Bounded Distortion Mapping and Shape Deformation

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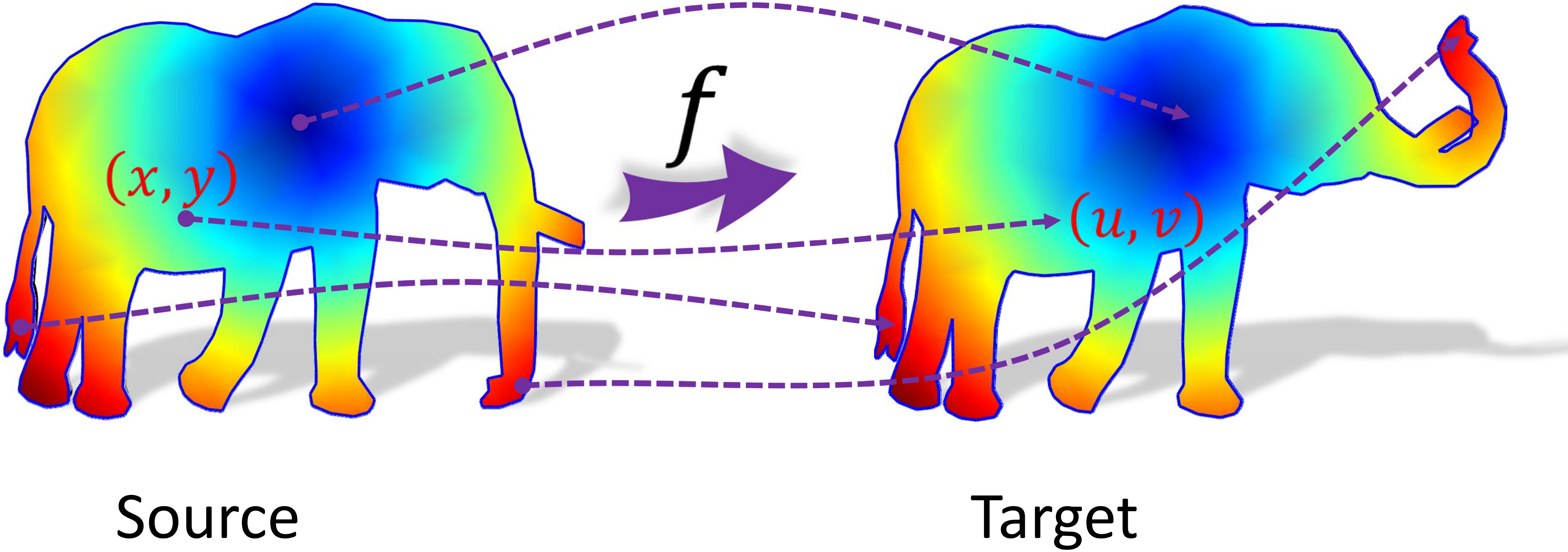
Outline

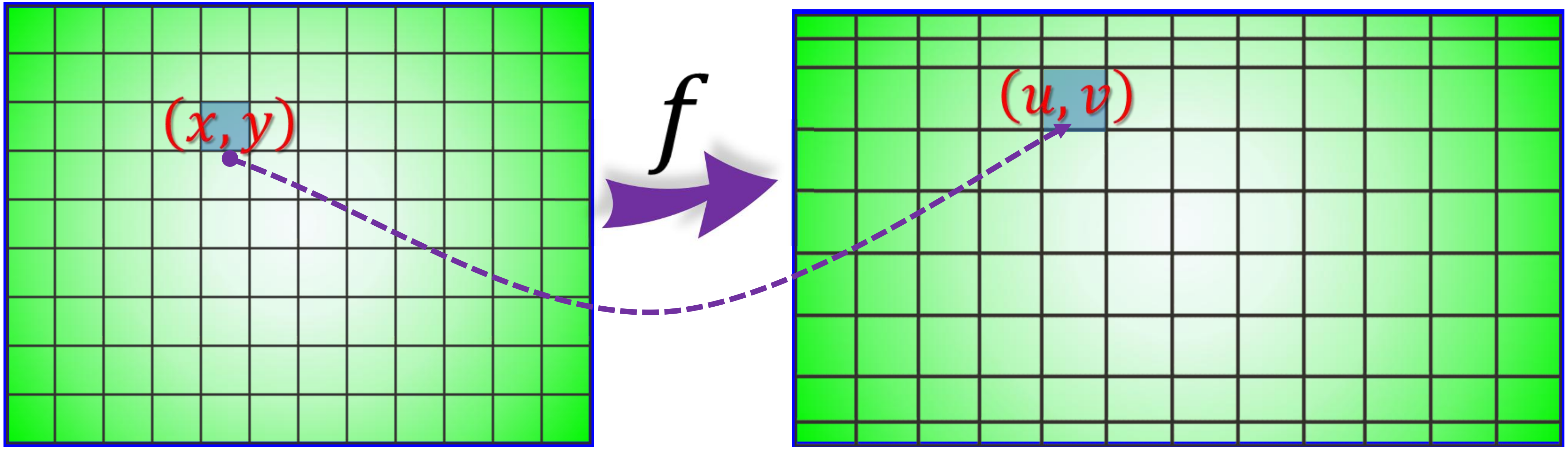
- Planar Mapping & Applications
- Bounded Distortion Mapping
- Harmonic Shape Deformation
- Shape Interpolation

Mapping – between planar shapes



$$f: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

Mapping – between images



Applications – keyframe animations

- Model key poses/frames

Shape deformation

- Fill in between key poses

Shape interpolation

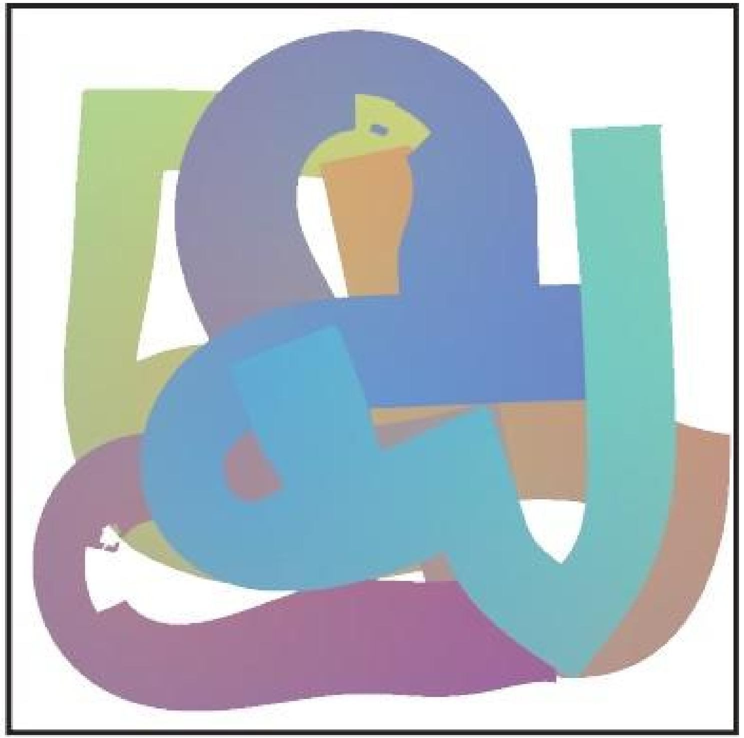
Applications – image editing

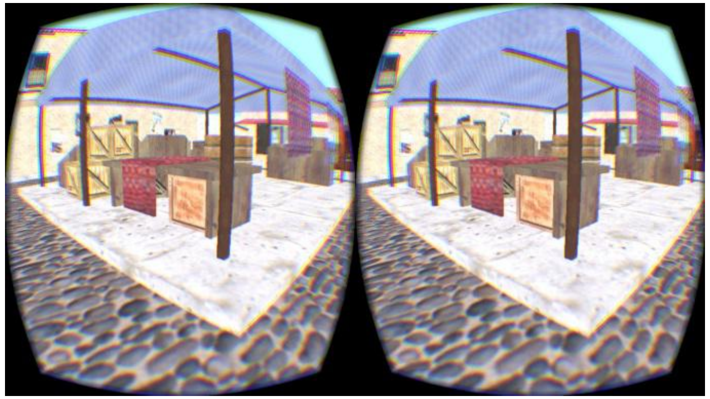
Aesthetic optimization

Content-aware resizing

Re-photography

Applications - virtual reality





[Sun et al. 2016]
[Dong et al. 2017]

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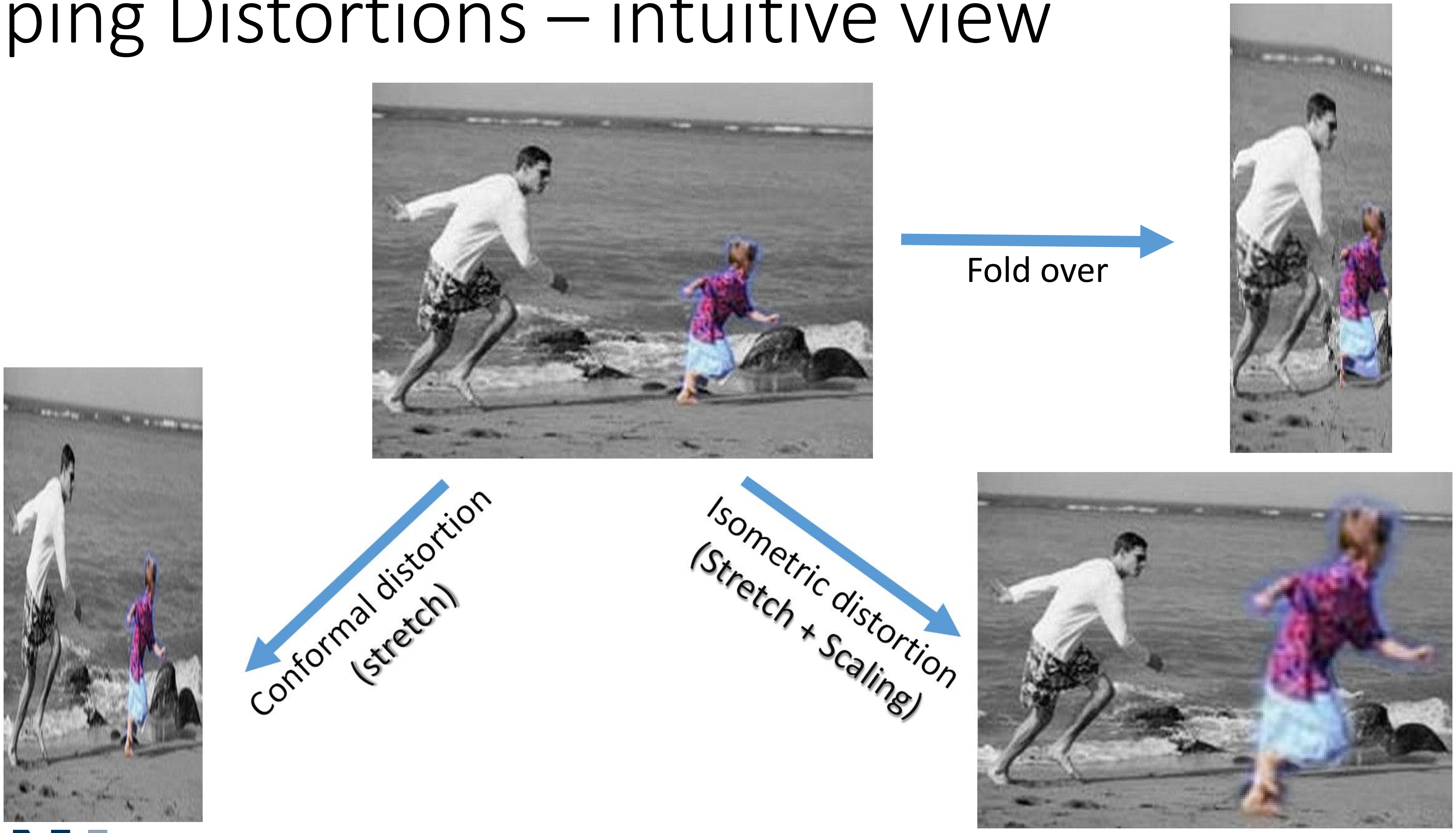
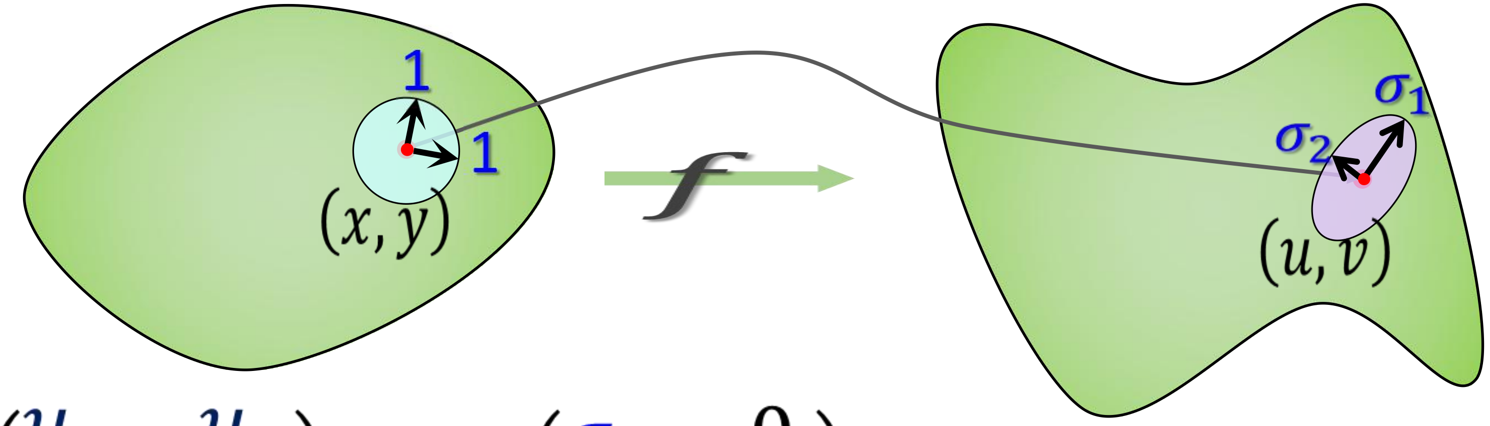
Mapping Distortions – intuitive view



Planar map – notations

$$f: \mathbb{R}^2 \mapsto \mathbb{R}^2$$
$$f(x, y) = (u, v)$$



$$J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = U \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} V \quad \sigma_1 \geq \sigma_2 \geq 0$$

$$f(z) = f(x + iy) \qquad f: \mathbb{C} \to \mathbb{C}$$

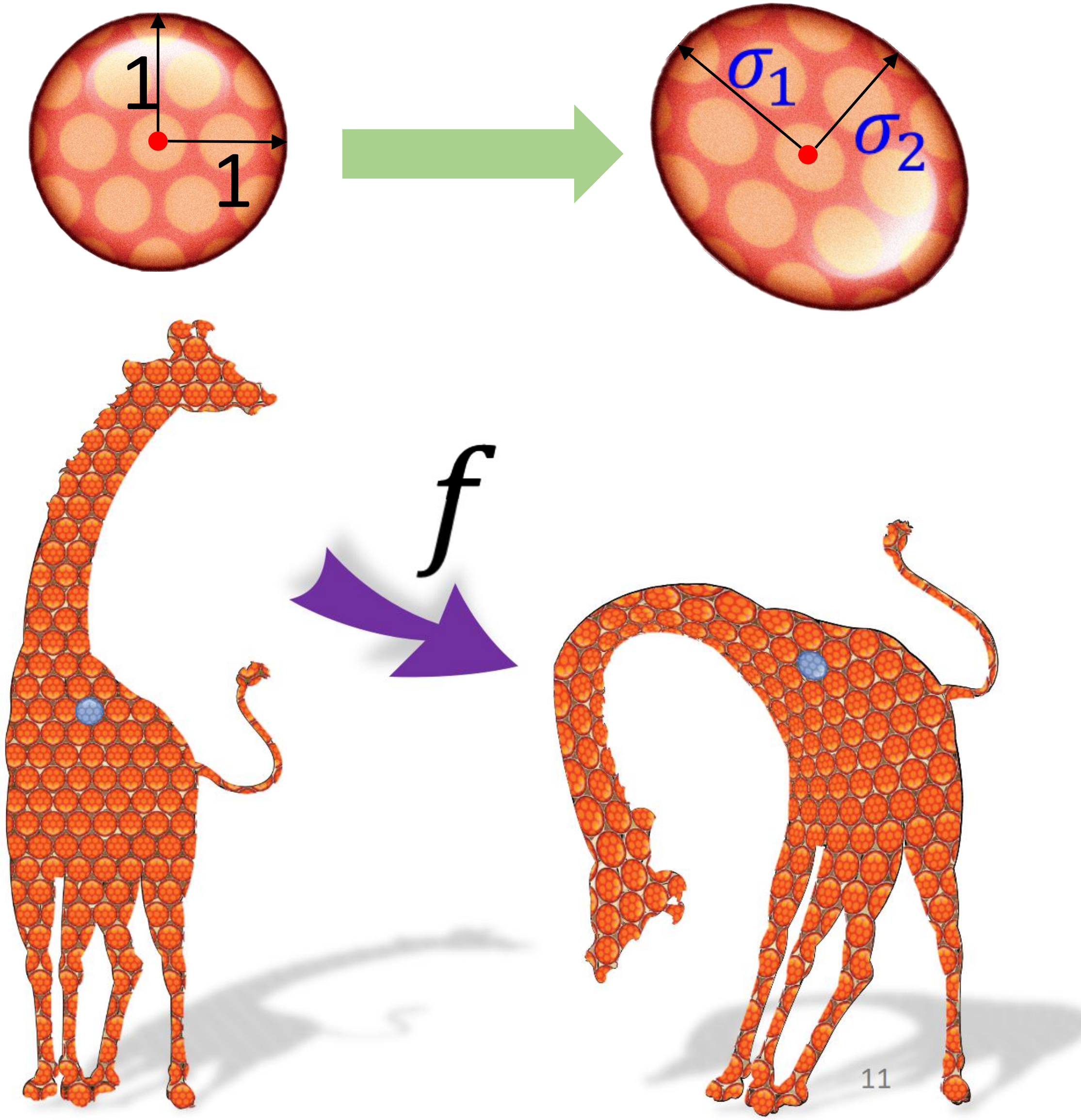
$$f_z = \frac{1}{2}\left(u_x + v_y + i(v_x - u_y)\right)$$
$$f_{\bar{z}} = \frac{1}{2}\left(u_x - v_y + i(v_x + u_y)\right)$$

$$\sigma_1 = |f_z| + |f_{\bar{z}}|$$
$$\sigma_2 = \big||f_z| - |f_{\bar{z}}|\big|$$

Mapping Distortions – formal definitions

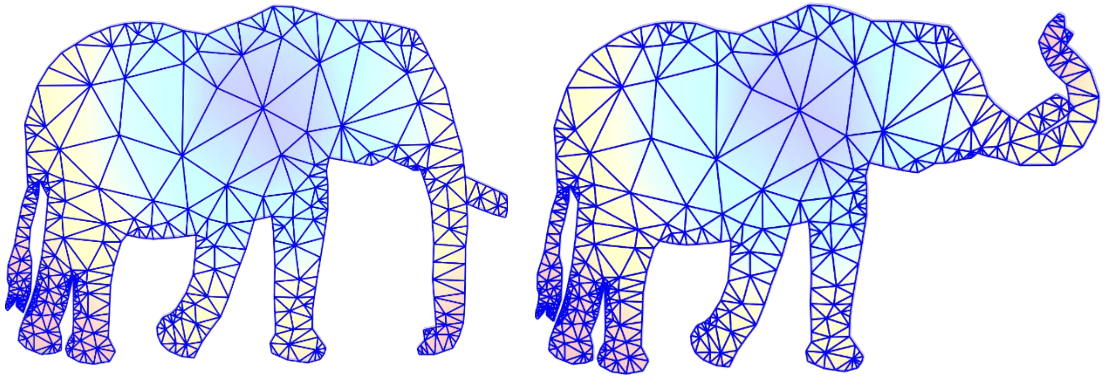
- Conformal
  - $|\sigma_1 - \sigma_2|$
- Isometric
  - $|\sigma_1 - 1| + |\sigma_2 - 1|$
- $D(\sigma_1, \sigma_2)$

$$D(f) = \iint_\Omega D^f(x)dx \searrow 0$$

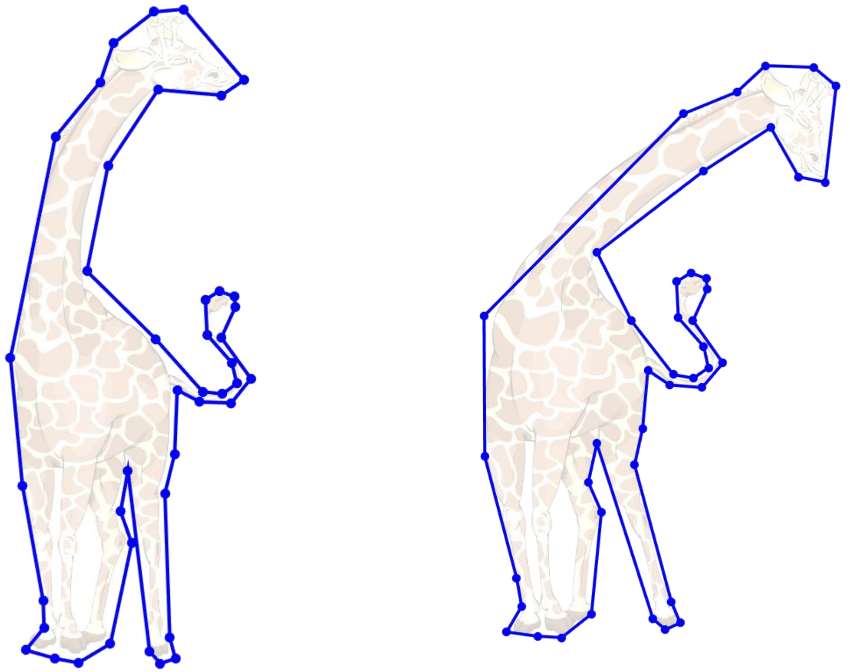


Mapping – discretization

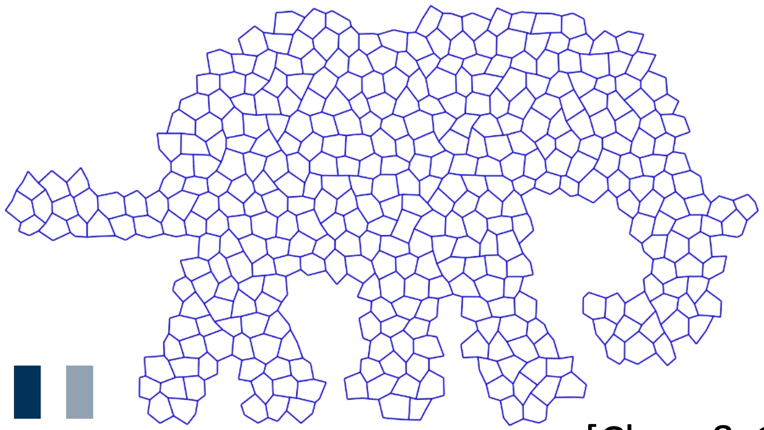
- Triangle mesh
  - Piecewise linear mapping



- Polygonal cage
  - Smooth barycentric mapping

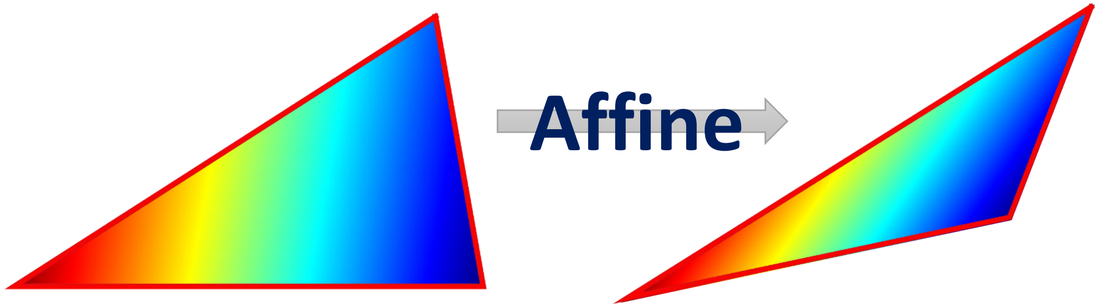


- Polygonal mesh + barycentric mapping



[Chen & Gotsman 2017]

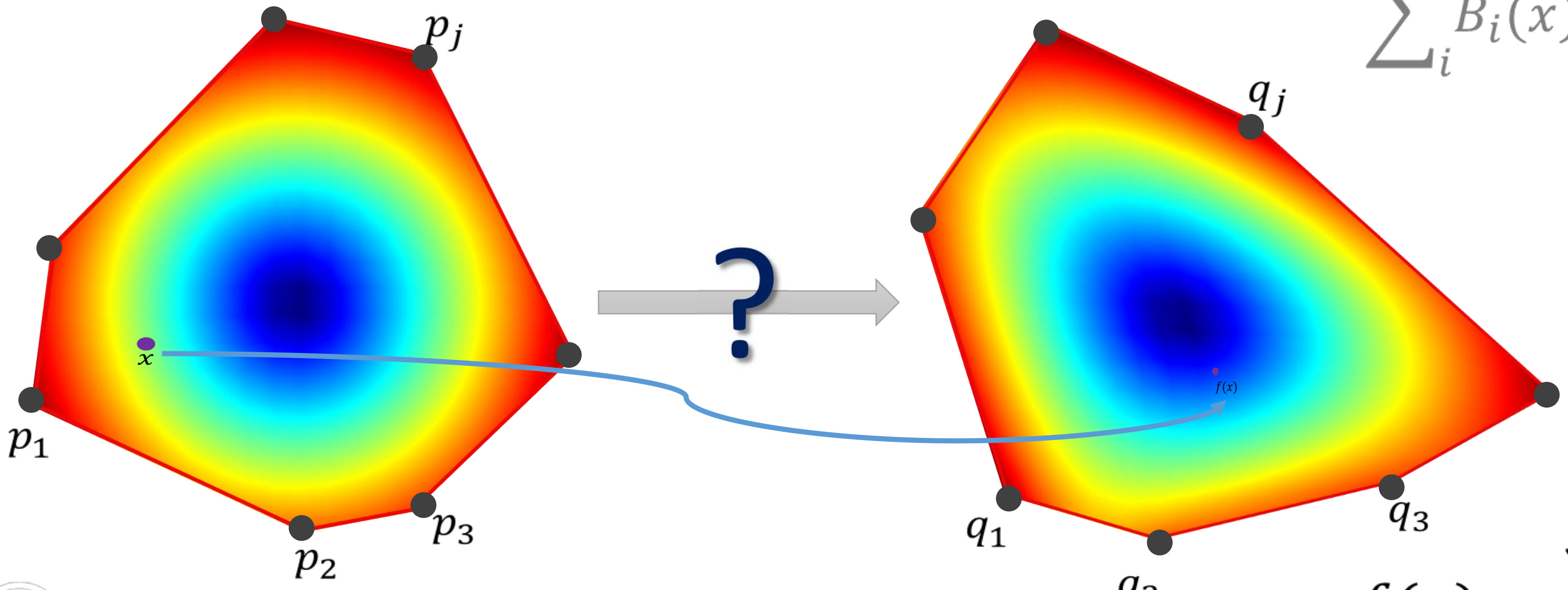
Barycentric Mapping a polygon



Barycentric Coordinates

$$x = \sum_i B_i(x) p_i$$

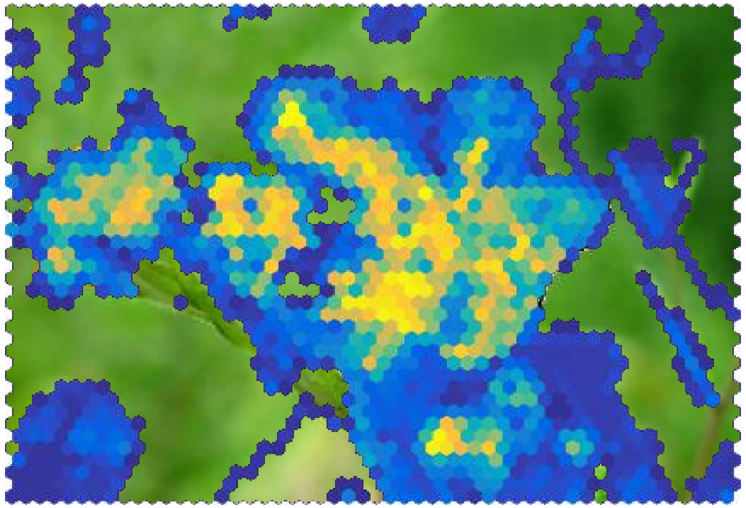
$$\sum_i B_i(x) = 1$$



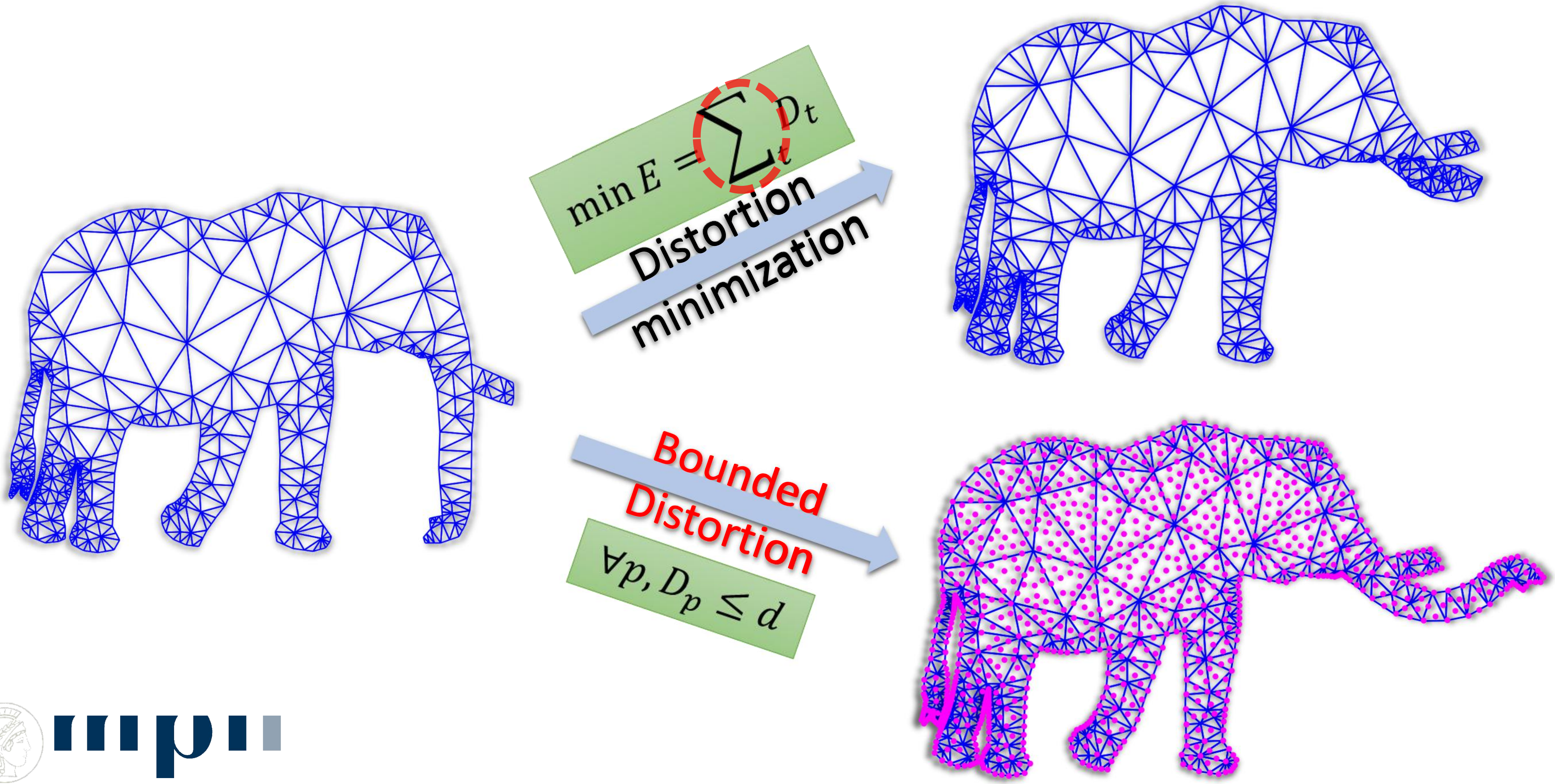
- Wachspress
- Cot/Laplace
- Mean-value
- *Harmonic*
- MLS
- …

$$f(x) = \sum_i B_i(x) q_i$$

Image resizing using hexmesh



Bounded Distortion Mapping



Outline

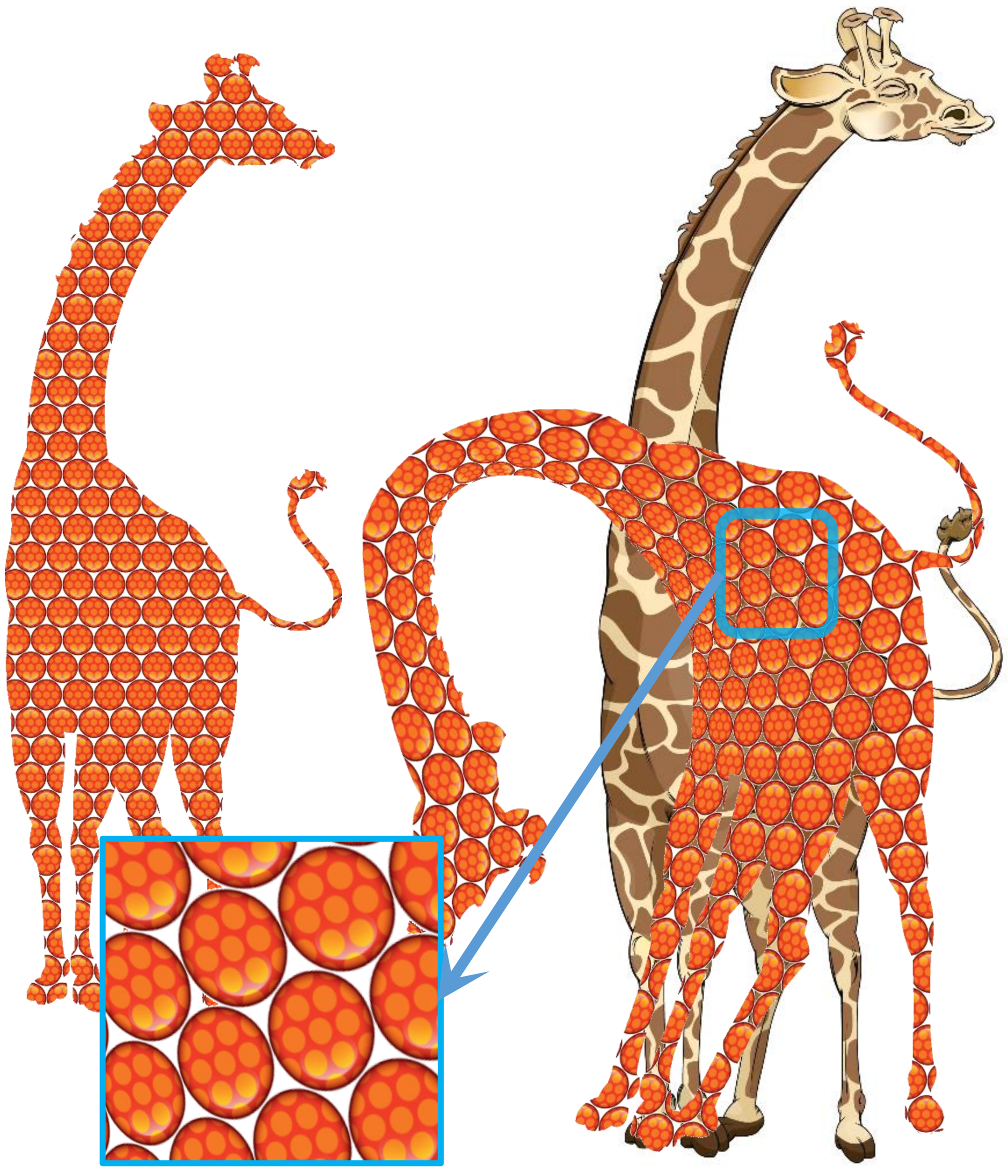
- Planar Mapping & Applications
- Bounded Distortion Mapping
- <u>Harmonic Shape Deformation</u>
- Shape Interpolation

Deformation – desirable properties

✓Intuitive user-interface
 ✓Drag and drop

✓Fast computation
 ✓Interactive

✓High quality
 ✓Smooth (C^{∞})
 ✓Locally injective (no foldovers)
 ✓Bounded conformal distortion
 ✓Bounded isometric distortion



Deformation – previous work

- Mesh-based
  - **Extremal quasiconformal** maps [Weber *et al.* 2012]
  - **Bounded distortion** mapping spaces [Lipman 2012]
  - **Locally injective** mappings [Schüller *et al.* 2013]
  - **Locally injective** parameterization [Weber & Zorin 2014]
  - Planar shape interpolation with **bounded distortion** [Chen *et al.* 2013]

P.W.L. → ✗ not smooth

- Meshless

✓ smooth

  - Generalized barycentric coordinates

✗ not locally injective
✗ no distortion bounds

  - Controllable **conformal** maps [Weber & Gotsman 2010]

✗ no positional constraints

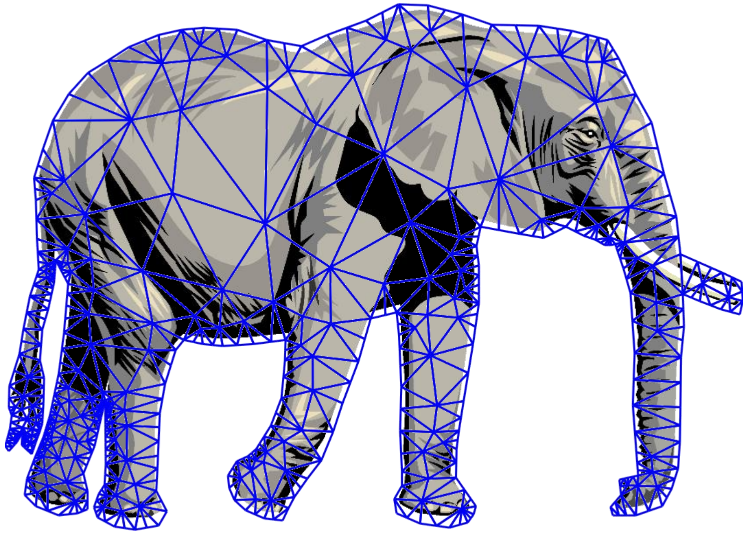
  - Provably good planar maps [Poranne & Lipman 2014]

R.B.F. → ✗ not shape aware

Mapping Space for Deformation

Piecewise linear map

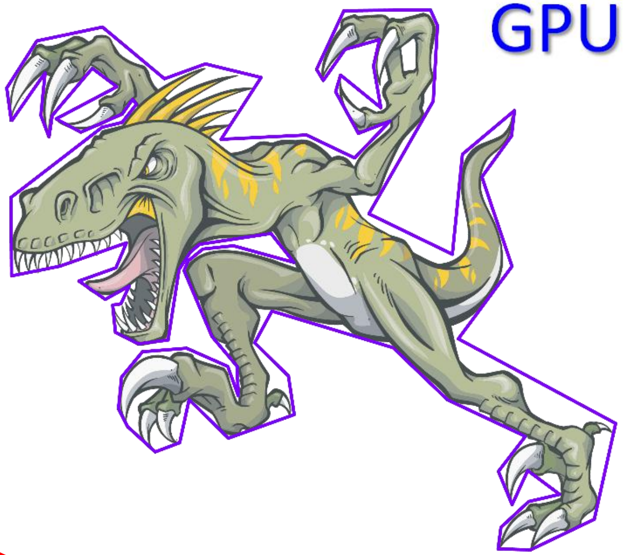
- Non-smooth
- Pointwise (facewise) constraints
- Sparse (**large**) linear algebra



Harmonic map

- Smooth
- Boundary constraints
- Dense (**small**) linear algebra

GPU friendly

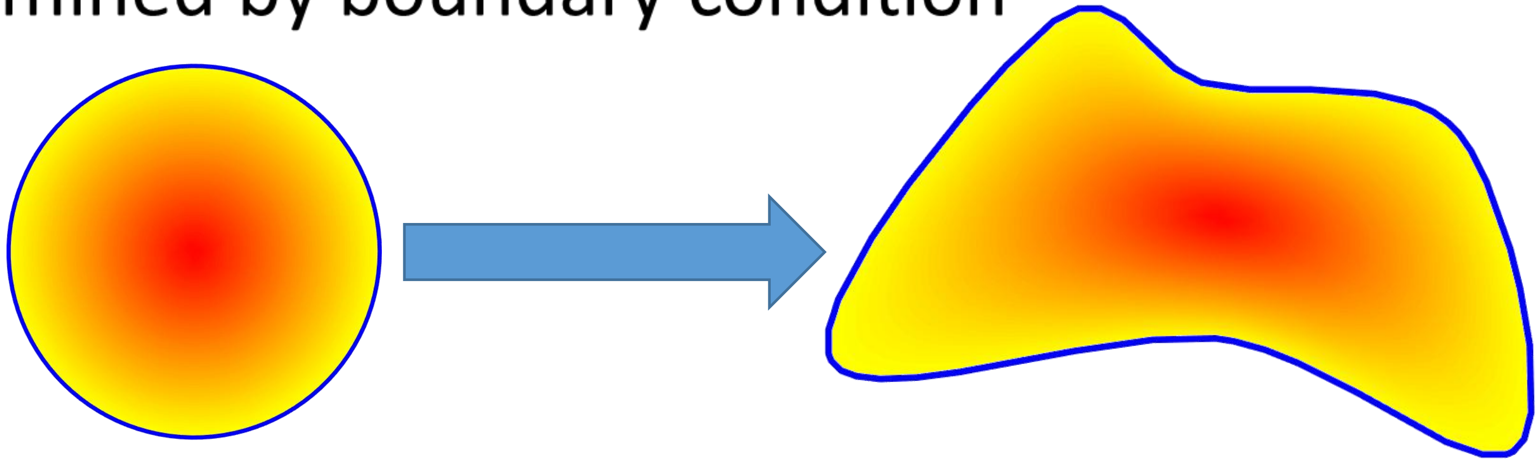


Harmonic Planar Mapping

$$f(x,y) = \big(u(x,y), v(x,y)\big) \qquad f\colon \Omega \to \mathbb{R}^2$$

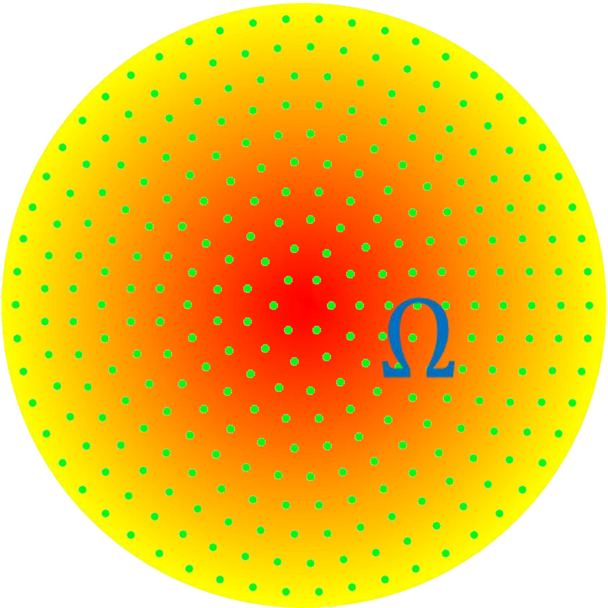
$$\Delta u = 0, \qquad \Delta v = 0$$

- C^∞ Smooth
- Maximum/minimum principle
- Uniquely determined by boundary condition



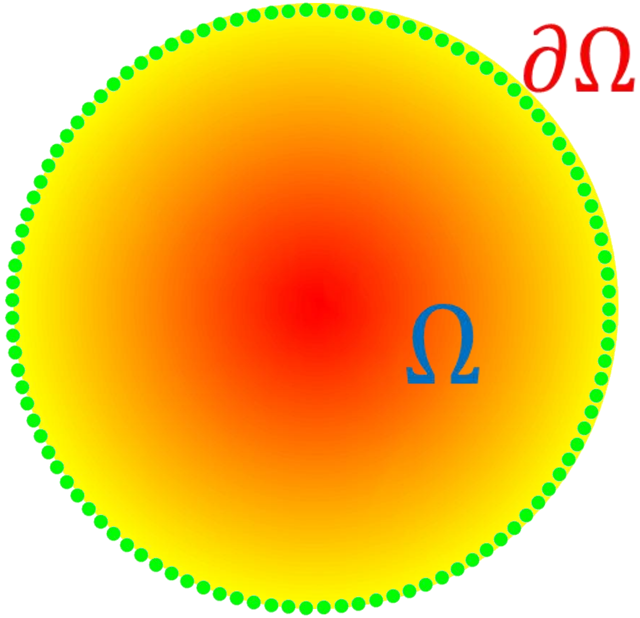
Bounded Distortion Harmonic Mapping

- Bound the distortion at every point

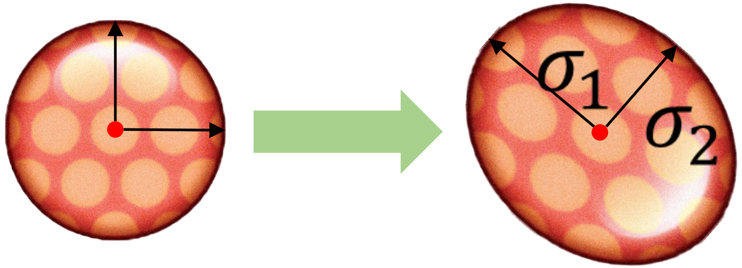


$\forall z \in \Omega,$ $\sigma_1(z) \leq \boldsymbol{\sigma_1}$
$\sigma_2(z) \geq \boldsymbol{\sigma_2}$

- Harmonic - Boundary only?



$\forall w \in \partial\Omega,$ $\sigma_1(w) \leq \boldsymbol{\sigma_1}$
$\sigma_2(w) \geq \boldsymbol{\sigma_2}$



Bounded Distortion Theorem

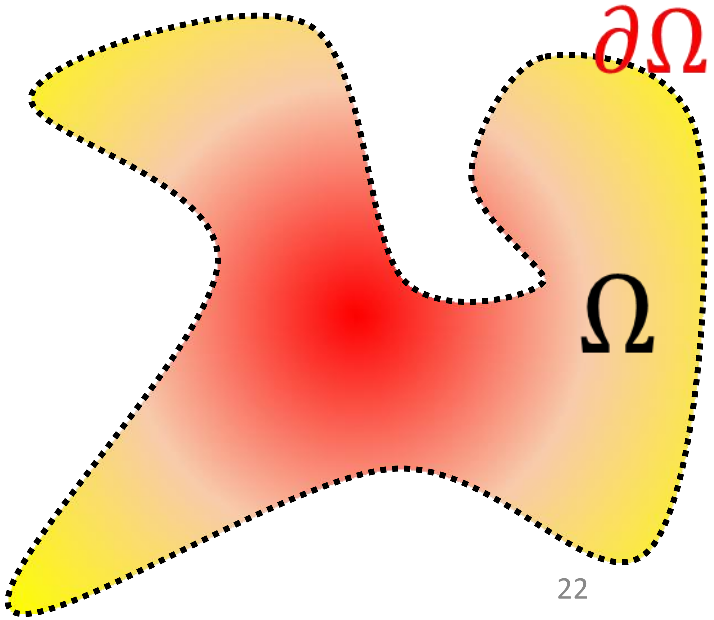
[Chen & Weber 2015]

A harmonic map f in a simply-connected domain Ω, is *locally injective* and has bounded distortion $(\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2})$ iff

$$\sigma_1^f(z) \leq \boldsymbol{\sigma_1} \qquad \forall z \in \Omega$$
$$\sigma_2^f(z) \geq \boldsymbol{\sigma_2}$$

$$\oint_{\partial\Omega} \frac{f_z'(z)}{f_z(z)} dz = 0$$

$$\sigma_1^f(w) \leq \boldsymbol{\sigma_1} \qquad \forall w \in \partial\Omega$$
$$\sigma_2^f(w) \geq \boldsymbol{\sigma_2}$$



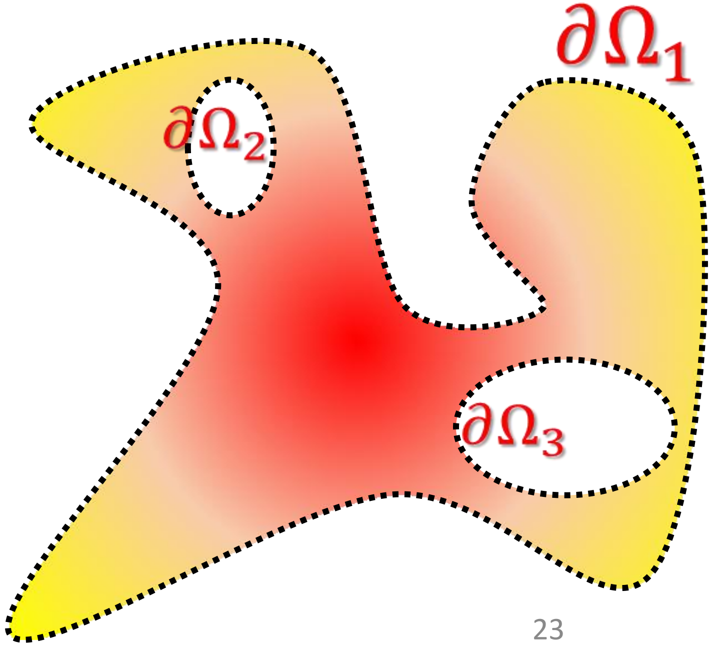
Bounded Distortion Theorem

[Chen & Weber 2017]

A harmonic map f in a multiply-connected domain Ω, is *locally injective* and has bounded distortion $(\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2})$ iff

$$\sum_i \oint_{\partial\Omega_i} \frac{f_z'(z)}{f_z(z)} dz = 0$$

$$\begin{aligned} \sigma_1^f(w) &\leq \boldsymbol{\sigma_1} \\ \sigma_2^f(w) &\geq \boldsymbol{\sigma_2} \end{aligned} \quad \forall w \in \bigcup \partial\Omega_i$$



Harmonic Shape Deformation

- Input:
  - User prescribed bounds $(\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2})$
  - Positional constraints
    - $\{p_i \rightarrow q_i, i = 1 \dots n\}$
- Output
  - Locally injective harmonic mapping
  - Bounded distortion $(\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2})$

$$\underset{f}{\text{minimize}}\, E_{ARAP}(f) + \lambda E_{p2p}(f)$$

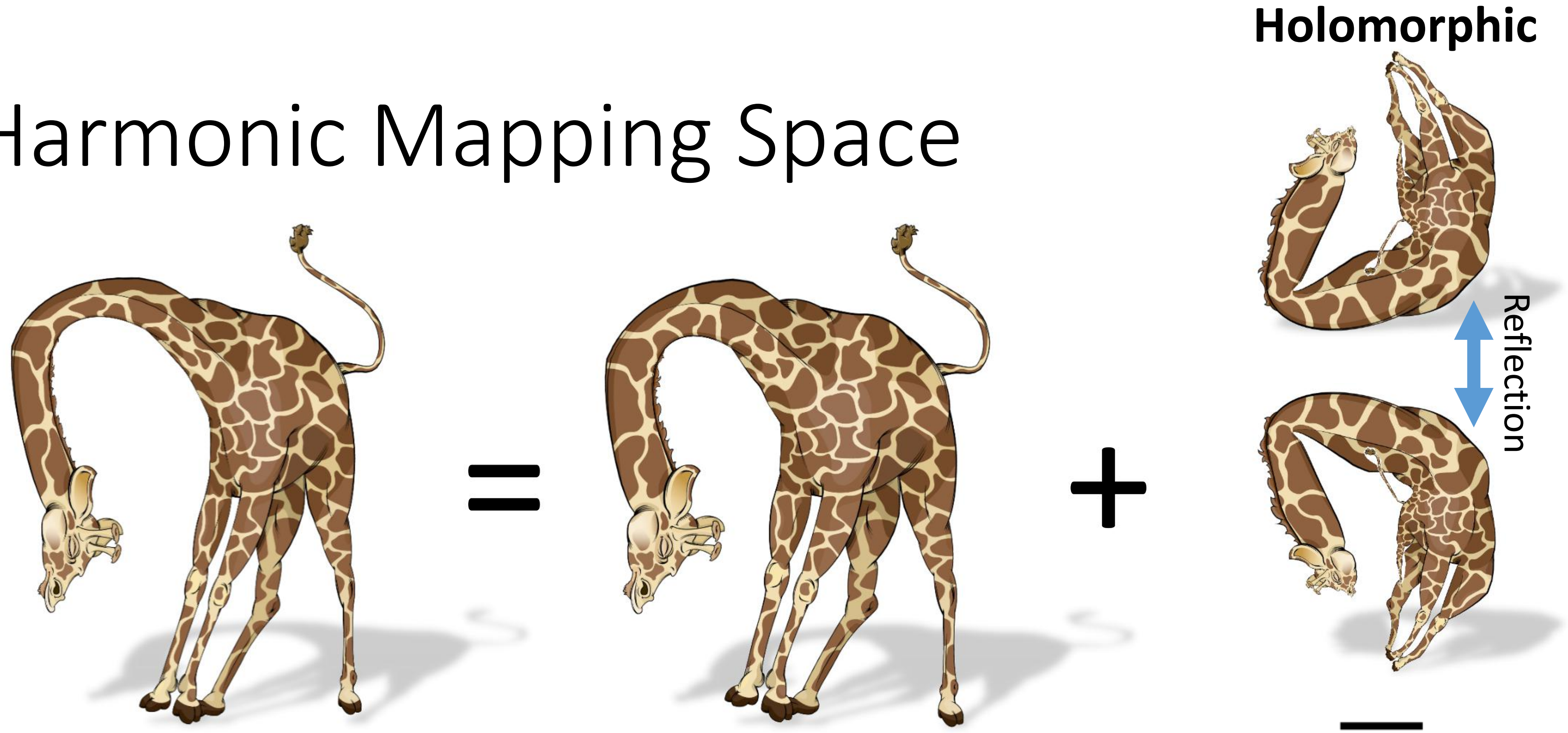
$$s.t. \quad f \; is \; harmonic$$

$$\oint_{\partial\Omega} \frac{f_z'(z)}{f_z(z)} dz = 0$$

$$\forall w \in \partial\Omega, \quad \begin{array}{l} \sigma_1(w) \le \boldsymbol{\sigma_1} \\ \sigma_2(w) \ge \boldsymbol{\sigma_2} \end{array}$$

Convexification [Lipman 2012]

Harmonic Mapping Space



$$f = \Phi + \bar{\Psi}$$

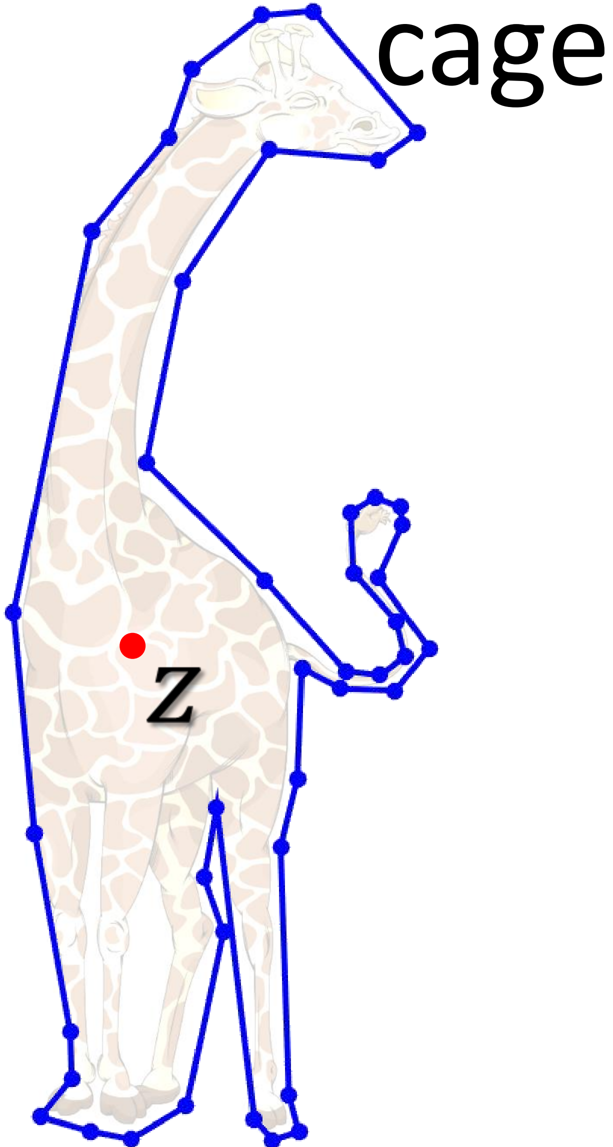
Harmonic

Holomorphic
(complex analytic)

Anti-
Holomorphic

Cauchy complex barycentric coordinates [Weber *et al*. 2009]

Cauchy Complex Barycentric Coordinate



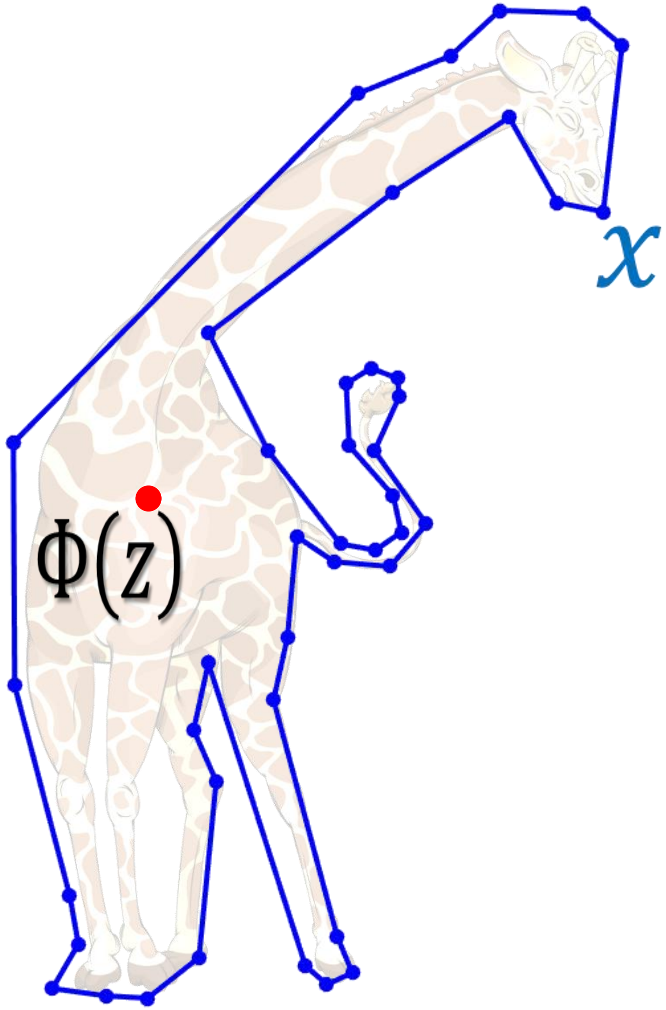
$$f = \Phi + \overline{\Psi} \leftrightarrow (x, y)$$

$$\Phi(z) = \sum_i C_i(z) x_i$$

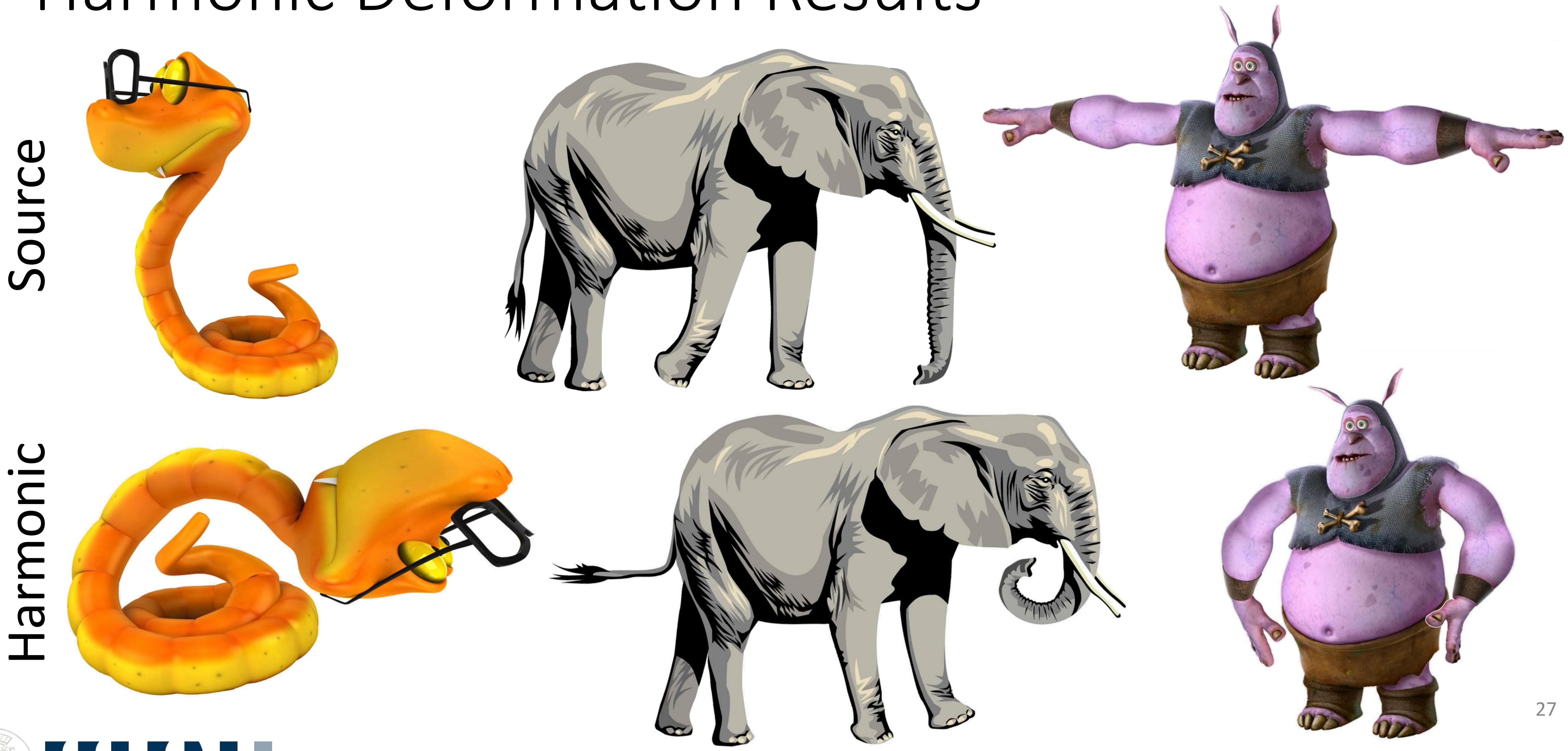
$$\Psi(z) = \sum_i C_i(z) y_i$$

$$f_z = \Phi'(z)$$

$$f_{\bar{z}} = \overline{\Psi'(z)}$$



Harmonic Deformation Results



An Interactive Session

Deformation – faster/better optimization?

[Chen & Weber 2015]

- Iterative convexification
  - Conic optimization
- User-specified bounds
  - Feasibility

$$
\begin{aligned}
&\underset{f}{\text{minimize}}\, \mathrm{E}_{\mathrm{ARAP}}(f) + \lambda \mathrm{E}_{\mathrm{p2p}}(f) \\
&s.t. \quad f \text{ is harmonic} \\
&\oint_{\partial\Omega} \frac{f_z'(z)}{f_z(z)} dz = 0 \\
&\forall w \in \partial\Omega, \quad \begin{array}{l} \sigma_1^f(w) \le \boldsymbol{\sigma_1} \\ \sigma_2^f(w) \ge \boldsymbol{\sigma_2} \end{array}
\end{aligned}
$$

Convexification [Lipman 2012]

[Chen & Weber 2017]

- Newton's method
  - GPU acceleration
- Smooth isometric energy
  - Automatic distortion bounds
  - Unconstrained optimization

Newton's Method

$$\textcolor{red}{Obj}: \text{ minimize } E(x)$$

- Taylor series

$$E(x) = \frac{1}{2}\Delta' H \Delta + g\Delta + E(x_0) \ldots, \qquad \Delta = x - x_0$$

- Iterative update

$$H(x_n)\Delta = -g(x_n)$$
$$x_{n+1} = x_n + \Delta$$

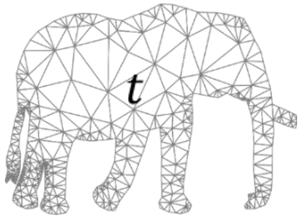
- Quadratic convergence

$$H^{+} = E\Lambda^{+}E^{T} > 0$$

$$H > 0 \Rightarrow E(x_{n+1}) < E(x_n)$$

- Per-element modification [Teran *et al*. 2005]

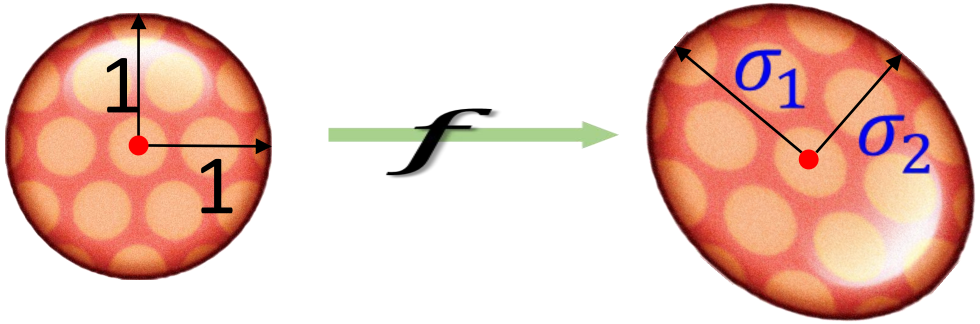
$$E(x) = \sum_t E_t(x) \Rightarrow H = \sum_t H_t \qquad H^{+} \sim \sum_t H_t^{+} > 0$$



Isometric Energy

$$E_{iso}(\sigma_1, \sigma_2)$$

$$\sigma_1 = |f_z| + |f_{\bar{z}}|, \qquad \sigma_2 = \big||f_z| - |f_{\bar{z}}|\big|$$



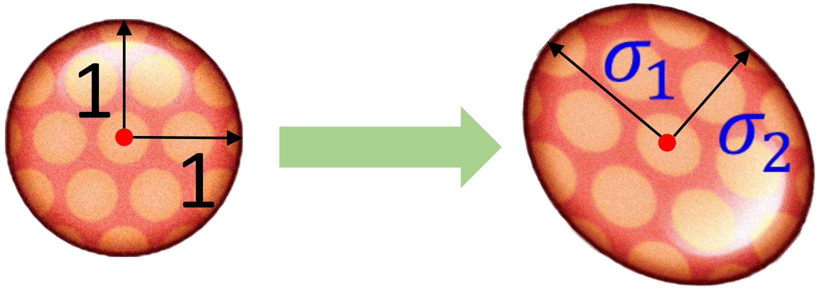
$\sigma_1 \geq \sigma_2 \geq 0$

- Capture Rigidity
  - $E_{iso}(1, 1) = 0$
- Barrier function
  - $E_{iso}(\sigma_1, 0) = \infty$
  - Local injectivity
- Smooth, differentiable

1. $E_{ARAP} = (\sigma_1 - 1)^2 + (\sigma_2 - 1)^2$ ✗ [Igarashi et al. 2007]
2. $\tau = \max\left(\sigma_1, \frac{1}{\sigma_2}\right)$ ✗ [Sorkine et al. 2002]
3. Symmetric Dirichlet $E_{iso} = \sigma_1^2 + \sigma_2^2 + \sigma_1^{-2} + \sigma_2^{-2}$ ✓ [Smith & Schafer 2015]
4. $E_{exp} = e^{sE_{iso}}$ ✓ [Rabinovich et al. 2017]
5. $E_{AMIPS} = e^{s\left(\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}\right) + \sigma_1\sigma_2 + \frac{1}{\sigma_1\sigma_2}}$ ✓ [Fu et al. 2015]

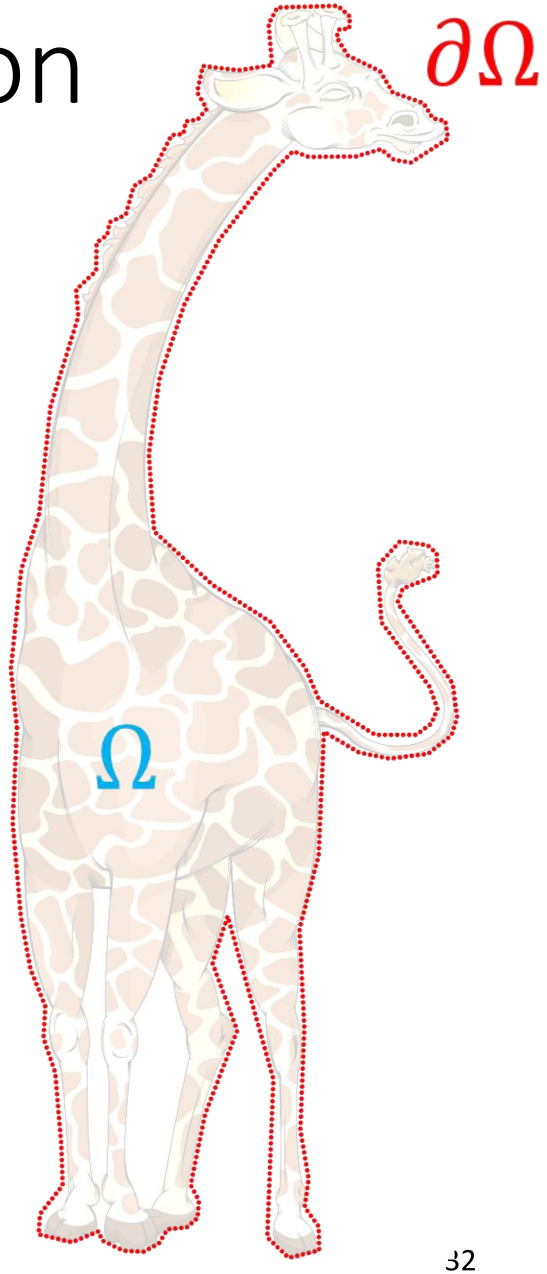
Locally Injective Harmonic Deformation

$$E_{iso}(\sigma_1, \sigma_2) = \sigma_1^2 + \sigma_2^2 + \sigma_1^{-2} + \sigma_2^{-2}$$



$$\text{minimize } E_{iso}^{f} = \int_{\partial\Omega} E_{iso}$$

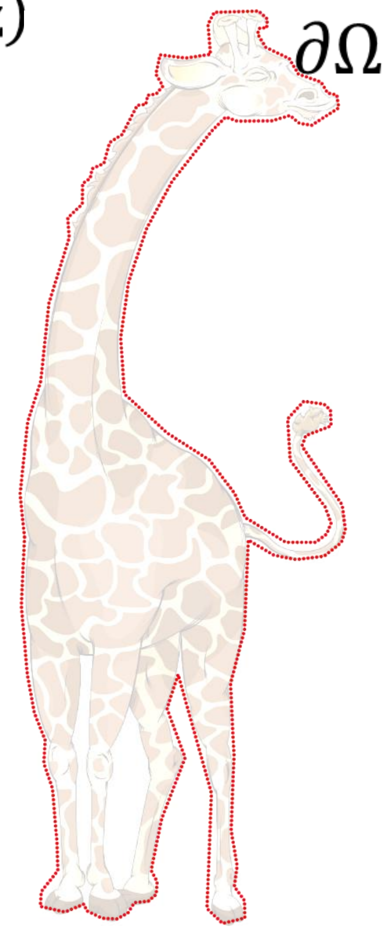
$$s.t.\ J(z) > 0, \forall z \in \partial\Omega$$



Locally Injective Harmonic Deformation

$$\text{minimize}\ \mathrm{E}^{f}_{\mathrm{iso}} + \lambda \mathrm{E}_{\mathrm{P2P}}$$
$$s.t.\ J(z) > 0, \forall z \in \partial\Omega$$

$$\mathrm{E}^{f}_{\mathrm{iso}} = \int_{\partial\Omega} \mathrm{E}_{\mathrm{iso}}(\mathrm{z})$$



- Newton's method
  1. $g = \nabla E,\ H = \nabla^2 E,\ \ H^{+} \rightarrow H$
  2. $H\Delta = -g$
  3. $x \leftarrow x + t\Delta$

- Local injective line search
  $J(z) = \sigma_1 \sigma_2 \rightarrow 0^{+} \Rightarrow \mathrm{E}_{\mathrm{iso}}(z) \rightarrow \infty \quad \Rightarrow \exists t > 0, s.t. J(z) > 0,$

Per-element SPD Hessian

$$\underbrace{H^+}_{4n \times 4n} \sim \sum_{z \in \partial\Omega} H(z)^+$$

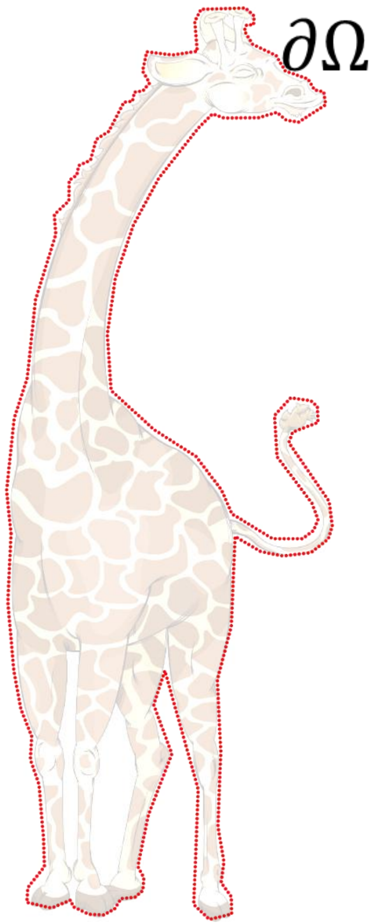
$$H(z)^+ = E\Lambda^+ E^T$$

$\mathrm{E_{iso}}(\sigma_1, \sigma_2)$

$$\Rightarrow \underbrace{H(z)}_{4n \times 4n} = \nabla^2 \mathrm{E_{iso}} = M^T \times \underbrace{K}_{4 \times 4} \times M \qquad M = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$$

$$MM^T = rI \quad \Rightarrow \quad H(z)^+ = M^T K^+ M$$

Analytical 4x4 SPD projection: $K^+ = E\Lambda^+ E^T$



Composite Majorization

[Shtengel et al. 2017]

- $e(x) = h\big(g(x)\big) = (h \circ g)(x)$
  - Convex-concave decomposition

$$h = h^+ + h^-$$
$$g = g^+ + g^-$$

- Convex majorizer

$$\bar{e} = \bar{h} \circ [g]$$
$$\nabla^2 \bar{e} \geq 0$$

$$H^+ = \left[\frac{\partial g}{\partial x}\right]^T \nabla^2 h^+ \left[\frac{\partial g}{\partial x}\right] + \left(\frac{\partial h}{\partial u}\right)_+ \nabla^2 g^+ + \left(\frac{\partial h}{\partial u}\right)_- \nabla^2 g^-$$

[Chen & Weber 2015]

$$H^+ = M^T K^+ M$$

Newton iteration on GPU

1. $g = \nabla \mathrm{E},\ H = \nabla^2 \mathrm{E},\ H^+ \to H$

$$g = D^T D(x \quad y)$$

$$H^+ = M^T K^+ M$$

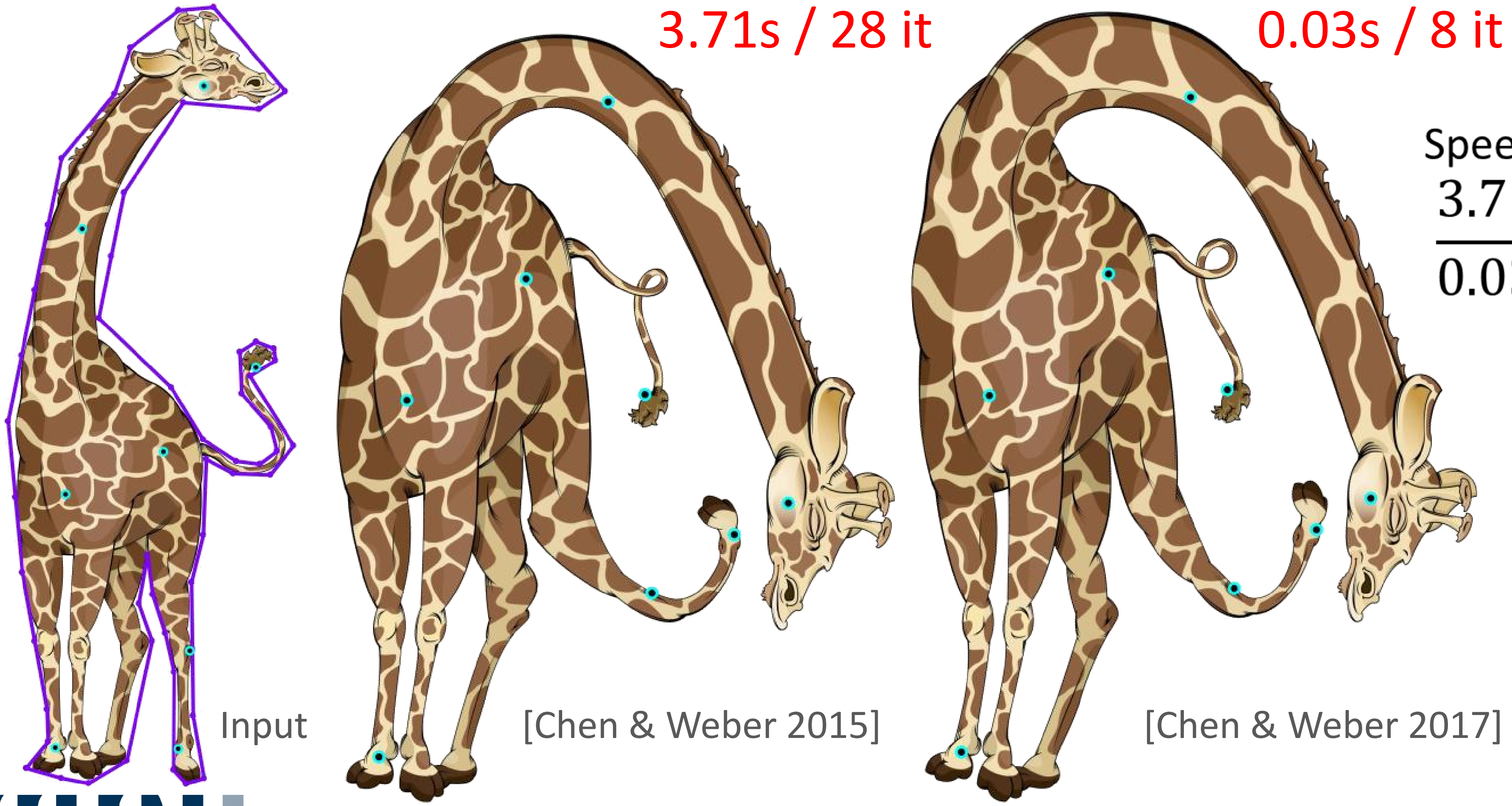
cuBLAS

2. $\Delta = -H^{-1} g$

cuSolver

3. $\begin{pmatrix} x \\ y \end{pmatrix} \leftarrow \begin{pmatrix} x \\ y \end{pmatrix} + t\Delta$

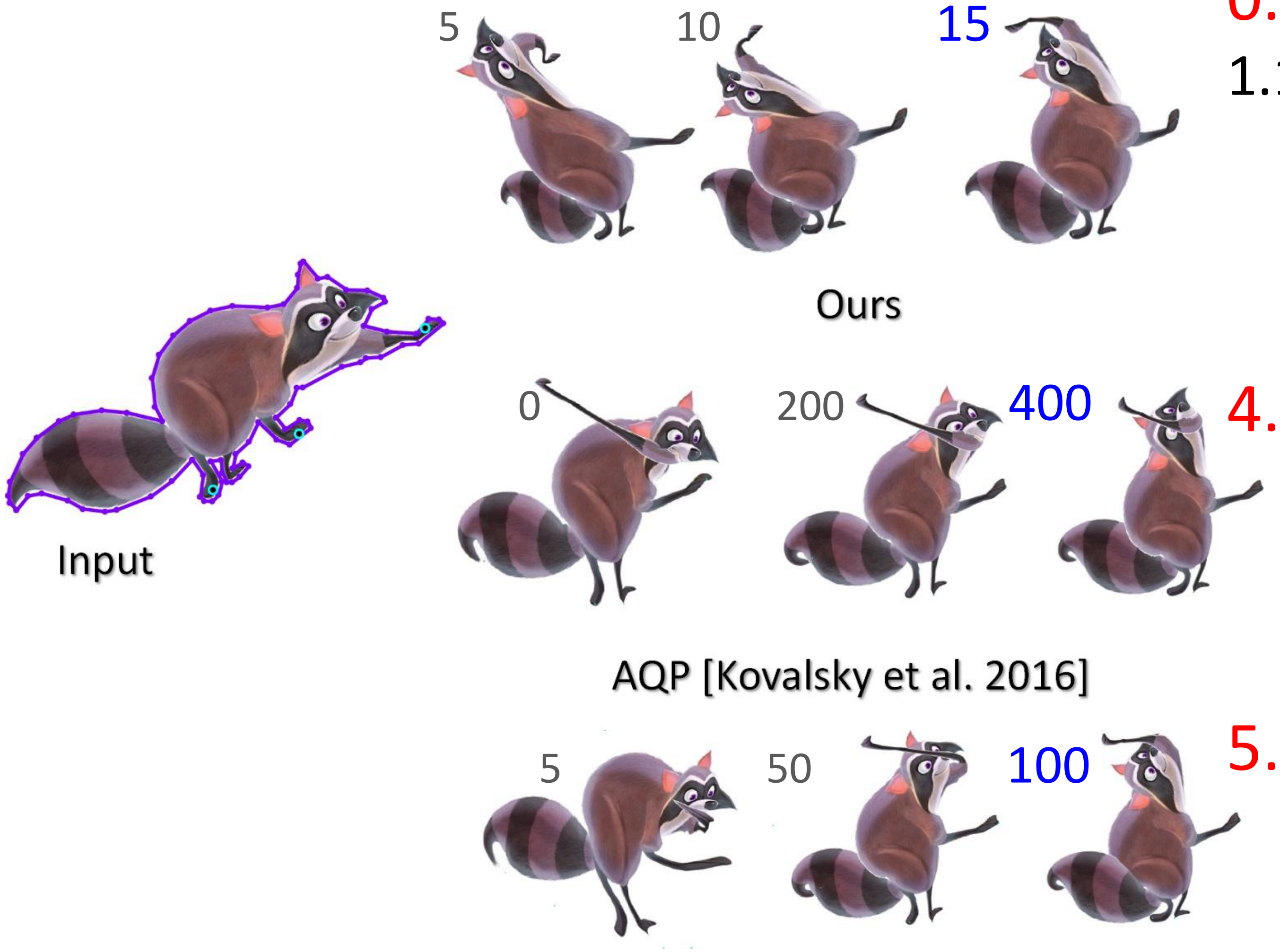
Results & Comparison



Speedup factor:

$$\frac{3.71}{0.03} = 125 \times$$

Results & Comparison



Ours

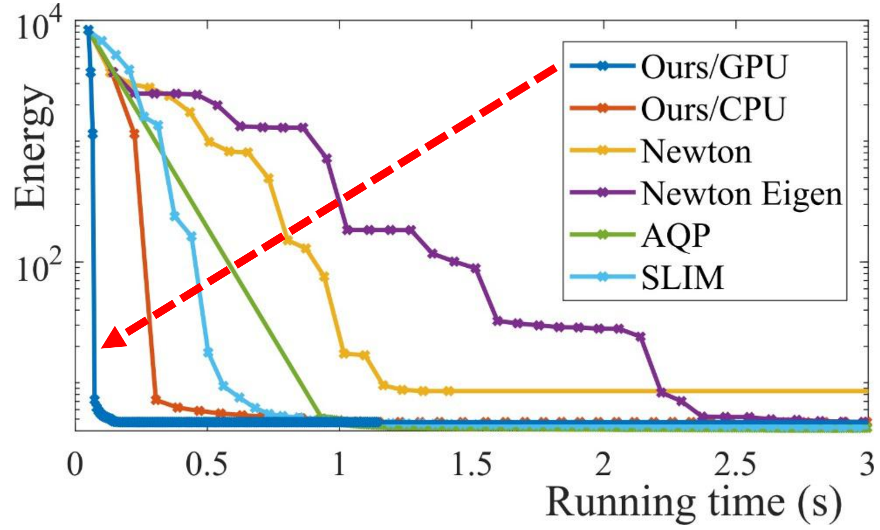
AQP [Kovalsky et al. 2016]

SLIM [Rabinovich et al. 2017]

0.11s GPU
1.18s CPU

4.32s

5.49s

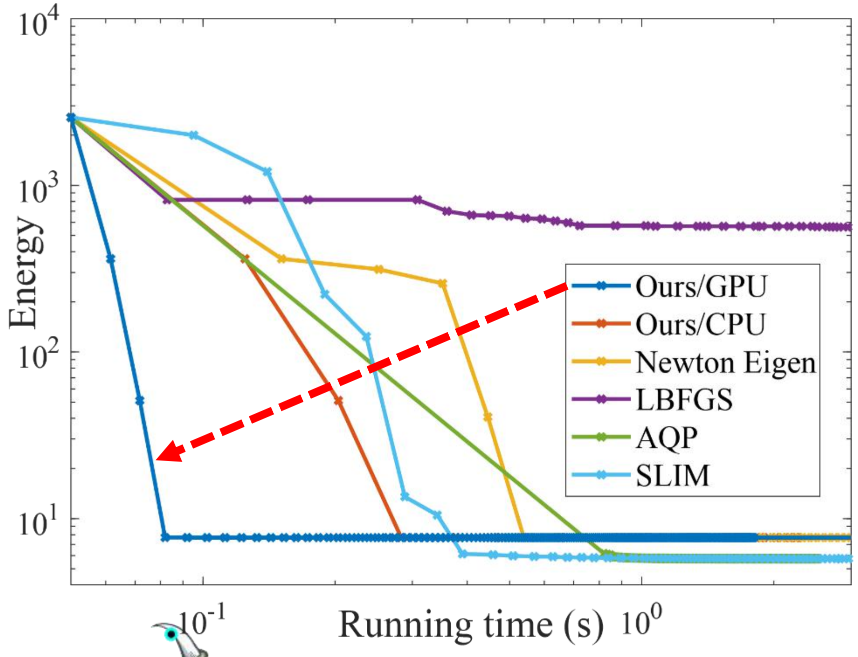


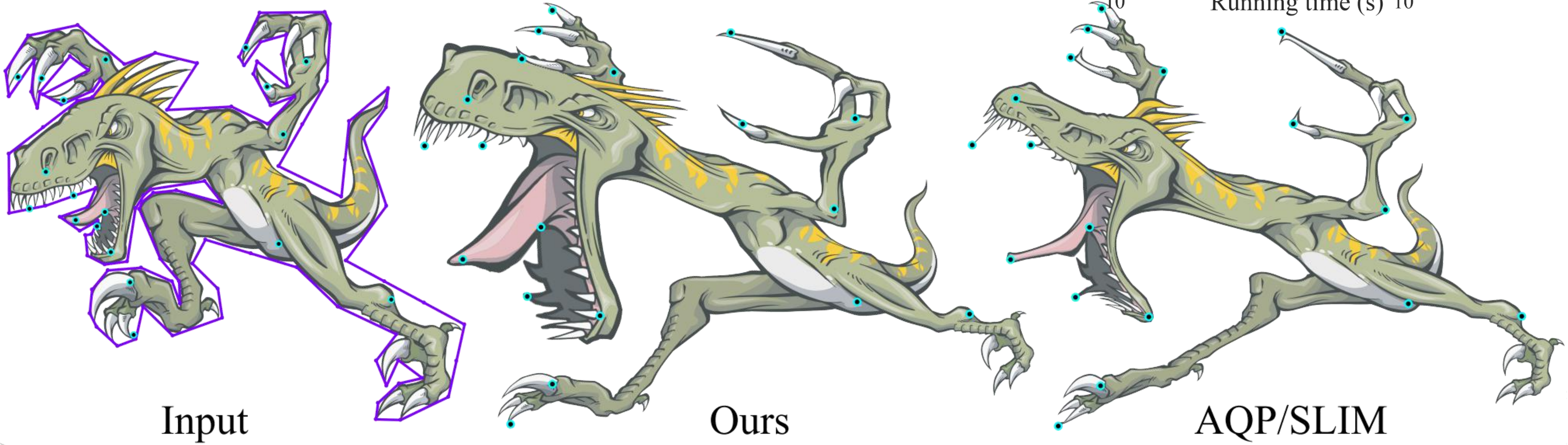
AQP

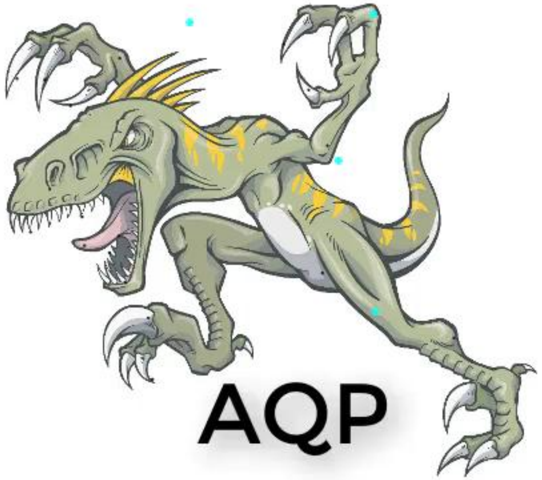
SLIM

Ours

Results & Comparison

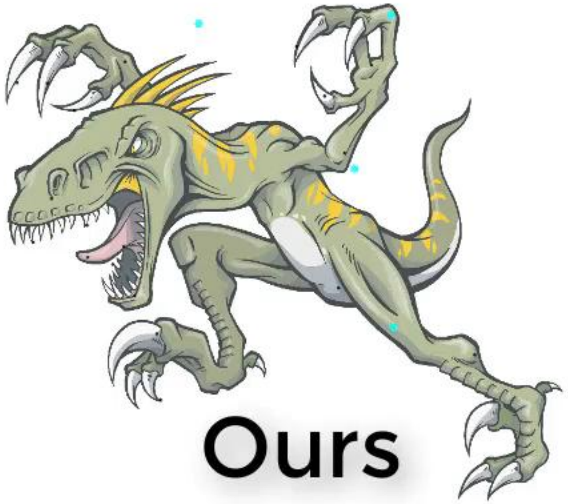






AQP

SLIM



Ours

SLIM

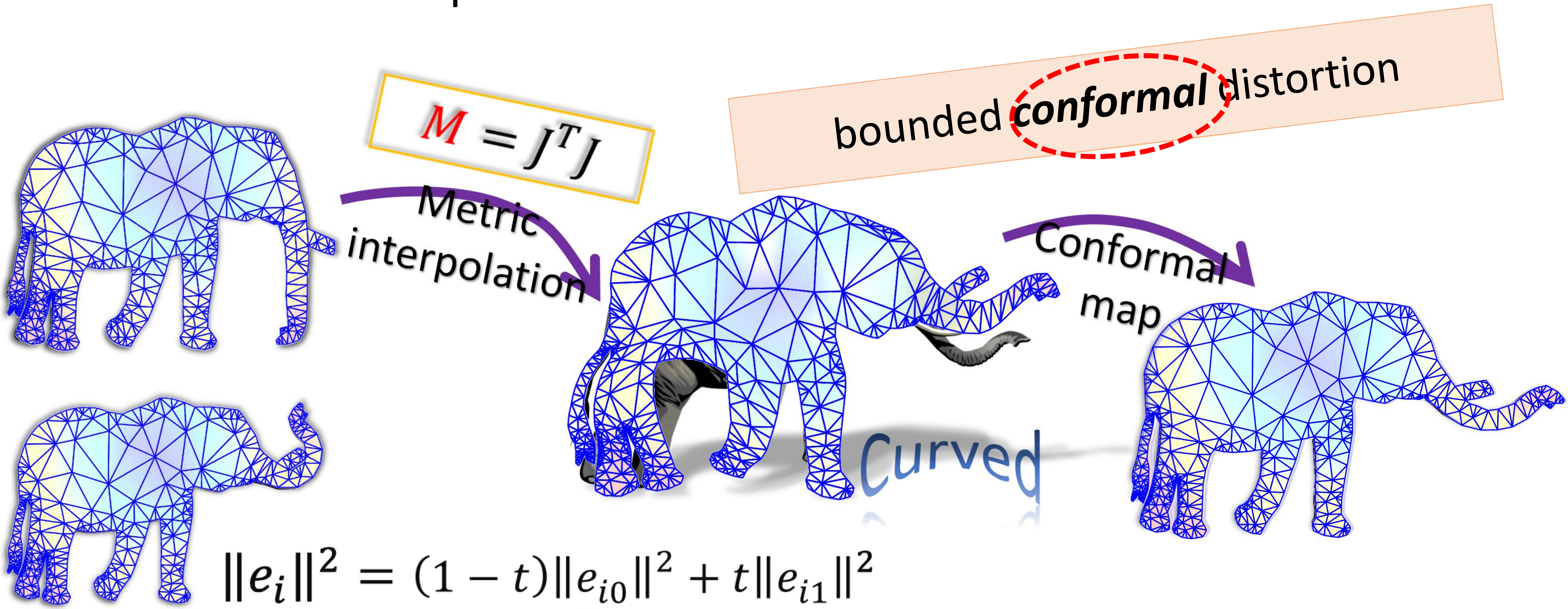
Ours

Outline

- Planar Mapping & Applications
- Bounded Distortion Mapping
- Harmonic Shape Deformation
- <u>Shape Interpolation</u>

Metric Interpolation

[Chen et al 2013]



$$\|e_i\|^2 = (1-t)\|e_{i0}\|^2 + t\|e_{i1}\|^2$$

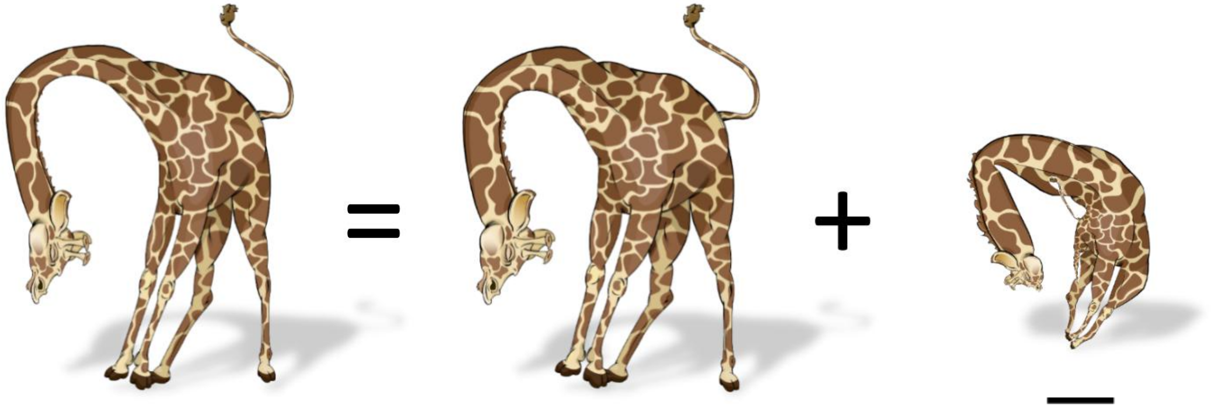
Square edge length

Harmonic Interpolation

[Chien et al 2016]

$f_0 = \Phi_0 + \overline{\Psi_0}$

$f_1 = \Phi_1 + \overline{\Psi_1}$



$$f = \Phi + \overline{\Psi}$$

- f: Harmonic
- Φ: Holomorphic
- $\overline{\Psi}$: Anti-Holomorphic

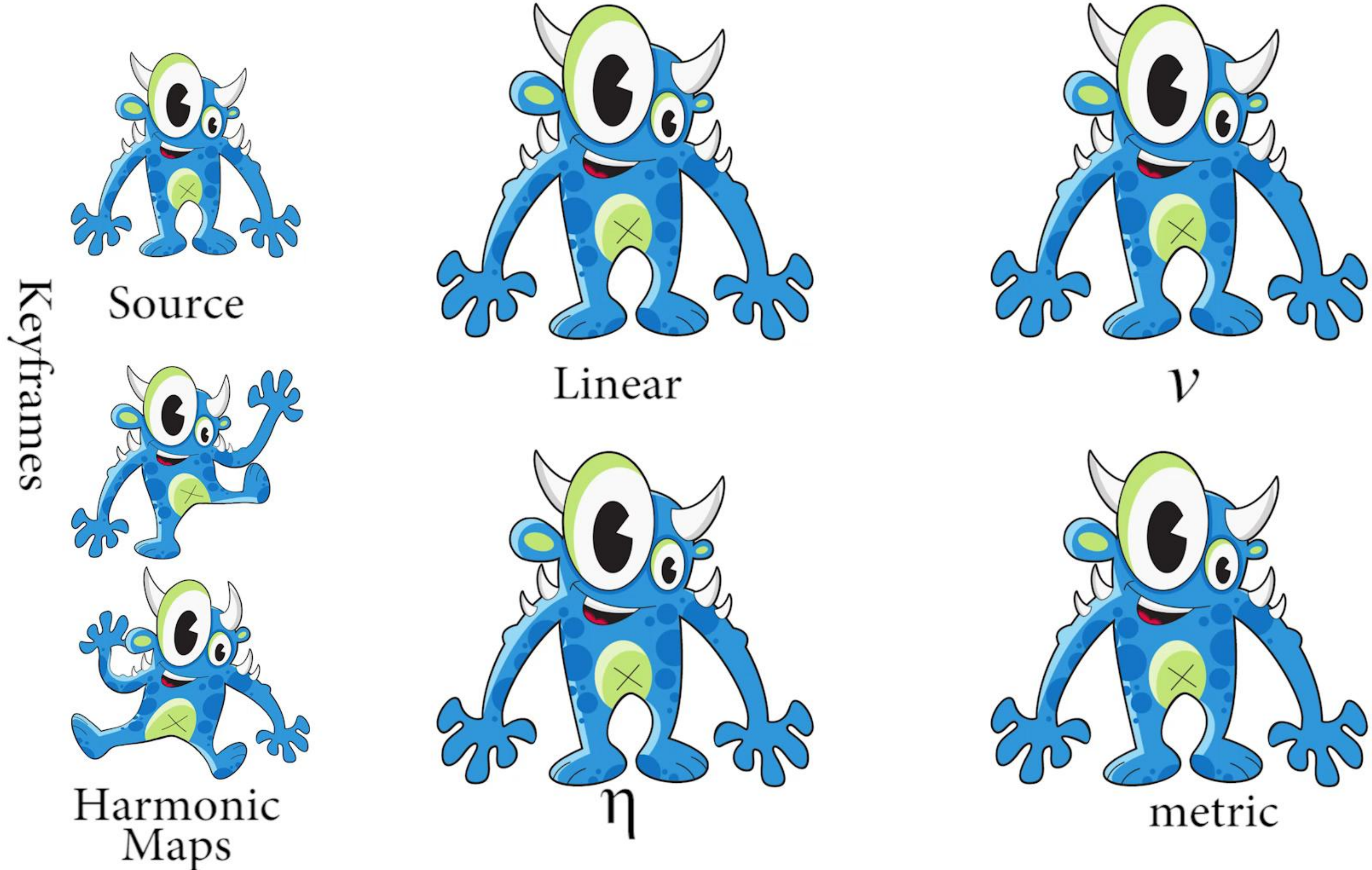
$$\log \Phi'(t) = (1-t)\log \Phi_0' + t \log \Phi_1'$$

- $\eta(t) = (1-t)\eta_0 + t\eta_1$ $\qquad \eta = \overline{\Psi'\Phi'}$
- $\nu(t) = (1-t)\nu_0 + t\nu_1$ $\qquad \nu = \Psi'/\Phi'$
- $M(t) = (1-t)M_0 + tM_1$

$f(t) = \Phi(t) + \overline{\Psi(t)}$

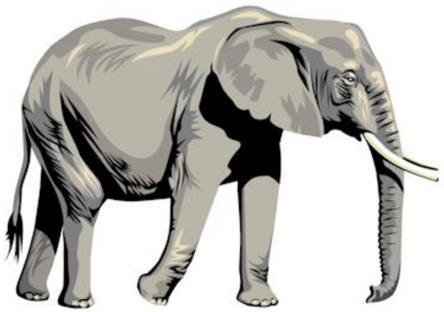
$$\sigma_1(t) \le \max\big(\sigma_1(0), \sigma_1(1)\big)$$
$$\sigma_2(t) \ge \min\big(\sigma_2(0), \sigma_2(1)\big)$$

Harmonic Interpolation



Interpolation comparison

Harmonic mapping animation



Conclusion

- Planar Mapping
    - Distortions
    - Discretization

- Harmonic Mapping
    - Bounded distortion theorem
    - Deformation
    - Interpolation

References

[1] R.Chen, O. Weber.
GPU-Accelerated Locally Injective Shape Deformation
SIGGRAPH Asia, 2017

[2] R. Chen, C. Gotsman.
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[3] E. Chien, R. Chen*, O. Weber.
Bounded Distortion Harmonic Shape Interpolation
SIGGRAPH 2016

[4] R. Chen, C. Gotsman.
Generalized As-Similar-As-Possible Warping with Applications in Digital Photography
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[5] R. Chen, O. Weber.
Bounded Distortion Harmonic Mappings in the Plane
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[6] R. Chen, O. Weber, D. Keren, M. Ben-Chen.
Planar Shape Interpolation with Bounded Distortion
SIGGRAPH 2013

Code available

http://people.mpi-inf.mpg.de/~chen/