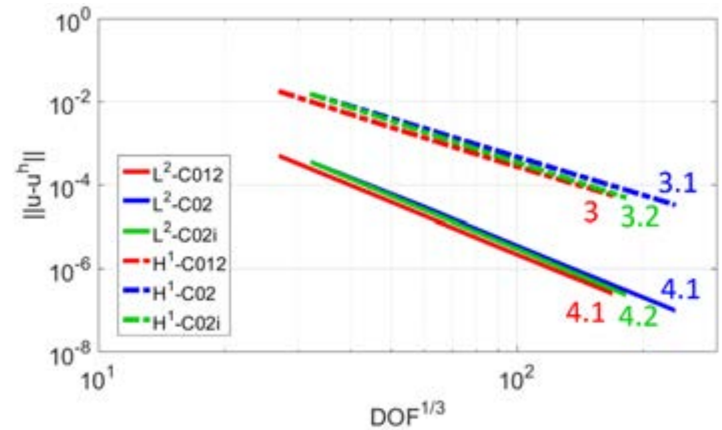
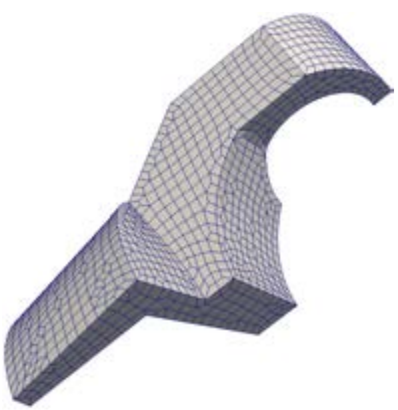
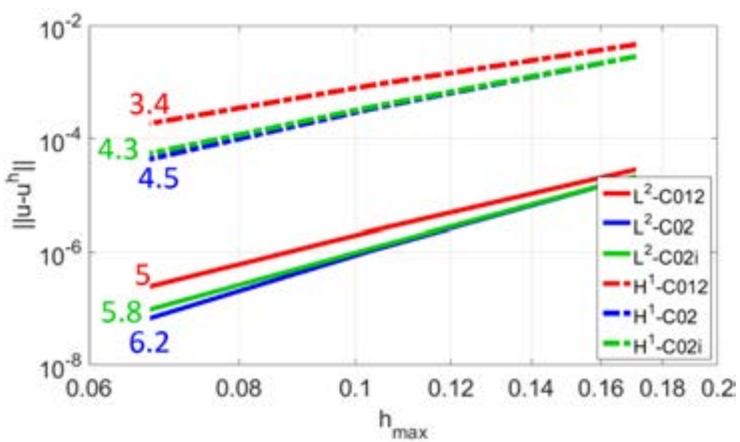
7.4. Optimal convergence rates in 3D

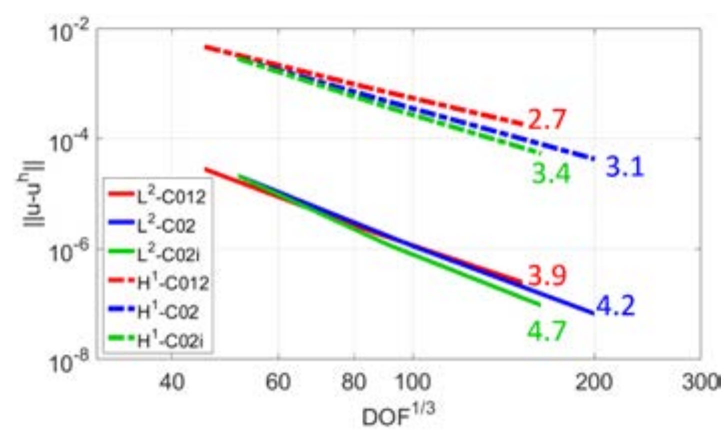












8. Contributions

The dissertation will have five main contributions:

1. Develop THCCS to support local refinement on arbitrary topologies, suitable for both geometric design and adaptive analysis
2. Develop eTHCCS to significantly improve the efficiency of local refinement in THCCS
3. Develop truncated T-splines to release the topological constraints in analysis-suitable T-splines in both regular domains and general 2-manifold domains
4. Develop TH-spline3D that support analysis-driven local refinement on unstructured hexahedral meshes
5. Develop multiple B-spline constructions that can achieve optimal convergence rates with 3D extraordinary vertices involved

8. Future work

- Apply THCCS (or eTHCCS) and truncated T-splines to Kirchhoff-Love shell
  - Global G^1 parameterization is necessary
- Build hierarchical structure based on the blended B-spline construction for local refinement
  - Disadvantage of truncated hierarchical splines in 3D: lack of refinability
- G^1 parameterization on unstructured hex meshes for high-order PDEs
  - Currently G^1 continuity is not even defined in literature

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Thank you very much!