Isogeometric Shape Optimization:

A brief introduction about shape sensitivity analysis and search direction normalization

Wang Zhenpei

NUS

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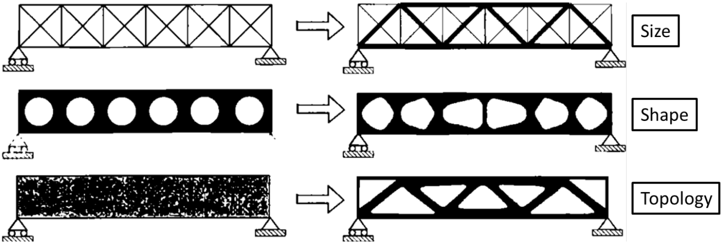
Size, shape and topology optimization



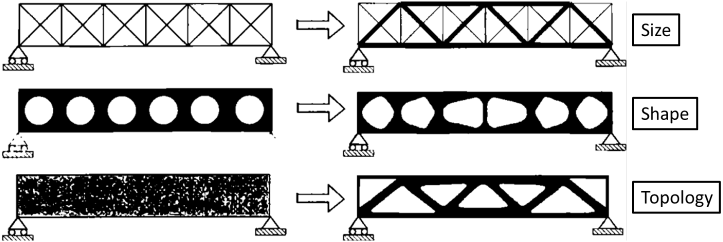
[Bendsøe and Sigmund(2004)]

Stiffness matrix:

$$\boldsymbol{K} = \sum_{e} \int_{\Omega^e} \boldsymbol{B}\mathbb{C}\boldsymbol{B} \mathrm{d}\Omega = \sum_{e} \int_{\Omega^e} \boldsymbol{B}\mathbb{C}\boldsymbol{B} |\boldsymbol{J}| \mathrm{d}\chi \quad (1)$$

Stiffness matrix variation: $\delta x \Rightarrow \delta \boldsymbol{K}$?

Size, shape and topology optimization



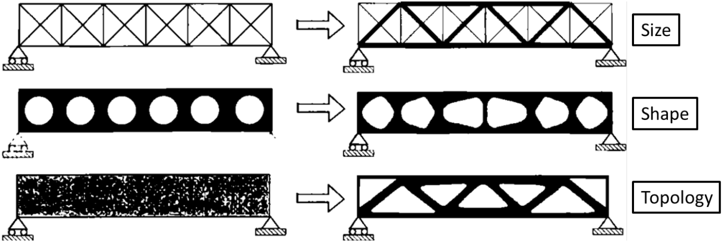
[Bendsøe and Sigmund(2004)]

Stiffness matrix:

$$\boldsymbol{K} = \sum_e \int_{\Omega^e} \boldsymbol{B}\mathbb{C}\boldsymbol{B}\mathrm{d}\Omega = \sum_e \int_{\Omega^e} \boldsymbol{B}\mathbb{C}\boldsymbol{B}|\boldsymbol{J}|\mathrm{d}\chi$$

Size optimization: $\delta x \Rightarrow \delta\mathbb{C} \Rightarrow \delta\boldsymbol{K}$ with $\mathbb{C} = h\bar{\mathbb{C}}$
Topology optimization: $\delta x \Rightarrow \delta\mathbb{C} \Rightarrow \delta\boldsymbol{K}$ with $\mathbb{C} = \rho\bar{\mathbb{C}}$

Size, shape and topology optimization



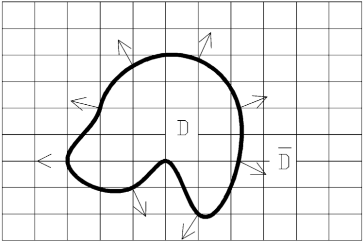
[Bendsøe and Sigmund(2004)]

Stiffness matrix:

$$\boldsymbol{K} = \sum_e \int_{\Omega^e} \boldsymbol{B}\mathbb{C}\boldsymbol{B}\mathrm{d}\Omega = \sum_e \int_{\Omega^e} \boldsymbol{B}\mathbb{C}\boldsymbol{B}|\boldsymbol{J}|\mathrm{d}\chi$$

Shape optimization: $\delta x \Rightarrow \{\delta \boldsymbol{B},\ \delta \boldsymbol{J}\} \Rightarrow \delta \boldsymbol{K}$

Topology optimization using shape optimization techniques



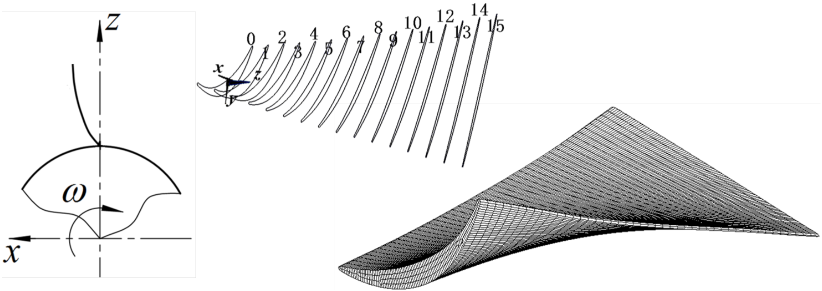
[Wang et al.(2003)Wang, Wang, and Guo]

Stiffness matrix:

$$\boldsymbol{K} = \sum_e \int_{\Omega^e} \boldsymbol{B}\mathbb{C}\boldsymbol{B} \mathrm{d}\Omega = \sum_e \int_{\Omega^e} \boldsymbol{B}\mathbb{C}\boldsymbol{B}|\boldsymbol{J}|\mathrm{d}\chi$$

Fixed background mesh: $\delta x \Rightarrow \delta\mathbb{C} \Rightarrow \delta\boldsymbol{K}$

Shape optimization by changing size parameters



[WANG et al.(2011)WANG, WANG, ZHU, and ZHANG]

Stiffness matrix:

$$\boldsymbol{K} = \sum_e \int_{\Omega^e} \boldsymbol{B}\mathbb{C}\boldsymbol{B}\mathrm{d}\Omega = \sum_e \int_{\Omega^e} \boldsymbol{B}\mathbb{C}\boldsymbol{B}|\boldsymbol{J}|\mathrm{d}\chi$$

Shape optimization: $\delta x \Rightarrow \{\delta \boldsymbol{B},\ \delta \boldsymbol{J}\} \Rightarrow \delta \boldsymbol{K}$

Outline

1. Structural optimization basics
2. IGA for shape optimization
3. Shape sensitivity analysis methods
4. Search directions related issues with NURBS parametrization
5. Research trends

IGA for shape optimization

Advantages:

- Seamless integration between CAD and CAE
  - Direct geometry updating
  - Meshing and re-meshing is easy
  - Curved features are preserved
- Enhanced sensitivity analysis
  - High order derivatives
  - More accurate structural response
  - Easily accessible geometry informations such as normal vector, curvature...
- Double levels discretization for design and analysis
  - e.g., coarse mesh for design & refined mesh for analysis

References: [Cho and Ha(2009)], [Qian(2010)], [Nagy et al.(2010)Nagy, Abdalla, and Gürdal].

Outline

Basic modules in a shape optimization problem

Optimizer:

- Update design variables
- GA, Steepest descent, SQP, MMA, GCMMA, ...

Sensitivity analysis:

- Compute the derivatives of the obj./cons. w.r.t. design variables
- Finite difference, Direct difference, Semi-analytical, adjoint method...

Supplementary processing:

- Search direction regularization/normalization
- Mesh updating
- Mesh regularization/smoothing
- ...

Finite difference and direct differential methods

Optimization problem

Obj. $\Psi[\boldsymbol{u}[x_i^j]]$ with n design variables: x_i^j, $i = 1,\ 2,\ 3,\ j = 1,\ 2,\ \cdots$

Finite difference

$$\frac{\mathrm{D}\Psi}{\mathrm{D}x_i^j} = \frac{\Psi[x_i^j + \Delta] - \Psi[x_i^j]}{\Delta}, \quad \text{by sovling } \boldsymbol{K}\boldsymbol{U} = \boldsymbol{F} \text{ for } n+1 \text{ times}$$

Direct differential method

$$\frac{\mathrm{D}\Psi}{\mathrm{D}x_i^j} = \Psi_{,\boldsymbol{U}}\mathring{\boldsymbol{U}}, \quad \text{by solving } \boldsymbol{K}\boldsymbol{U} = \boldsymbol{F} \text{ once}$$

$$\mathring{\boldsymbol{U}} = \frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}x_i^j}$$

Semi-analytical methods

$$\frac{\mathrm{D}\Psi}{\mathrm{D}x_i^j} = \Psi_{,\boldsymbol{U}}\mathring{\boldsymbol{U}}$$

$$\mathring{\boldsymbol{U}} = \frac{\mathrm{D}\boldsymbol{U}}{\mathrm{D}x_i^j} = \boldsymbol{K}^{-1}\left[\frac{\Delta \boldsymbol{F}}{\Delta x_i^j} - \frac{\Delta \boldsymbol{K}}{\Delta x_i^j}\boldsymbol{U}\right], \quad \text{by sovling } \boldsymbol{K}^{-1} \text{ once}$$

Remark: spatial and material design derivatives of strain/stress

$$\epsilon'[\boldsymbol{u}] = (\nabla \boldsymbol{u})' = \nabla(\boldsymbol{u}') = \epsilon[\boldsymbol{u}'];$$

$$\mathring{\epsilon}[\boldsymbol{u}] = \mathring{\overline{\nabla \boldsymbol{u}}} = (\nabla \boldsymbol{u})' + \nabla(\nabla \boldsymbol{u})\boldsymbol{v} = \nabla \mathring{\boldsymbol{u}} - (\nabla \boldsymbol{u})(\nabla \boldsymbol{v}) = \epsilon[\mathring{\boldsymbol{u}}] - (\nabla \boldsymbol{u})(\nabla \boldsymbol{v}).$$

$$\mathring{\boldsymbol{U}} \xrightarrow{\boldsymbol{B}} \nabla \mathring{\boldsymbol{u}}$$

Adjoint method

Optimization problem statement:

Objetive function Ψ

$$s.\ t.\ \begin{cases} \boldsymbol{c}[\boldsymbol{u}] := \operatorname{div} \mathbb{C}\nabla \boldsymbol{u} + \boldsymbol{f} = \boldsymbol{0} & \text{in } \Omega \\ (\mathbb{C}\nabla \boldsymbol{u})\,\boldsymbol{n} - \hat{\boldsymbol{t}} = \boldsymbol{0} & \text{on } \Gamma \text{ or } \boldsymbol{K}\boldsymbol{U} = \boldsymbol{F} \\ \boldsymbol{u} - \hat{\boldsymbol{u}} = \boldsymbol{0} & \text{on } \Gamma \end{cases}$$

Discrete approach

Discretize the problem first, then derive the formulation:

$$\Psi[\boldsymbol{U}],\ \boldsymbol{K}\boldsymbol{U} = \boldsymbol{F}$$

Continuous approach

Derive the formulation first as a continuum, then distretize the formulation and compute:

$$\Psi[\boldsymbol{u}],\ \text{BVP formulation}$$

Adjoint method – discrete approach

Optimization problem statement:

Objetive function Ψ

$s.\ t.\ \boldsymbol{KU} = \boldsymbol{F}$

Augmented formulation

$$\tilde{\Psi} = \Psi = \Psi + \boldsymbol{U}^{*\mathsf{T}}(-\boldsymbol{KU} + \boldsymbol{F})$$

Note that $\mathring{\boldsymbol{U}}^{*}(\boldsymbol{KU} - \boldsymbol{F}) = 0$,

$$\begin{aligned}\mathring{\tilde{\Psi}} &= \Psi_{,\boldsymbol{U}}\mathring{\boldsymbol{U}} + \boldsymbol{U}^{*\mathsf{T}}(-\mathring{\boldsymbol{K}}\boldsymbol{U} - \boldsymbol{K}\mathring{\boldsymbol{U}} + \mathring{\boldsymbol{F}}) \\ &= (\Psi_{,\boldsymbol{U}} - \boldsymbol{U}^{*\mathsf{T}}\boldsymbol{K})\mathring{\boldsymbol{U}} + \mathring{\boldsymbol{F}} - \boldsymbol{U}^{*\mathsf{T}}\mathring{\boldsymbol{K}}\boldsymbol{U}\end{aligned}$$

Introducing an adjoint problem with $\boldsymbol{U}^{*}$ that satisfies $\boldsymbol{KU}^{*} = \Psi_{,\boldsymbol{U}}$ we have

$$\mathring{\tilde{\Psi}} = \mathring{\boldsymbol{F}} - \boldsymbol{U}^{*\mathsf{T}}\mathring{\boldsymbol{K}}\boldsymbol{U}$$

Adjoint method – discrete approach

Example: minimizing structure compliance

$$\min\ \Psi := \boldsymbol{F}\boldsymbol{U}$$
$$s.\ t.\ \boldsymbol{K}\boldsymbol{U} = \boldsymbol{F} \text{ with } \mathring{\boldsymbol{F}} = \boldsymbol{0} \text{ (Design-independent load)}$$

Adjoint problem

$$\boldsymbol{K}\boldsymbol{U}^* = \Psi_{,\boldsymbol{U}} = \boldsymbol{F}$$
$$\Rightarrow \boldsymbol{U}^* = \boldsymbol{U} \quad \text{(self-adjoint problem)}$$

Shape sensitivity

$$\mathring{\Psi} = -\boldsymbol{U}^{*\mathsf{T}}\mathring{\boldsymbol{K}}\boldsymbol{U} = -\boldsymbol{U}^{\mathsf{T}}\mathring{\boldsymbol{K}}\boldsymbol{U}$$

Adjoint method – continuous approach

Objective function[Wang and Turteltaub(2015)]:

$$\Psi[s] := \int_{\Omega^s} \psi_\omega \left[\boldsymbol{u}[\boldsymbol{x}; s] \right] \mathrm{d}\Omega + \int_{\Gamma^s} \psi_\gamma \left[\boldsymbol{t}[\boldsymbol{x}; s], \boldsymbol{u}[\boldsymbol{x}; s] \right] \mathrm{d}\Gamma$$

BVP constraint:

$$\begin{cases} \boldsymbol{c}[\boldsymbol{u}] := \operatorname{div} \mathbb{C} \nabla \boldsymbol{u} + \boldsymbol{f} = \boldsymbol{0} & \text{in } \Omega \\ (\mathbb{C} \nabla \boldsymbol{u}) \, \boldsymbol{n} - \hat{\boldsymbol{t}} = \boldsymbol{0} & \text{on } \Gamma \\ \boldsymbol{u} - \hat{\boldsymbol{u}} = \boldsymbol{0} & \text{on } \Gamma \end{cases}$$

$$\Downarrow\Downarrow\Downarrow$$

$$\langle \boldsymbol{c}[\boldsymbol{u}], \boldsymbol{u}^* \rangle_{\Omega^s} = - \int_{\Omega^s} \mathbb{C} \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u}^* \, \mathrm{d}\Omega + \int_{\Omega^s} \boldsymbol{f} \cdot \boldsymbol{u}^* \, \mathrm{d}\Omega + \int_{\Gamma_{\mathrm{t}}^s} \hat{\boldsymbol{t}} \cdot \boldsymbol{u}^* \, \mathrm{d}\Gamma + \int_{\Gamma_{\mathrm{u}}^s} \boldsymbol{t} \cdot \boldsymbol{u}^* \, \mathrm{d}\Gamma = 0$$

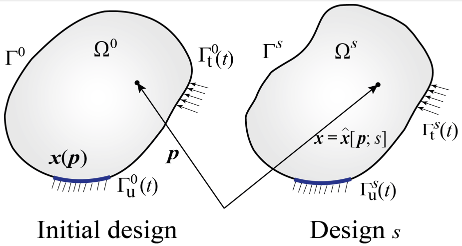
Adjoint method – continuous approach

Material and spatial derivatives

Material/full derivative: $\mathring{\boldsymbol{h}}[\boldsymbol{p}; s] := \frac{\partial \boldsymbol{h}}{\partial s}[\boldsymbol{p}; s]\Big|_{\boldsymbol{p}} = \frac{\mathrm{D}\boldsymbol{h}}{\mathrm{D}s}$

Spatial/partial derivative: $\boldsymbol{h}'[\boldsymbol{x}; s] := \frac{\partial \boldsymbol{h}}{\partial s}[\boldsymbol{x}; s]\Big|_{\boldsymbol{x}} = \frac{\partial \boldsymbol{h}}{\partial s}$

Design velocity: $\boldsymbol{\nu}[\boldsymbol{p}; s] := \mathring{\hat{\boldsymbol{x}}}[\boldsymbol{p}; s] = \frac{\partial \hat{\boldsymbol{x}}}{\partial s}[\boldsymbol{p}; s]\Big|_{\boldsymbol{p}}$



Adjoint method – continuous approach

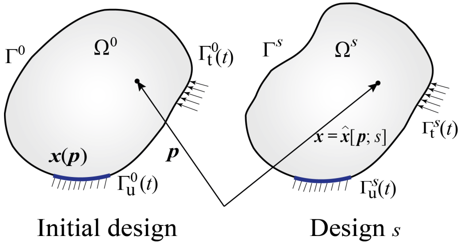
Transport relations

Volume

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{\Omega^s} f \,\mathrm{d}\Omega = \int_{\Omega^s} f' \mathrm{d}\Omega + \int_{\Gamma^s} f \boldsymbol{\nu}\cdot\boldsymbol{n}\,\mathrm{d}\Gamma$$

Boundary

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_{\Gamma^s} h \mathrm{d}\Gamma = \int_{\Gamma^s}\left(\mathring{h} - \kappa h \boldsymbol{\nu}\cdot\boldsymbol{n}\right)\mathrm{d}\Gamma \qquad \kappa := -\mathrm{div}_\Gamma \boldsymbol{n}$$



Adjoint method – continuous approach

Objective function[Wang and Turteltaub(2015)]:

$$\Psi[s] := \int_{\Omega^s} \psi_\omega\left[\boldsymbol{u}[\boldsymbol{x}; s]\right] \mathrm{d}\Omega + \int_{\Gamma^s} \psi_\gamma\left[\boldsymbol{t}[\boldsymbol{x}; s], \boldsymbol{u}[\boldsymbol{x}; s]\right] \mathrm{d}\Gamma$$

BVP constraint:

$$\begin{aligned}\langle \boldsymbol{c}[\boldsymbol{u}], \boldsymbol{u}^* \rangle_{\Omega^s} = &- \int_{\Omega^s} \mathbb{C}\nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u}^* \, \mathrm{d}\Omega + \int_{\Omega^s} \boldsymbol{f} \cdot \boldsymbol{u}^* \, \mathrm{d}\Omega \\ &+ \int_{\Gamma^s_\mathrm{t}} \hat{\boldsymbol{t}} \cdot \boldsymbol{u}^* \, \mathrm{d}\Gamma + \int_{\Gamma^s_\mathrm{u}} \boldsymbol{t} \cdot \boldsymbol{u}^* \, \mathrm{d}\Gamma = 0\end{aligned}$$

Augmented function

$$\tilde{\Psi} := \Psi + \langle \boldsymbol{c}[\boldsymbol{u}], \boldsymbol{u}^* \rangle_{\Omega^s}$$

Adjoint method – continuous approach

Derivatives:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}s} = \int_{\Omega^s} \psi_{\omega,\boldsymbol{u}} \boldsymbol{u}' \,\mathrm{d}\Omega + \int_{\Gamma^s} \psi_\omega \nu_n \,\mathrm{d}\Gamma + \int_{\Gamma^s} (\nabla \psi_\gamma \cdot \boldsymbol{n} \nu_n - \psi_\gamma \kappa \nu_n) \,\mathrm{d}\Gamma$$
$$+ \int_{\Gamma^s_{\mathrm{t}}} \psi_{\gamma,\boldsymbol{u}} \boldsymbol{u}' \,\mathrm{d}\Gamma + \int_{\Gamma^s_{\mathrm{u}}} \psi_{\gamma,\boldsymbol{u}} \hat{\boldsymbol{u}}' \,\mathrm{d}\Gamma + \int_{\Gamma^s_{\mathrm{t}}} \psi_{\gamma,\hat{\boldsymbol{t}}} \hat{\boldsymbol{t}}' \,\mathrm{d}\Gamma + \int_{\Gamma^s_{\mathrm{u}}} \psi_{\gamma,\boldsymbol{t}} \boldsymbol{t}' \,\mathrm{d}\Gamma$$

$$\frac{\partial}{\partial s} \langle \boldsymbol{c}[\boldsymbol{u}], \boldsymbol{u}^* \rangle = \int_{\Omega^s} \left(-\mathbb{C} \nabla \boldsymbol{u}' \cdot \nabla \boldsymbol{u}^* - \mathbb{C} \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u}^{*\prime} + \boldsymbol{f}' \cdot \boldsymbol{u}^* + \boldsymbol{f} \cdot \boldsymbol{u}^{*\prime} \right) \mathrm{d}\Omega$$
$$- \int_{\Gamma^s} (\mathbb{C} \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u}^* - \boldsymbol{f} \cdot \boldsymbol{u}^*) \nu_n \,\mathrm{d}\Gamma$$
$$+ \int_{\Gamma^s_{\mathrm{t}}} \left(\hat{\boldsymbol{t}}' \cdot \boldsymbol{u}^* + \hat{\boldsymbol{t}} \cdot \boldsymbol{u}^{*\prime} \right) \mathrm{d}\Gamma + \int_{\Gamma^s_u} \left(\boldsymbol{t}' \cdot \boldsymbol{u}^* + \boldsymbol{t} \cdot \boldsymbol{u}^{*\prime} \right) \mathrm{d}\Gamma$$
$$+ \int_{\Gamma^s} \left(\nabla \left(\boldsymbol{t} \cdot \boldsymbol{u}^* \right) \boldsymbol{n} \nu_n - \left(\boldsymbol{t} \cdot \boldsymbol{u}^* \right) \kappa \nu_n \right) \mathrm{d}\Gamma$$

Derivatives:

Note $\langle \boldsymbol{c}[\boldsymbol{u}], \boldsymbol{u}^{*\prime} \rangle = 0$,
$\int_{\Omega^s} \mathbb{C}\nabla \boldsymbol{u}' \cdot \nabla \boldsymbol{u}^* \, \mathrm{d}\Omega = \int_{\Gamma^s} \boldsymbol{t}^* \cdot \boldsymbol{u}' \mathrm{d}\Gamma - \int_{\Omega^s} \mathbb{C}\nabla^2 \boldsymbol{u}^* \cdot \boldsymbol{u}' \, \mathrm{d}\Omega$, we have
$\frac{\mathrm{D}\tilde{\Psi}}{\mathrm{D}s} = \Phi_1 + \Phi_2$, where

$$\Phi_1 = \int_{\Omega^s} \left(\psi_{\omega,\boldsymbol{u}} + \mathbb{C}\nabla^2 \boldsymbol{u}^*\right) \cdot \boldsymbol{u}' \, \mathrm{d}\Omega + \int_{\Gamma^s_{\mathrm{t}}} \left(\psi_{\gamma,\boldsymbol{u}} - \boldsymbol{t}^*\right) \cdot \boldsymbol{u}' \, \mathrm{d}\Gamma + \int_{\Gamma^s_{\mathrm{u}}} \left(\psi_{\gamma,\boldsymbol{t}} + \boldsymbol{u}^*\right) \cdot \boldsymbol{t}' \, \mathrm{d}\Gamma$$

$$\begin{aligned}\Phi_2 = & \int_{\Gamma^s} \left(\psi_\omega - \mathbb{C}\nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u}^* + \boldsymbol{f} \cdot \boldsymbol{u}^*\right)\nu_n \, \mathrm{d}\Gamma + \int_{\Omega^s} \boldsymbol{f}' \cdot \boldsymbol{u}^* \mathrm{d}\Omega \\ & + \int_{\Gamma^s} \Big(\left(\nabla \psi_\gamma \cdot \boldsymbol{n}\nu_n - \psi_\gamma \kappa \nu_n\right) + \nabla \left(\boldsymbol{t} \cdot \boldsymbol{u}^*\right) \cdot \boldsymbol{n}\nu_n - \left(\boldsymbol{t} \cdot \boldsymbol{u}^*\right) \kappa \nu_n \Big) \, \mathrm{d}\Gamma \\ & + \int_{\Gamma^s_{\mathrm{u}}} \left(\psi_{\gamma,\boldsymbol{u}} - \boldsymbol{t}^*\right) \cdot \hat{\boldsymbol{u}}' \, \mathrm{d}\Gamma + \int_{\Gamma^s_{\mathrm{t}}} \left(\psi_{\gamma,\hat{\boldsymbol{t}}} + \boldsymbol{u}^*\right) \cdot \hat{\boldsymbol{t}}' \, \mathrm{d}\Gamma\end{aligned}$$

Adjoint method – continuous approach

Adjoint model:

Introducing

$$\begin{aligned} \mathbb{C}\nabla^2 \boldsymbol{u}^* + \boldsymbol{f}^* = 0 \quad &\text{with} \quad \boldsymbol{f}^* = \psi_{\omega,\boldsymbol{u}} && \text{in} \quad \Omega^s; \\ \boldsymbol{u}^* = \hat{\boldsymbol{u}}^* \quad &\text{with} \quad \hat{\boldsymbol{u}}^* = -\psi_{\gamma,\boldsymbol{t}} && \text{on} \quad \Gamma_u^s; \\ \boldsymbol{t}^* = (\mathbb{C}\nabla \boldsymbol{u}^*)^{\mathsf{T}} \boldsymbol{n} = \hat{\boldsymbol{t}}^* \quad &\text{with} \quad \hat{\boldsymbol{t}}^* = \psi_{\gamma,\boldsymbol{u}} && \text{on} \quad \Gamma_t^s. \end{aligned}$$

such that

$$\Phi_1 = 0$$

Eventually,

$$\frac{\mathrm{D}\tilde{\Psi}}{\mathrm{D}s} = \frac{\mathrm{D}\Psi}{\mathrm{D}s} = \Phi_2$$

Adjoint method – continuous approach

Shape sensitivity:

$$\frac{\mathrm{D}\Psi}{\mathrm{D}s} = \Phi_2 = \int_{\Gamma^s} \left(\psi_\omega - \mathbb{C}\nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u}^* + \boldsymbol{f} \cdot \boldsymbol{u}^*\right) \nu_n \, \mathrm{d}\Gamma + \int_{\Omega^s} \boldsymbol{f}' \cdot \boldsymbol{u}^* \mathrm{d}\Omega$$
$$+ \int_{\Gamma^s} \Big(\left(\nabla \psi_\gamma \cdot \boldsymbol{n}\nu_n - \psi_\gamma \kappa \nu_n\right) + \nabla \left(\boldsymbol{t} \cdot \boldsymbol{u}^*\right) \cdot \boldsymbol{n}\nu_n - \left(\boldsymbol{t} \cdot \boldsymbol{u}^*\right) \kappa \nu_n \Big) \, \mathrm{d}\Gamma$$
$$+ \int_{\Gamma^s_{\mathrm{u}}} \left(\psi_{\gamma,\boldsymbol{u}} - \boldsymbol{t}^*\right) \cdot \hat{\boldsymbol{u}}' \, \mathrm{d}\Gamma + \int_{\Gamma^s_{\mathrm{t}}} \left(\psi_{\gamma,\hat{\boldsymbol{t}}} + \boldsymbol{u}^*\right) \cdot \hat{\boldsymbol{t}}' \, \mathrm{d}\Gamma$$

$$\boldsymbol{\nu} = \mathring{\hat{\boldsymbol{x}}} = \sum_I R^I \frac{\mathrm{d}\boldsymbol{x}^I[s]}{\mathrm{d}s}$$

$$\frac{\mathrm{D}\Psi}{\mathrm{D}\boldsymbol{x}^I} = \int_{\Gamma^s} \left(\psi_\omega - \mathbb{C}\nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u}^* + \boldsymbol{f} \cdot \boldsymbol{u}^*\right) \boldsymbol{n} R^I \, \mathrm{d}\Gamma + \int_{\Omega^s} \boldsymbol{f}' \cdot \boldsymbol{u}^* \mathrm{d}\Omega$$
$$+ \int_{\Gamma^s} \Big(\left(\nabla \psi_\gamma \cdot \boldsymbol{n} - \psi_\gamma \kappa\right) + \nabla \left(\boldsymbol{t} \cdot \boldsymbol{u}^*\right) \cdot \boldsymbol{n} - \left(\boldsymbol{t} \cdot \boldsymbol{u}^*\right) \kappa \Big) \boldsymbol{n} R^I \, \mathrm{d}\Gamma$$

Adjoint method – continuous approach

Shape sensitivity:

$$\frac{D\Psi}{Ds} = \Phi_2 = \int_{\Gamma^s} \left(\psi_\omega - \mathbb{C}\nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u}^* + \boldsymbol{f} \cdot \boldsymbol{u}^*\right) \nu_n \, d\Gamma + \int_{\Omega^s} \boldsymbol{f}' \cdot \boldsymbol{u}^* d\Omega$$
$$+ \int_{\Gamma^s} \Big((\nabla \psi_\gamma \cdot \boldsymbol{n} \nu_n - \psi_\gamma \kappa \nu_n) + \nabla (\boldsymbol{t} \cdot \boldsymbol{u}^*) \cdot \boldsymbol{n} \nu_n - (\boldsymbol{t} \cdot \boldsymbol{u}^*) \kappa \nu_n \Big) \, d\Gamma$$
$$+ \int_{\Gamma^s_{\mathrm{u}}} (\psi_{\gamma,\boldsymbol{u}} - \boldsymbol{t}^*) \cdot \hat{\boldsymbol{u}}' \, d\Gamma + \int_{\Gamma^s_{\mathrm{t}}} (\psi_{\gamma,\hat{\boldsymbol{t}}} + \boldsymbol{u}^*) \cdot \hat{\boldsymbol{t}}' \, d\Gamma$$

$$\boldsymbol{\nu} = \mathring{\hat{\boldsymbol{x}}} = \sum_I R^I \frac{d\boldsymbol{x}^I[s]}{ds}$$

Adjoint method – continuous approach

Shape sensitivity:

$$\frac{D\Psi}{D\boldsymbol{x}^l} = \int_{\Gamma^s} (\psi_\omega - \mathbb{C}\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^* + \boldsymbol{f}\cdot\boldsymbol{u}^*)\boldsymbol{n}R^l \,\mathrm{d}\Gamma + \int_{\Omega^s} \boldsymbol{f}'\cdot\boldsymbol{u}^*\mathrm{d}\Omega$$

$$+ \int_{\Gamma^s} \Big((\nabla\psi_\gamma\cdot\boldsymbol{n} - \psi_\gamma\kappa) + \nabla(\boldsymbol{t}\cdot\boldsymbol{u}^*)\cdot\boldsymbol{n} - (\boldsymbol{t}\cdot\boldsymbol{u}^*)\,\kappa\Big)\boldsymbol{n}R^l \,\mathrm{d}\Gamma$$

$$+ \int_{\Gamma^s_{\mathrm{u}}} (\psi_{\gamma,\boldsymbol{u}} - \boldsymbol{t}^*)\cdot\hat{\boldsymbol{u}}' \,\mathrm{d}\Gamma + \int_{\Gamma^s_{\mathrm{t}}} (\psi_{\gamma,\hat{\boldsymbol{t}}} + \boldsymbol{u}^*)\cdot\hat{\boldsymbol{t}}' \,\mathrm{d}\Gamma$$

Adjoint method – continuous approach

Example: minimize structural compliance

Obj: $\Psi[s] := \int_{\Gamma_t^s} \psi_\gamma \, d\Gamma$ with $\psi_\gamma = \boldsymbol{t} \cdot \boldsymbol{u}$

$\boldsymbol{t} = \hat{\boldsymbol{t}}$ is design-independent, i.e., $\hat{\boldsymbol{t}}' = \mathbf{0}$.

BVP constraint:

$$\begin{cases} \boldsymbol{c}[\boldsymbol{u}] := \operatorname{div} \mathbb{C} \nabla \boldsymbol{u} + \boldsymbol{f} = \mathbf{0} \quad \text{with} \quad \boldsymbol{f} = \mathbf{0} & \text{in } \Omega^s \\ \boldsymbol{t} = \hat{\boldsymbol{t}} \neq \mathbf{0} & \text{on } \Gamma_t^s \\ \boldsymbol{u} = \hat{\boldsymbol{u}} = \mathbf{0} & \text{on } \Gamma_u^s \end{cases}$$

Adjoint model = primary model (self-adjoint)

$$\begin{cases} \mathbb{C} \nabla^2 \boldsymbol{u}^* + \boldsymbol{f}^* = 0 \quad \text{with} \quad \boldsymbol{f}^* = \psi_{\omega,\boldsymbol{u}} = \mathbf{0} & \text{in} \quad \Omega^s; \\ \boldsymbol{t}^* = (\mathbb{C} \nabla \boldsymbol{u}^*)^\mathsf{T} \boldsymbol{n} = \hat{\boldsymbol{t}}^* \quad \text{with} \quad \hat{\boldsymbol{t}}^* = \psi_{\gamma,\boldsymbol{u}} = \hat{\boldsymbol{t}} & \text{on} \quad \Gamma_t^s \\ \boldsymbol{u}^* = \hat{\boldsymbol{u}}^* \quad \text{with} \quad \hat{\boldsymbol{u}}^* = -\psi_{\gamma,\boldsymbol{t}} = \mathbf{0} & \text{on} \quad \Gamma_u^s. \end{cases}$$

Adjoint method – continuous approach

Example: minimizing the structural compliance

Shape sensitivity:

$$\frac{D\Psi}{Dx^l} = -\int_{\Gamma^s} \mathbb{C}\nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u}^* \boldsymbol{n} R^l \, d\Gamma + \int_{\Gamma^s_t} (\nabla \psi_\gamma \cdot \boldsymbol{n} - \psi_\gamma \kappa) \boldsymbol{n} R^l \, d\Gamma$$

$$+ \int_{\Gamma^s} \Big(\nabla \left(\boldsymbol{t} \cdot \boldsymbol{u}^* \right) \cdot \boldsymbol{n} - \left(\boldsymbol{t} \cdot \boldsymbol{u}^* \right) \kappa \Big) \boldsymbol{n} R^l \, d\Gamma$$

$$= -\int_{\Gamma^s_t} \mathbb{C}\nabla \boldsymbol{u} \cdot \nabla \boldsymbol{u}^* \boldsymbol{n} R^l \, d\Gamma + 2 \int_{\Gamma^s_t} \Big(\nabla \left(\boldsymbol{t} \cdot \boldsymbol{u} \right) \cdot \boldsymbol{n} - \left(\boldsymbol{t} \cdot \boldsymbol{u} \right) \kappa \Big) \boldsymbol{n} R^l \, d\Gamma$$

Compared with the discrete approach:

$$\mathring{\Psi} = -\boldsymbol{U}^{*\mathsf{T}} \mathring{\boldsymbol{K}} \boldsymbol{U} = -\boldsymbol{U}^{\mathsf{T}} \mathring{\boldsymbol{K}} \boldsymbol{U}$$

Which one is easier for you to compute??

Discrete approach *vs* Continuous approach

For problems with design-dependent boundary conditions, which approach is easier ??

Discrete approach *vs* Continuous approach

For problems with design-dependent boundary conditions, which approach is easier ??

The answers can be different for different people.

In general, just choose the one you like.

Adjoint method – continuous approach

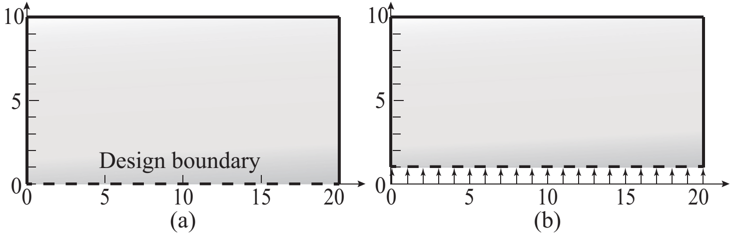
Some additional references about continuous adjoint method:

- [Dems and Mroz(1984)]: Variational approach by means of adjoint systems to structural optimization and sensitivity analysis—II: Structure shape variation, IJSS, 1984.
- [Choi and Kim(2005)]: Structural sensitivity analysis and optimization 1: Linear systems, 2006.
- [Arora(1993)]: An exposition of the material derivative approach for structural shape sensitivity analysis, CMAME, 1993.
- [Tortorelli and Haber(1989)]:First-order design sensitivities for transient conduction problems by an adjoint method, IJNME, 1989.
- [Wang and Turteltaub(2015)]: Isogeometric shape optimization for quasi-static processes, IJNME, 2015.
- [Wang et al.(2017c)Wang, Turteltaub, and Abdalla]: Shape optimization and optimal control for transient heat conduction problems using an isogeometric approach, C&S, 2017.

Outline

1. Structural optimization basics
2. IGA for shape optimization
3. Shape sensitivity analysis methods
4. Search directions related issues with NURBS parametrization
5. Research trends

Parameterization-dependency of the search directions



Example: volume reduction

Volume: $\Sigma = \int_\Omega \mathrm{d}\Omega$

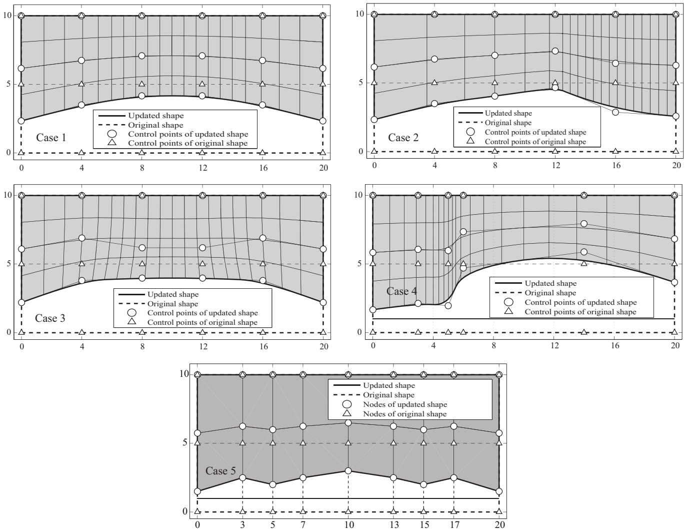
Gradient (continuous):

$$\boldsymbol{g} = \boldsymbol{n}$$

Gradient (NURBS discretization):

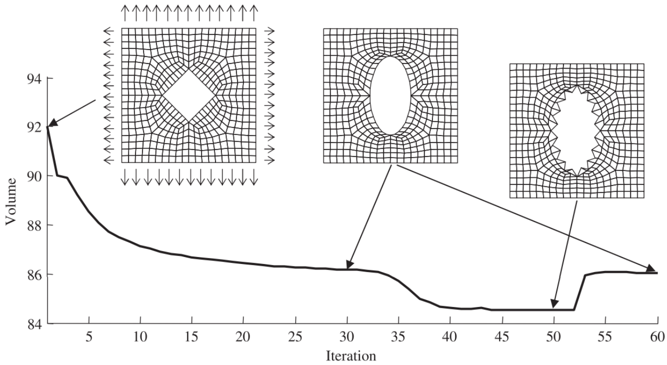
$$\boldsymbol{g}_d^I = \int_\Gamma \boldsymbol{n} R^I \, \mathrm{d}\Gamma$$

Parameterization-dependency of the search directions



$$\left(\boldsymbol{x}^{l}\right)^{(s+1)} = \left(\boldsymbol{x}^{l}\right)^{(s)} + \alpha \boldsymbol{d}_{d}^{l} = \left(\boldsymbol{x}^{l}\right)^{(s)} - \alpha \boldsymbol{g}_{d}^{l}$$

Parameterization-dependency of the search directions



Parameterization-free approach for FE-based shape optimization [Le et al.(2011)Le, Bruns, and Tortorelli]

A simple example about the quadratic norm induced by discretization

A (squared) L^2 norm

$$f\left[\boldsymbol{x}\right] := \boldsymbol{x}\cdot\boldsymbol{x},$$

gradient: $\boldsymbol{g} = f_{,\boldsymbol{x}} = 2\boldsymbol{x}$

steepest search direction: $\boldsymbol{d} = -\boldsymbol{g}$

Quadratic norm induced by discretization $\boldsymbol{x} = \boldsymbol{R}^{\mathrm{T}}\boldsymbol{X}$

$\boldsymbol{R}$ is a vector of shape functions, $\boldsymbol{X}$ is a vector of discrete variables $\boldsymbol{x}^I$

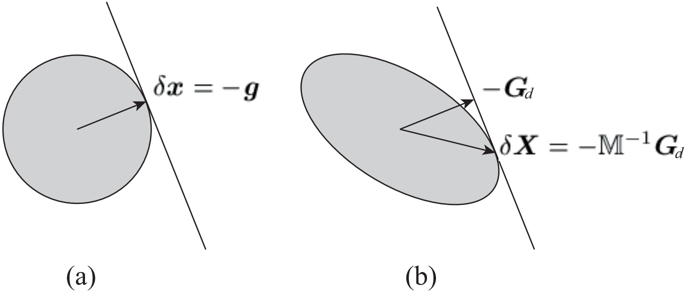
$$f = \boldsymbol{X}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{X}, \quad \text{with} \quad \boldsymbol{M} = \boldsymbol{R}\boldsymbol{R}^{\mathrm{T}}$$

gradient: $\boldsymbol{g}^I = f_{,\boldsymbol{x}^I} = 2M_{IJ}\boldsymbol{x}^J$, $\boldsymbol{G} = f_{,\boldsymbol{X}} = 2\boldsymbol{M}\boldsymbol{X}$

steepest search direction: $\boldsymbol{D}_n = -\boldsymbol{M}^{-1}\boldsymbol{G}$, $\boldsymbol{D}_n = [\boldsymbol{d}_n^1,\ \boldsymbol{d}_n^2, \cdots]$

!! $\boldsymbol{D}_d = -\boldsymbol{G}$ is NOT the steepest search direction !!

Steepest search directions of quadratic and (squared) L^2 norms



Reproduced from [Boyd and Vandenberghe(2009)]: Convex Optimization

Consistency

The normalized search direction of a discrete form is consistent with the steepest search direction of a continuous form.

Normalization approaches

1. Standard approach

$$\boldsymbol{D}_n = -\boldsymbol{M}^{-1}\boldsymbol{G}$$

2. DLMM normalization approach

The *diagonally lumped mapping matrix* (DLMM)

$$\bar{\boldsymbol{M}}_{II} := \sum_J \boldsymbol{M}_{IJ} = \bar{\boldsymbol{M}}_{II} = \int_D R^I \,\mathrm{d}D, \quad \text{with } \sum_J R^J = 1$$

$$\boldsymbol{d}_n^I = -\frac{\boldsymbol{g}_d^I}{\bar{\boldsymbol{M}}_{II}} = -\frac{\int_D \boldsymbol{g} R^I \,\mathrm{d}D}{\int_D R^I \,\mathrm{d}D}$$

"Sensitivity weighting" method in [Kiendl et al.(2014)Kiendl, Schmidt, WWüchner, and Bletzinger].

Normalization approaches

3. B-Spline space ($\bar{D}$) normalization

$$\boldsymbol{d}_n^I \approx -\frac{\int_{\bar{D}} \boldsymbol{g} N^I \,\mathrm{d}\bar{D}}{\int_{\bar{D}} N^I \,\mathrm{d}\bar{D}}$$

4. Simplified DLMM approach

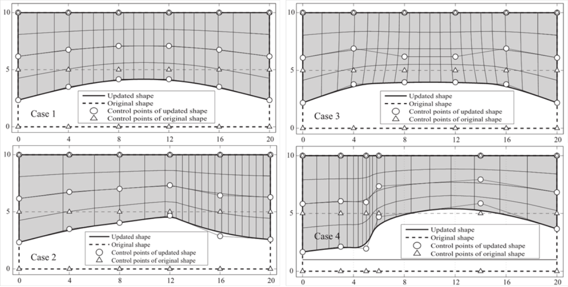
Unity of integral property of B-spline basis

$$\frac{\int N^{i,p} \mathrm{d}\xi}{\xi_{i+p+1} - \xi_i} = \frac{1}{p+1},$$

$$\boldsymbol{d}_n^I = -\frac{(p+1) \int_{\bar{D}} \boldsymbol{g} N^I \,\mathrm{d}\bar{D}}{\xi_{i+p+1} - \xi_i},$$

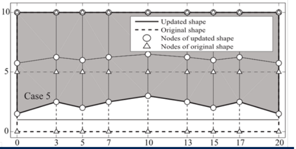
More information in [Wang et al.(2017a)Wang, Abdalla, and Turteltaub].

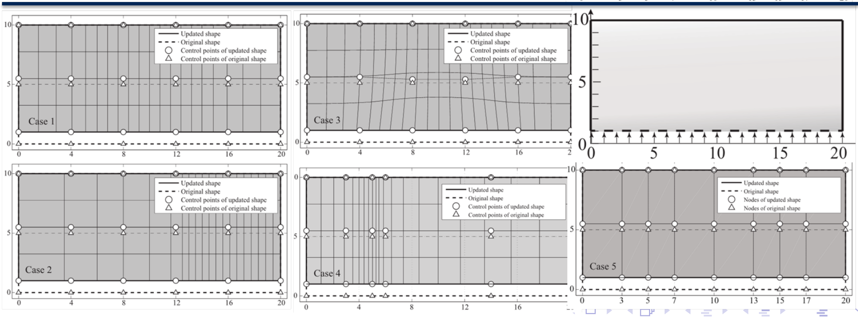
Effectiveness of the simplified DLMM approach



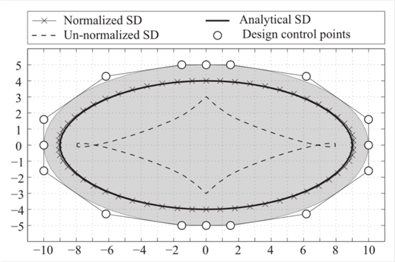
Discrete search direction

$$D_d^I = \delta \boldsymbol{x}^I = -\boldsymbol{g}^I$$



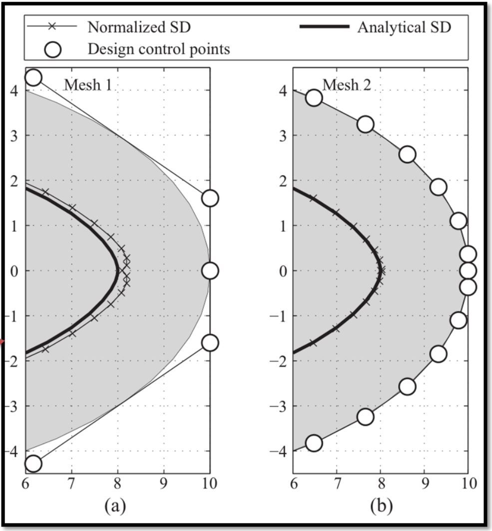


Effectiveness of the simplified DLMM approach



L2 norm error

Mesh	Approach		
	Standard	DLMM	Simplified DLMM
1	0.02121	0.02597	0.02476
2	0.02087	0.02140	0.02129



Normalization approaches

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- [Boyd and Vandenberghe(2009)]: Convex optimization, Cambridge University Press, 2009.
- [Wang and Kumar(2017)]:On the numerical implementation of continuous adjoint sensitivity for transient heat conduction problems using an isogeometric approach, SMO, 2017.

Outline

Research trends

- Shape optimization techniques

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- Shape optimization techniques
- Special applications of isogeometric shape optimization, e.g.,
  - Auxetic structures design
    [Wang et al.(2017b)Wang, Poh, Dirrenberger, Zhu, and Forest]
  - Curved (laminated) shells
    [Kiendl et al.(2014)Kiendl, Schmidt, WWüchner, and Bletzinger, Nagy et al.(2013)Nagy, IJsselmuiden, and Abdalla]

Research trends

- Shape optimization techniques
- Special applications of isogeometric shape optimization, e.g.,
  - Auxetic structures design [Wang et al.(2017b)Wang, Poh, Dirrenberger, Zhu, and Forest]
  - Curved (laminated) shells [Kiendl et al.(2014)Kiendl, Schmidt, WWüchner, and Bletzinger, Nagy et al.(2013)Nagy, IJsselmuiden, and Abdalla]
- Shape optimization using new analysis techniques, e.g.,
  - Trimmed spline surface [Seo et al.(2010)Seo, Kim, and Youn]
  - Bézier triangle based isogeometric shape optimization [Wang et al.(2018)Wang, Xia, Wang, and Qian]
  - Level set-based topology optimization [Cai et al.(2014)Cai, Zhang, Zhu, and Gao]

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Thank you for your attention!

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This note may contain errors because of my limited knowledge about related topics. Please feel free to contact me if you find any mistakes/errors in it. Thank you.

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