Hexahedral Mesh Generation Based on Surface Foliation Theory

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IGA Online Tutorial

2018-03-09 @ GAMES

Thanks

Thanks for the invitation.

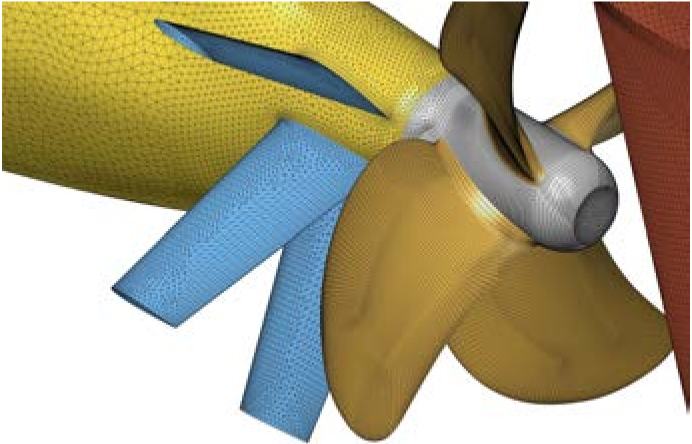
Outline

Motivation

Simulation

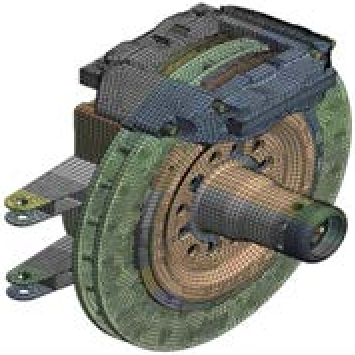
Numerical simulation is one of the most important techniques.

Simulation



Conventionally, finite element method is applied using variational principle.

Simulation

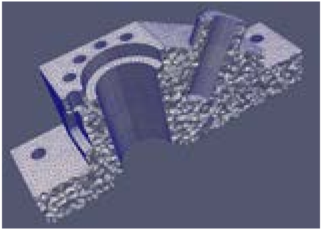


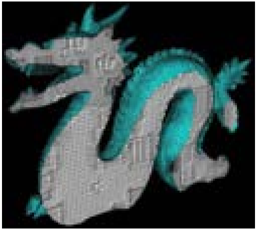
Meshing step costs 70% time and cost for manufacture industry, such as Boeing.

Holy Grid

Volumetric mesh

- Tetrahedral mesh
- Hexahedral mesh
  - unstructural hexahedral mesh
  - structural hexahedral mesh (Holy Grid)





Motivation

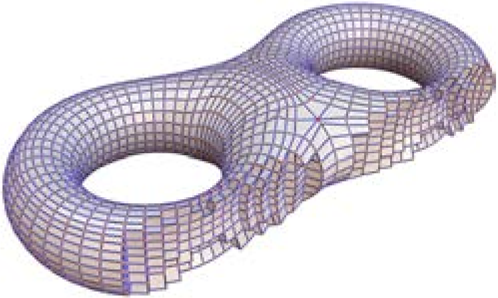
- Iso-Geometric Analysis is widely used in computational mechanics, CAD, CAM, CAG, manufacture industries and so on.
- IGA requires the geometric objects to be represented as solid Splines, such as NURBS, T-Splines, U-Splines.
- The construction of Splines requires hexahedral meshing with high qualities, such as
  - tensor product structure,
  - minimal number of singular lines/points,
  - conforming to the geometric features,
  - automatic.

In order to tackle these challenges, we propose a novel framework with solid theoretic foundation, which satisfies the above requirements.

Holy Grid

Main Problem

Given a closed surface S, with minimal user input, automatically construct a quadrilateral mesh $\mathscr{Q}$ on S, and extend $\mathscr{Q}$ to a hexahedral mesh of the enclosed volume. Both the quadrilateral and hexahedral meshes are with local tensor product structure, and with the least number of singular vertices or singular lines, which is the so-called "Holy grid" problem.



Previous Works

Theorem (Thurston 93)

For a genus zero closed surface, a quadrilateral mesh admits a hexahedral mesh of the enclosed volume if and only if it has even number of cells.

- W. Thurston, Hexadedral decomposition of polyhedra, posting to Sci.Math. (25 October 1993).

Previous Works

Theorem (Mitchell 96)

For a genus g closed surface in $\mathbb{R}^3$, with a quad-mesh,

1. *A compatible hex-mesh exists if one can find g disjoint topological disks in the interior body, each bounded by an cycle of even length in the quad-mesh, that cut the interior body into a ball.*
2. *A compatible hex-mesh does not exist if there is a topological disk in the interior whose boundary is a cycle of odd length in the quad-mesh.*

- S. A. Mitchell, A characterization of the quadrilateral meshes of a surface which admit a compatible hexahedral mesh of the enclosed volume, proceeding of STACS 96, pp. 465 – 476.

Previous Works

Theorem (Erickson 2014)

Let Ω be a compact connected subset of $\mathbb{R}^3$ whose boundary $\partial\Omega$ is a (possibly disconnected) 2-manifold, and let $\mathscr{Q}$ be a topological quad-mesh on $\partial\Omega$ with an even number of facets. The following conditions are equivalent:

1. *$\mathscr{Q}$ is the boundary of a topological hex-mesh of Ω.*
2. *Every subgraph of $\mathscr{Q}$ that is null-homologous in Ω has an even number of edges.*
3. *The dual of $\mathscr{Q}$ is null-homologous in Ω.*

- J. Erickson, Efficiently Hex-Meshing Things with Topology, Discrete and Computational Geometry 52(3):427-449,2014.

Generalization of Thurston and Mitchell's works.

Previous Works

These theoretic works consider general unstructured hex-meshes, which do not have local tensor product structure.

Open Problem

Which kind of quadrilateral mesh admits a structural hexahedral mesh?

Previous Works

The “advancing front” approach generates a hex-mesh from the boundary of the surface mesh inward.

1. Pastering method: T. D. Blacker, R. J. Meyers, Seams and wedges in plastering: A 3d hexahedral mesh generation algorithm, Engineering with Computers 9(2) (1993) 83 – 93.
2. Harmonic Field method: M. Li, R. Tong, All-hexahedral mesh generation via inside-out advancing front based on harmonic fields, The Visual Computer 28(6) (2012) 839 – 847.

The singularities might be propagated to the medial axes, which might lead to non-hexahedron shaped elements.

Previous Works

The “whisker weaving” approach is a kind of “advancing front” method, which is very popular.

1. T. J. Tautges, T. Blacker, S. A. Mitchell, The whisker weaving algorithm: A connectivitybased method for constructing all-hexahedral finite element meshes (1995).
2. F. Ledoux, J.-C. Weill, An extension of the reliable whisker weaving algorithm, in: 16th International Meshing Roundtable, 2007.

The hex-mesh has no local tensor product structure.

Previous Works

The “Frame field” method constructs smooth frame field, the hex-mesh is extracted from the field.

1. J. Huang, Y. Tong, H. Wei, H. Bao, Boundary aligned smooth 3d cross- frame field, ACM Trans. Graph. 30 (6) (2011) 143.
2. Y. Li, Y. Liu, W. Xu, W. Wang, B. Guo, All-hex meshing using singularity-restricted field, ACM Trans. Graph. 31 (6) (2012).
3. M. Nieser, U. Reitebuch, K. Polthier, Cubecover- parameterization of 3d volumes, Comput. Graph. Forum 30(5) (2011), 1397 – 1406.

The automatic generation of frame fields with prescribed singularity structure is unsolved.

Previous Works

The “Octree” method decomposes the domain into octree structure.

1. M. A. Awad, A. A. Rushdib, M. A. Abbas, S. A. Mitchell, ,A. H. Mahmoud, C. L. Bajaj, M. S. Ebeida, All-Hex Meshing of Multiple-Region Domains without Cleanup, in Proceeding of 25th International Meshing Roundtable, 2016.

All the singular lines are on the surface.

Our Approach

We have

- proved the equivalence among three fundamental concepts:

  $$\{\text{Colorable Quad-Mesh}\} \leftrightarrow \{\text{Finite Measured Foliation}\} \leftrightarrow \{\text{Strebel Differential}\}.$$

- lay down the theoretical foundation for the existence of structural hexahedral mesh of three manifold with complex topology.
- designed the algorithm to automatically generate the “holy grid”.

Colorable Quadrilateral Mesh

Colorable Quad-Mesh

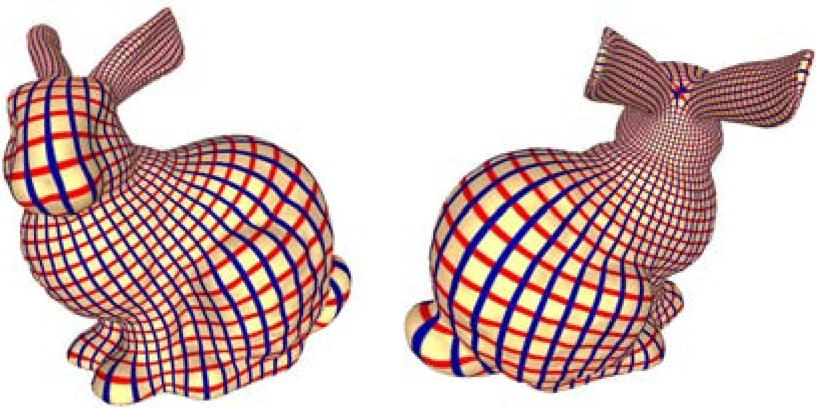
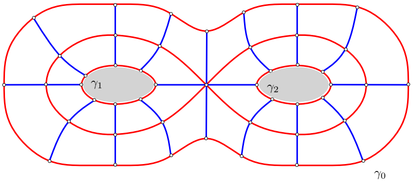


Figure: A red-blue (colorable) Quad-Mesh.

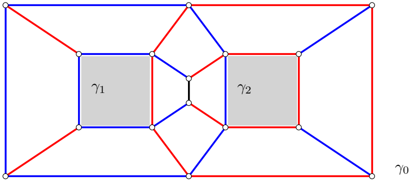
Colorable Quad-Mesh

Definition (Colorable Quad Mesh)

Suppose Q is a quadrilateral mesh on a surface S, if there is a coloring scheme $\imath : E \to \{red, blue\}$, which colors each edge either red or blue, such that each quadrilateral face includes two opposite red edges and two opposite blue edges, then Q is called a colorable (red-blue) quadrilateral mesh.



(a) Colorable quad-mesh.



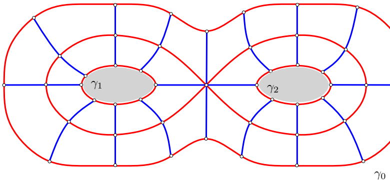
(b) Non-colorable quad-mesh

Figure: Quadrilateral meshes of a multiply connected planar domain.

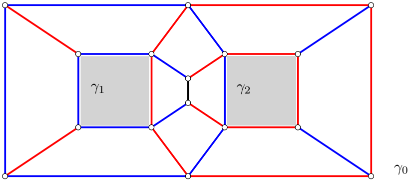
Colorable Quad-Mesh

Lemma

Suppose S is an oriented closed surface, $\mathcal{Q}$ is a quadrilateral mesh on S. $\mathcal{Q}$ is colorable if and only if the valences of all vertices are even.



(a) Colorable quad-mesh.



(b) Non-colorable quad-mesh

Figure: Quadrilateral meshes of a multiply connected planar domain.

Measured Foliations

Measured Foliations

A surface foliation is a decomposition of the surface as a union of parallel curves. Each curve is called a leaf of the foliation.

Figure: A finite measured foliation on a genus three surface.

Foliations

Figure: Finite measured foliations on high genus surfaces generated by Strebel differentials.

Colorable Quad-Mesh

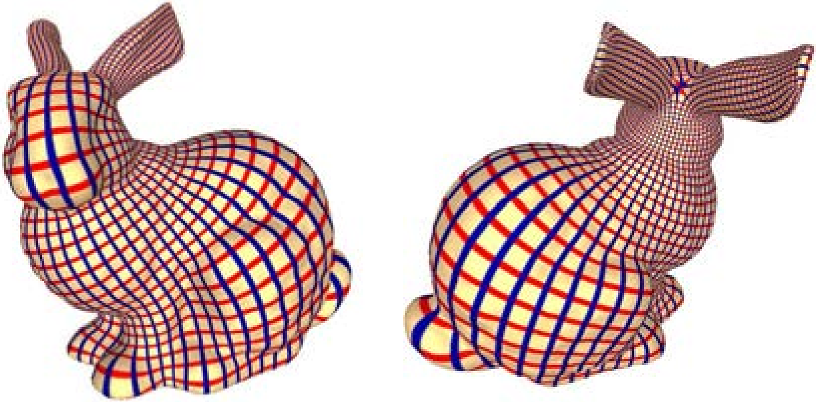


Figure: A red-blue (colorable) Quad-Mesh.

Hubbard-Masur Theorem

Theorem (Hubbard-Masur)

If $(\mathscr{F}, \mu)$ *is a measured foliation on a compact Riemann surface S, then there is a unique holomorphic quadratic differential* Φ *on S, whose horizontal trajectory is equivalent to* $(\mathscr{F}, \mu)$.

Strebel Differential

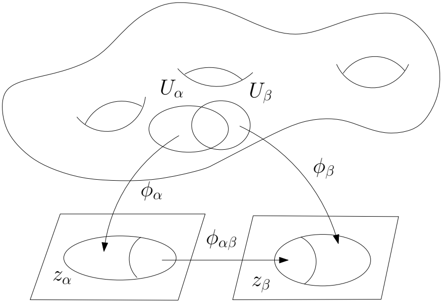
Holomorphic Quadratic Differential

Definition (Holomorphic Quadratic Differentials)

Suppose S is a Riemann surface. Let Φ be a complex differential form, such that on each local chart with the local complex parameter $\{z_\alpha\}$,

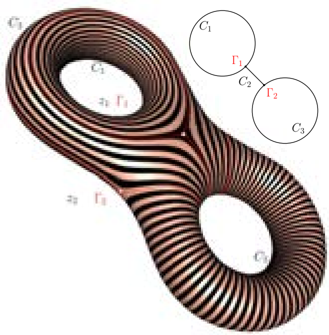
$$\Phi = \varphi_\alpha(z_\alpha) dz_\alpha^2,$$

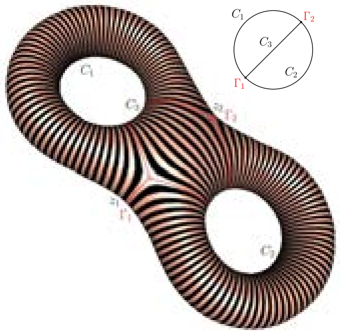
where $\varphi_\alpha(z_\alpha)$ is a holomorphic function.



Definition (Zeros)

Given a holomorphic quadratic differential $\Phi = \varphi(z_\alpha)dz_\alpha^2$, it has $4g-4$ *zeros*, where φ vanishes.





Trajectories

Definition (Trajectories)

For any point away from zero, we can define local coordinates

$$\zeta(p) := \int^{p} \sqrt{\varphi(z)}dz. \quad (1)$$

which are the so-called *natural coordinates* induced by Φ. The iso-parametric curves are called horizontal and vertical *trajectories*. The trajectories through the zeros are called the *critical trajectories*.

Holomorphic Quadratic Differentials

All the holomorphic quadratic differentials form a linear space. The dimension is 0 for genus 0 closed surface, 1 for genus 1 surface, and $3g - 3$ for genus $g > 1$ surface.

(a) (b) (c) (d)

Figure: The holomorphic quadratic differentials of the foliations form a linear space. (c) equals to (a) plus (b), (d) equals to 0.4 (a) plus 1.6 (b).

(a) non-Strebel (b) Strebel

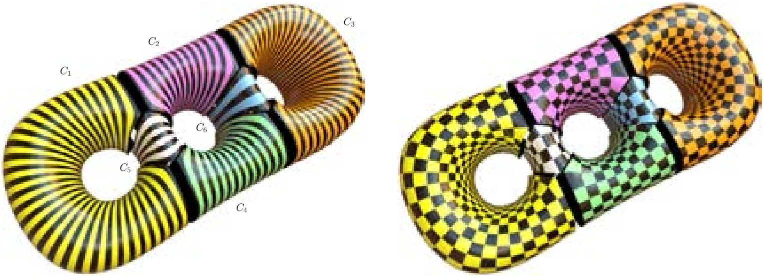
Figure: A non-Strebel (a) and a Strebel differential (b).

Strebel Differential

Definition (Strebel 84)

Given a holomorphic quadratic differential Φ on a Riemann surface S, if all of its horizontal trajectories are finite, then Φ is called a Strebel differential.

The critical horizontal trajectories of a Strebel differential form a finite graph, which divides the surface into cylinders.



Existence of Strebel Differential

Definition (Admissible Curve System)

On a genus $g > 1$ surface S, a set of disjoint, pairwise not homotopic, homotopically nontrivial simple loops $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$, where $n \le 3g - 3$ is called an *admissible curve system*.

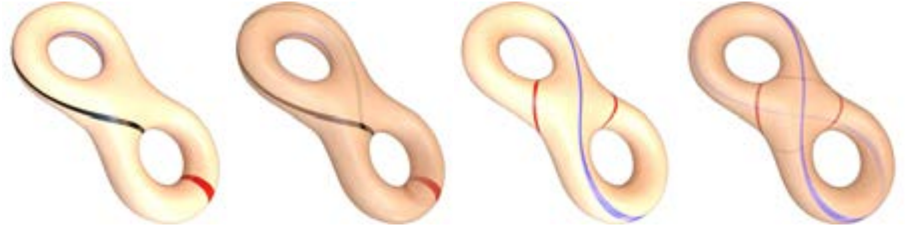


Figure: Admissible Curve Systems on a genus 2 surface.

Existence of Strebel Differential

Theorem (Jenkin 1957 and Strebel 1984)

Given an admissible curve system $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$, $n \leq 3g - 3$, *and positive numbers (heights)* $\mathbf{h} = \{h_1, h_2, \cdots, h_n\}$, *there exists a unique holomorphic quadratic differential* Φ, *satisfying the following :*

1. *The critical graph of* Φ *partitions the surface into n cylinders* $\{C_1, C_2, \cdots, C_n\}$, *s.t.* γ_k *is the generator of* C_k,
2. *The height of each cylinder* (C_k, d_Φ) *equals to* h_k, $k = 1, 2, \cdots, n$.

Figure: Admissible curve systems and the horizontal trajectories generated by them on a genus 2 surface

Trinity

Trinity

Theorem (Trinity)

Suppose S is a closed Riemann surface with a genus greater than 1*. Given an colorable quad-mesh* $\mathscr{Q}$*, there is a finite measured foliation* $(\mathscr{F}_{\mathscr{Q}}, \mu_{\mathscr{Q}})$ *induced by* $\mathscr{Q}$*, and there exits a unique Strebel different* Φ*, such that the horizontal measured foliation induced by* Φ*,* $(\mathscr{F}_{\Phi}, \mu_{\Phi})$ *is equivalent to* $(\mathscr{F}_{\mathscr{Q}}, \mu_{\mathscr{Q}})$*.*
Inversely, given a Strebel differential Φ*, it is associated with a finite measured foliation* $(\mathscr{F}_{\Phi}, \mu_{\Phi})$*, and induces a colorable quad-mesh* $\mathscr{Q}$*.*

$$\{\text{Colorable Quad-Mesh}\} \leftrightarrow \{\text{Finite Measured Foliation}\} \leftrightarrow \{\text{Strebel Differential}\}.$$

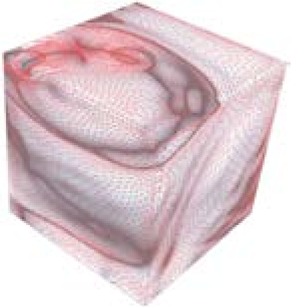
Algorithmic Pipeline

Genus Zero Case

Genus Zero Case

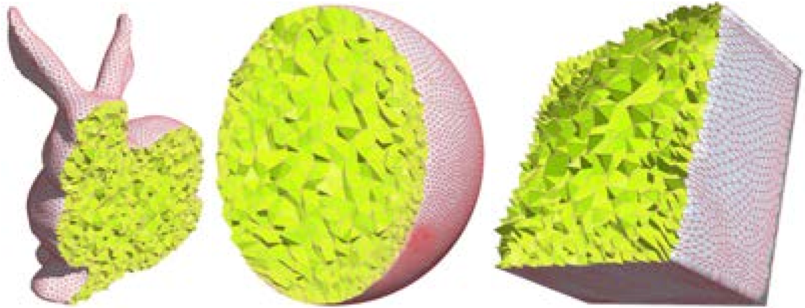
(a) Stanford bunny

(b) Spherical mapping



(c) Cube mapping

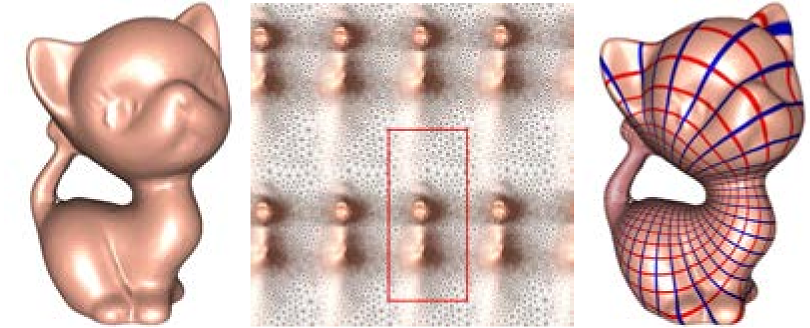
Genus Zero Case



(d) Solid bunny (e) Solid ball mapping (f) Solid cube mapping

Genus One Case

Genus One Case



(a) Kitten surface (b) Flat torus (c) Quad-mesh

Figure: A genus one closed surface can be conformally and periodically mapped onto the plane, each fundamental domain is a parallelogram. The subidvision of the parallelogram induces a quad-mesh of the surface.

Genus One Case

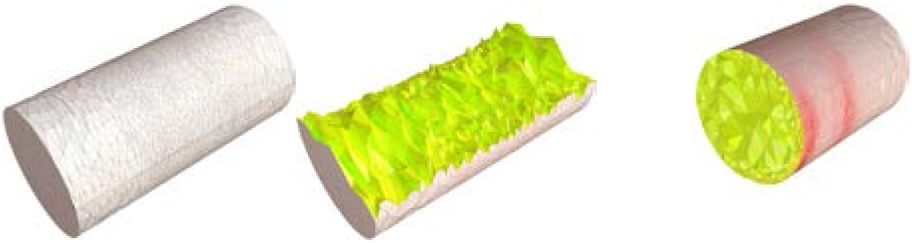


Figure: The interior of the kitten surface is mapped onto a canonical solid cylinder.

High Genus Case

Admissible curve system

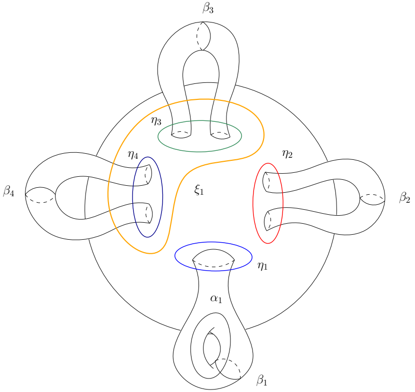


Figure: Admissible curve system.

1. Boundaries of cutting disks β_k, $k = 1, \cdots, g$
2. $\eta_k = \alpha_k \beta_k \alpha_k^{-1} \beta_k^{-1}$, $k = 1, \cdots, g$
3. ξ_k, $k = 1, \cdots, g-3$

We obtain an admissible curve system:

$$\Gamma = \{\beta_i, \eta_j, \xi_k\}$$

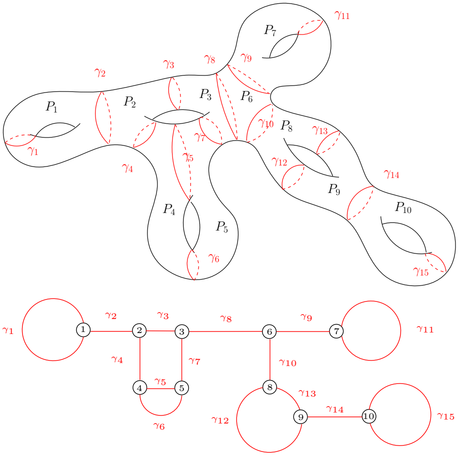
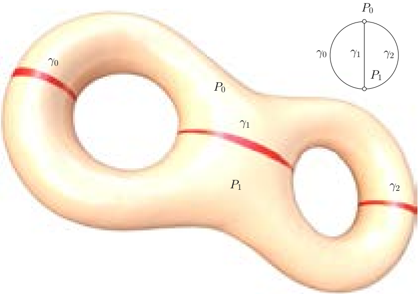
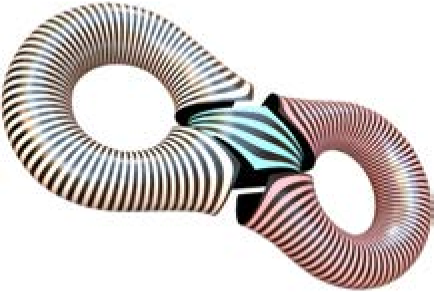
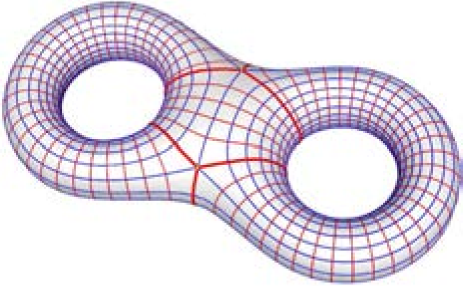


Figure: Pants decomposition and the pants decomposition graph.

High Genus Case







High Genus Case

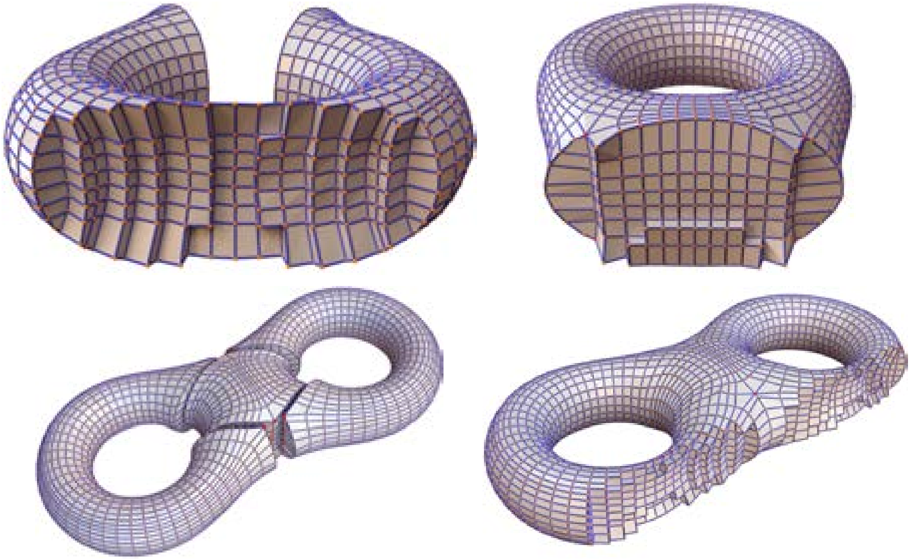
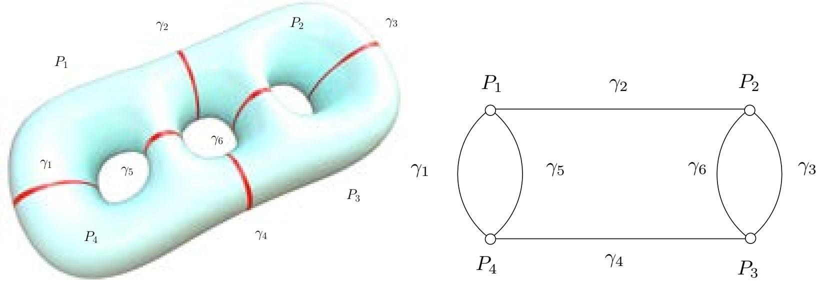


Figure: Hexahedral meshing pipeline.

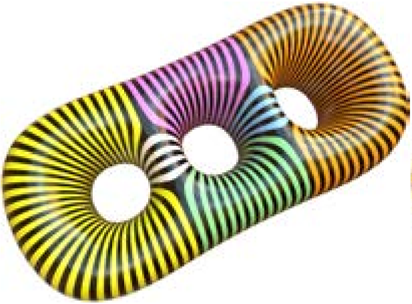
Experiments

Genus Three Example

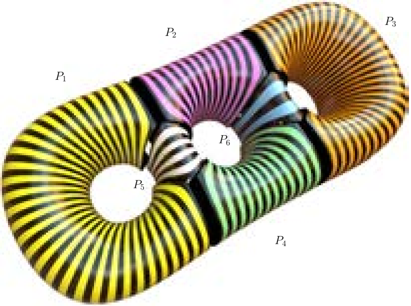


(a) Admissible curve system (b) Pants decomposition graph

Genus Three Example

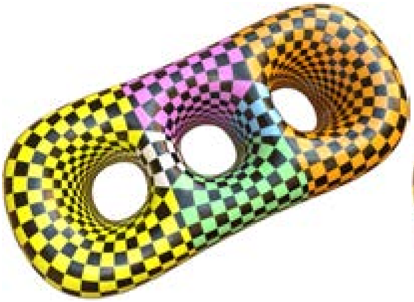


(c) Foliation induced by the pants decomposition

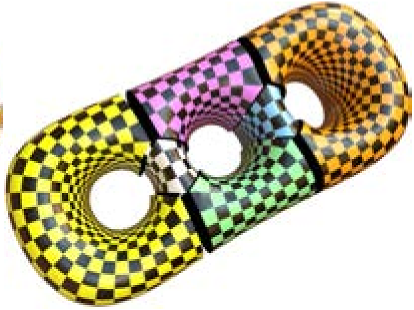


(d) Cylindrical decomposition

Genus Three Example

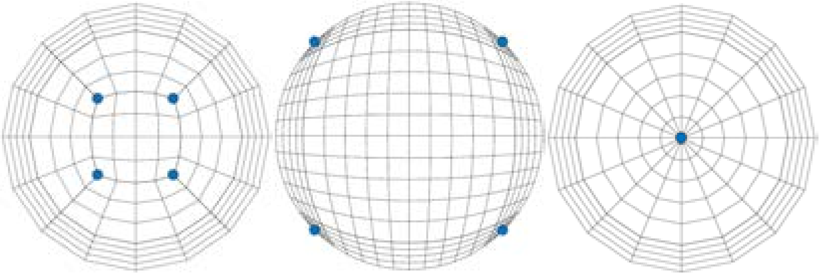


(e) Strebel differential

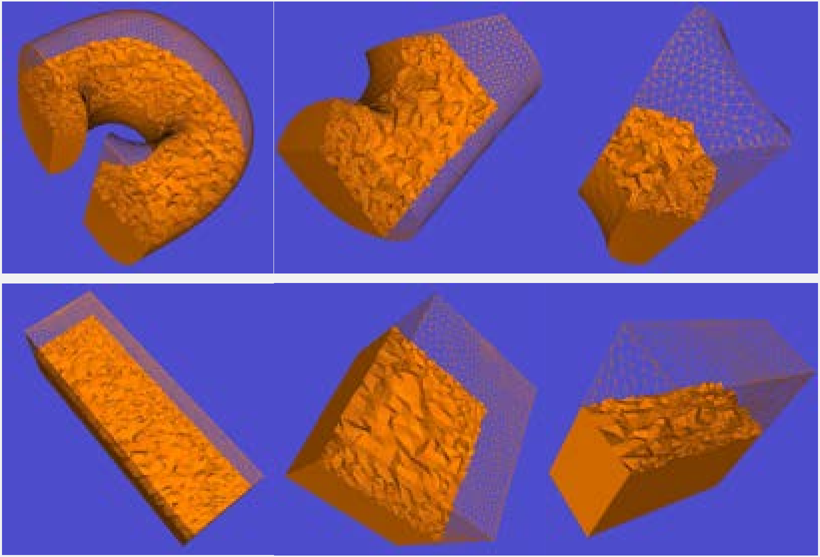


(f) Cylindrical decomposition

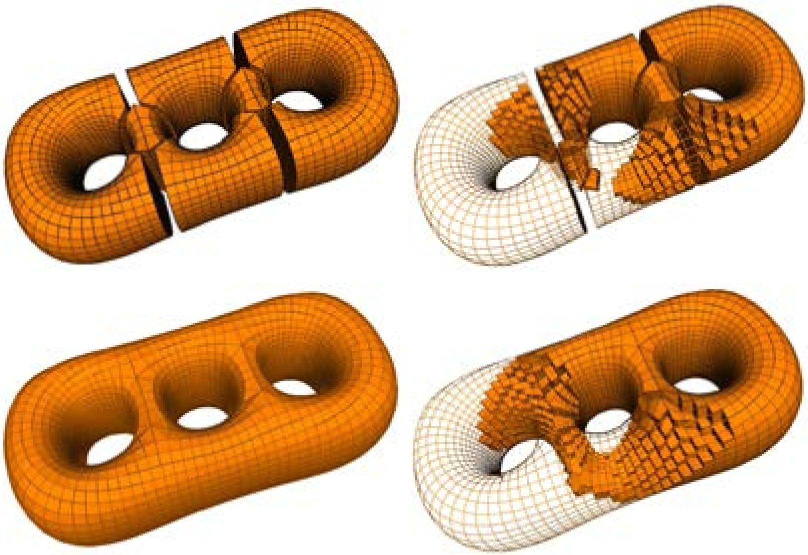
Genus Three Example



Genus Three Example



Genus Three Example



Decocube Example

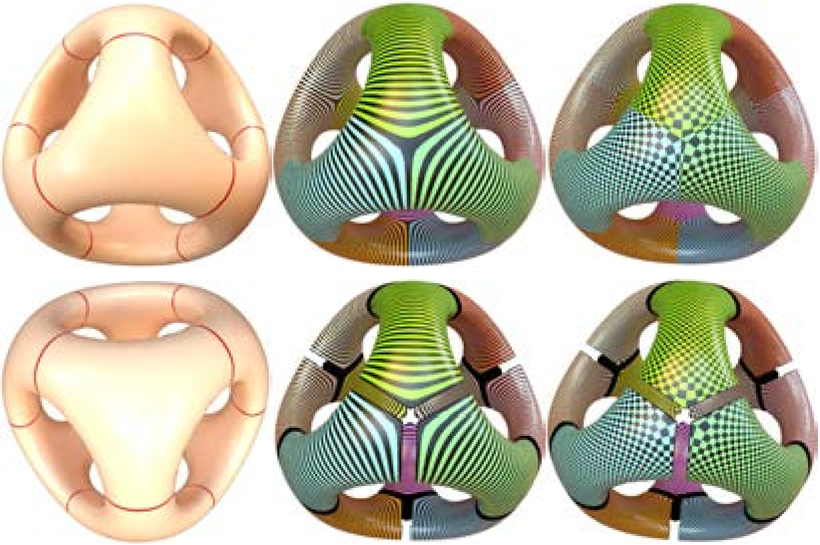
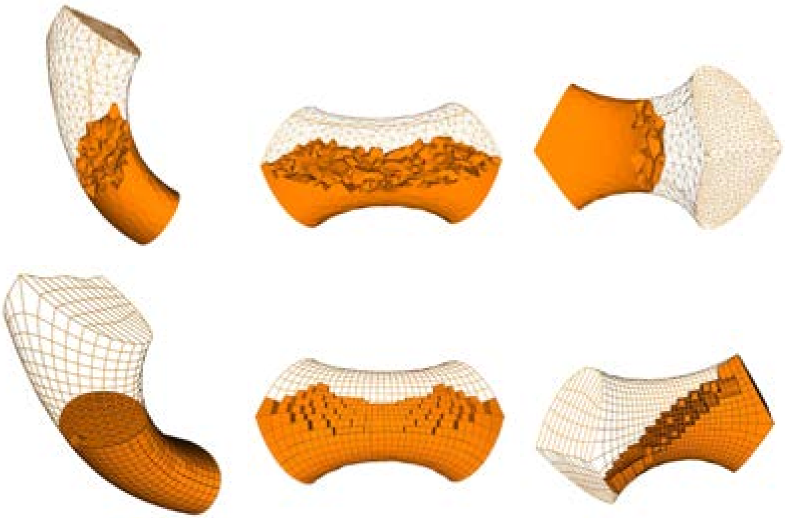
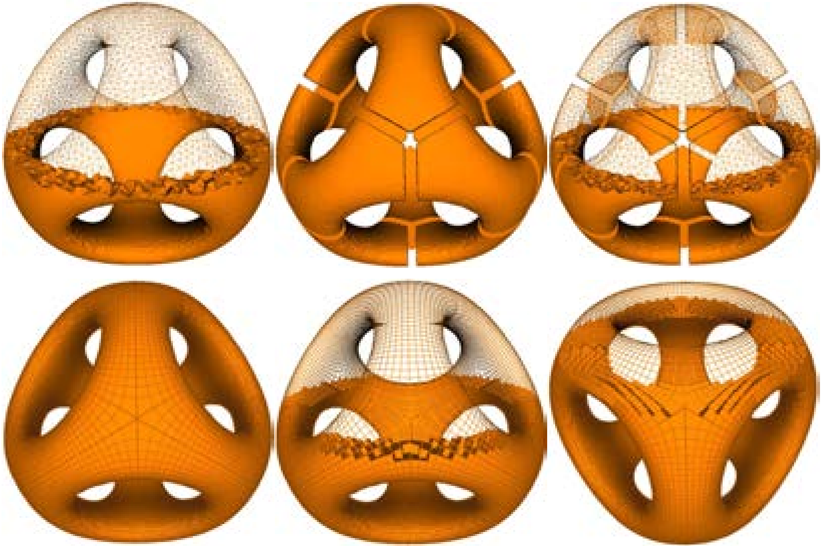


Figure: The admissible curve system is shown in (a) (front view) and (c) back

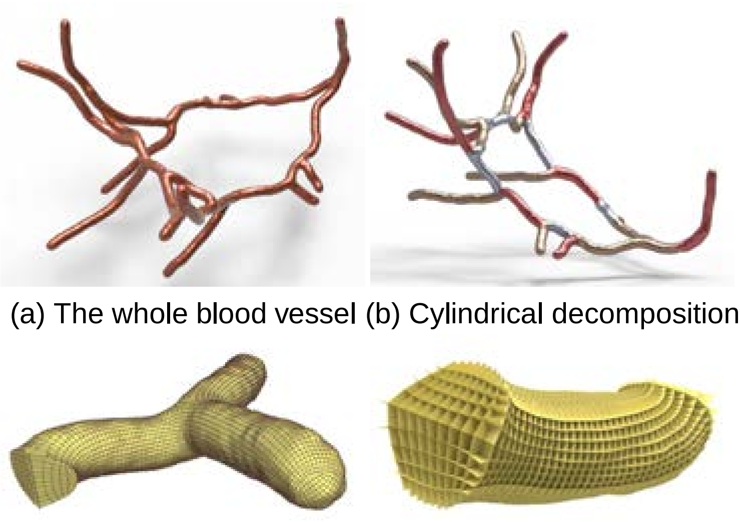
Decocube Example



Decocube Example



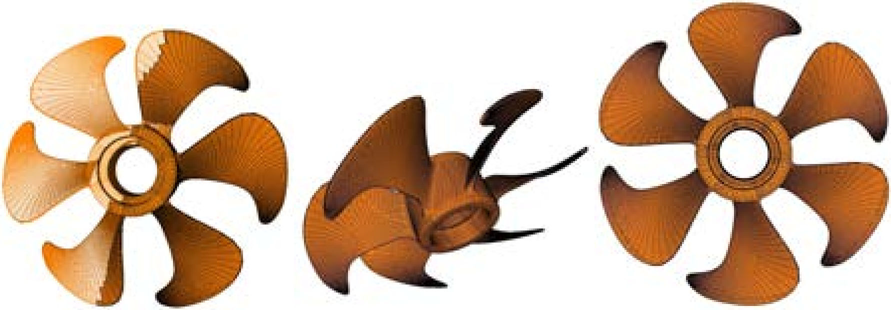
Blood Vessel



(a) The whole blood vessel (b) Cylindrical decomposition

(c) Quadrilateral mesh (d) Hexahedral mesh

Propeller



Summary

The merits of our algorithm:

- complicated topology,
- globally structured hex-mesh,
- minimal number of singular lines/points,
- automatically,
- completeness: the solutions form a finite dimensional space, we offer the basis of the solution space,
- solid theoretic foundation.

Future Work

- Take care of the sharp features on the surface.
- Generalize the theoretic framework to include singularities with odd valences.
- Develop practical software system for “holy grid” generation.

Thanks

For more information, please email to nalei@dlut.edu.cn.

Thank you!

Sharp Feature