Singularity-Constrained Octahedral Fields for Hexahedral Meshing

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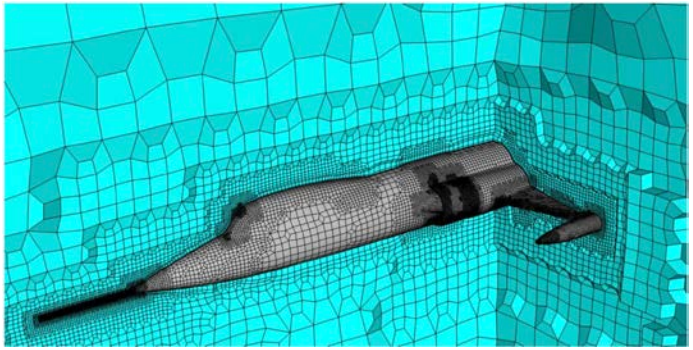
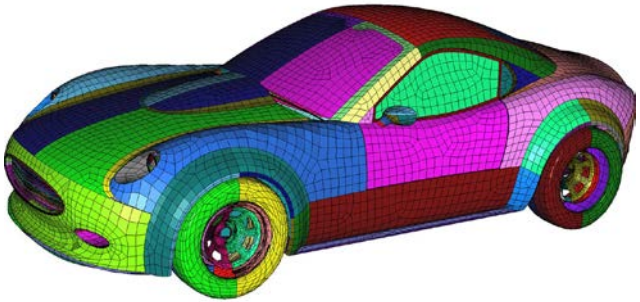
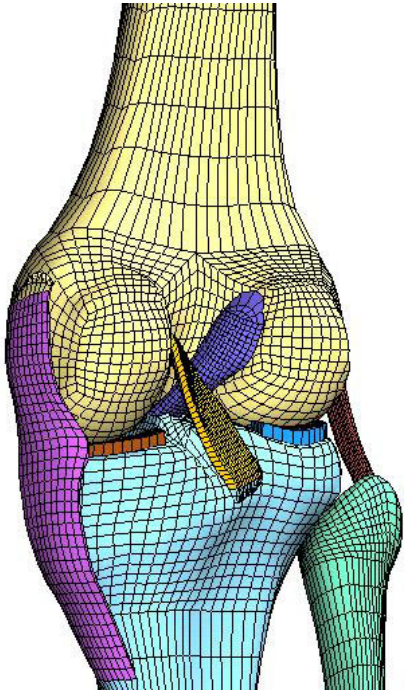
Why Hexahedral Meshes?



Aerospace [MeshGems]

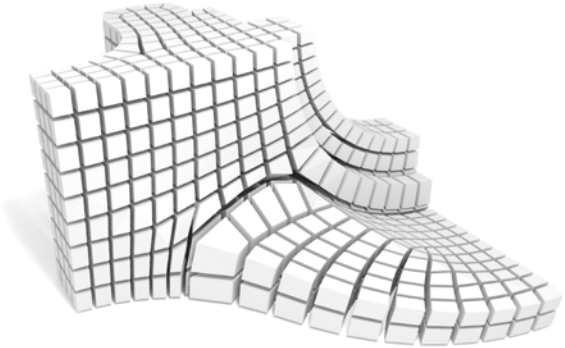


Automotive [CM2 MeshTools]



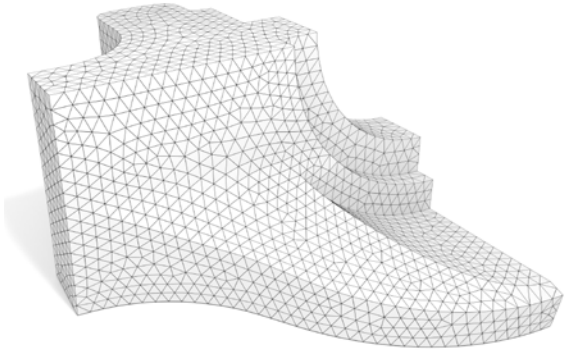
Biomechanics [TrueGrid]

Motivation



hexahedral mesh

VS



tetrahedral mesh

- **Advantage:**

superior numerical properties,

e.g. performance, accuracy

- **Problem:**

difficult to generate good quality hex meshes

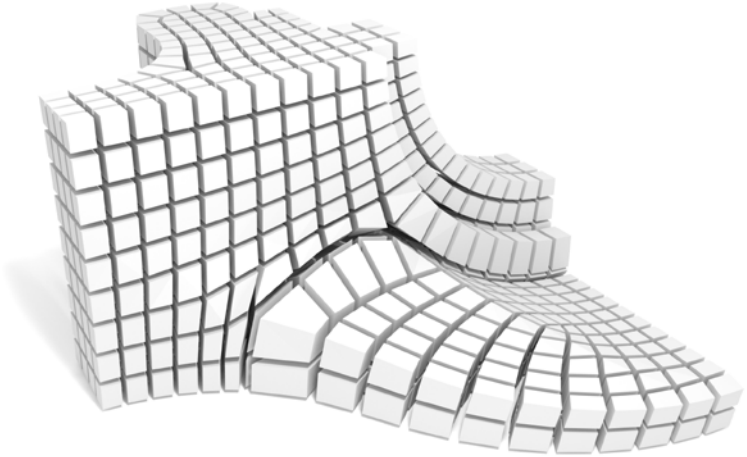
Motivation

- **Problem:**

difficult to generate good quality hex meshes.

- **Good quality:**
  - approximation
    - → faithful & boundary aligned
  - regularity
    - → few singularities/irregularities
  - element quality
    - → scaled Jacobians close to 1

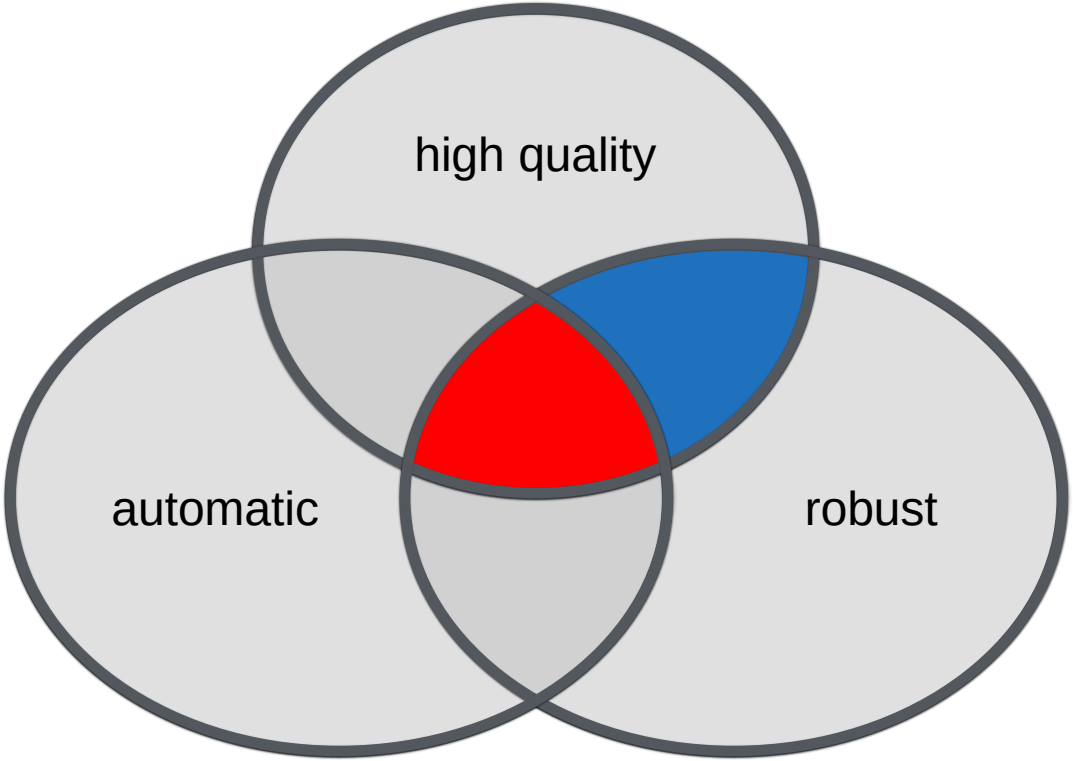
…



hexahedral mesh

Motivation

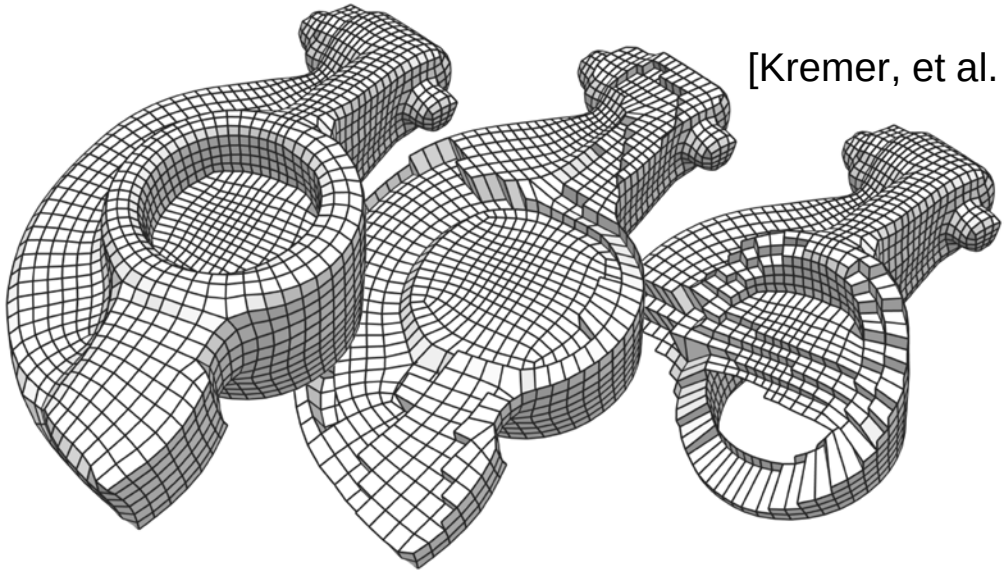
- **Hex Meshing**
  - automation
  - robustness
  - high quality



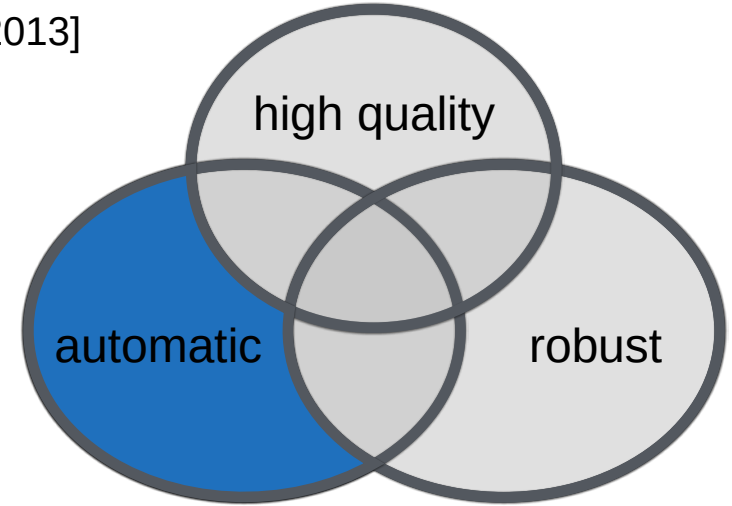
No robust & automatic algorithm!

State-of-the-art

- **Hex Meshing**
  - Advancing fronts

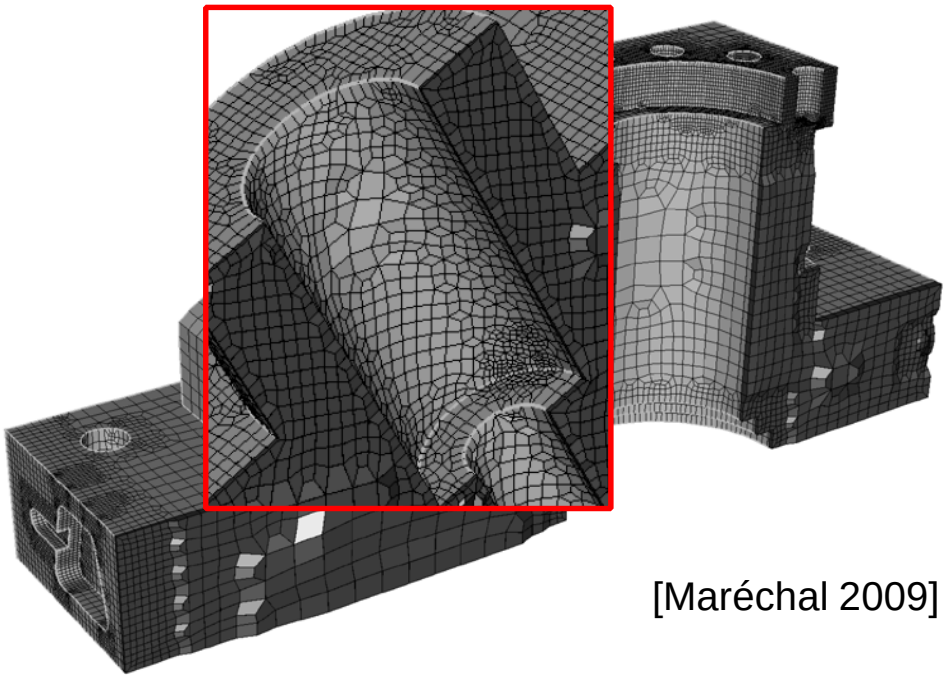


[Kremer, et al. 2013]

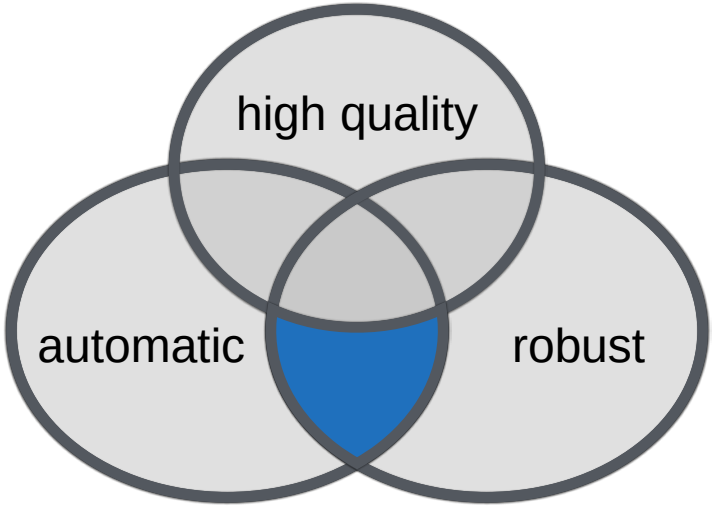


State-of-the-art

- **Hex Meshing**
  - Grid based methods

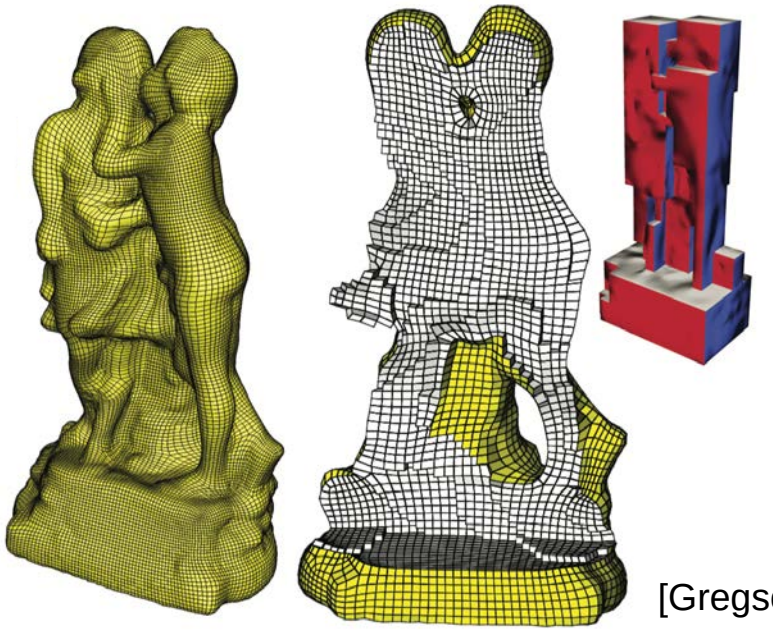


[Maréchal 2009]

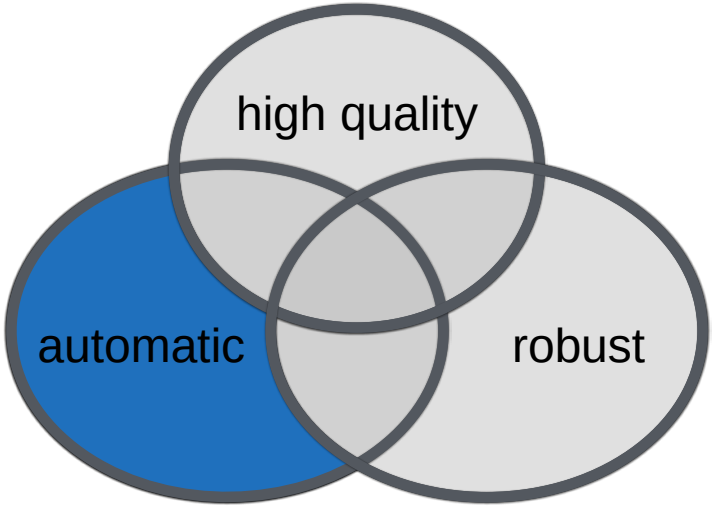


State-of-the-art

- **Hex Meshing**
  - Polycube based: Gregson et al. 2011, Livesu et al. 2013, Huang et al. 2014, Fu et al. 2016, Fang et al. 2016, Xu et al. 2017…



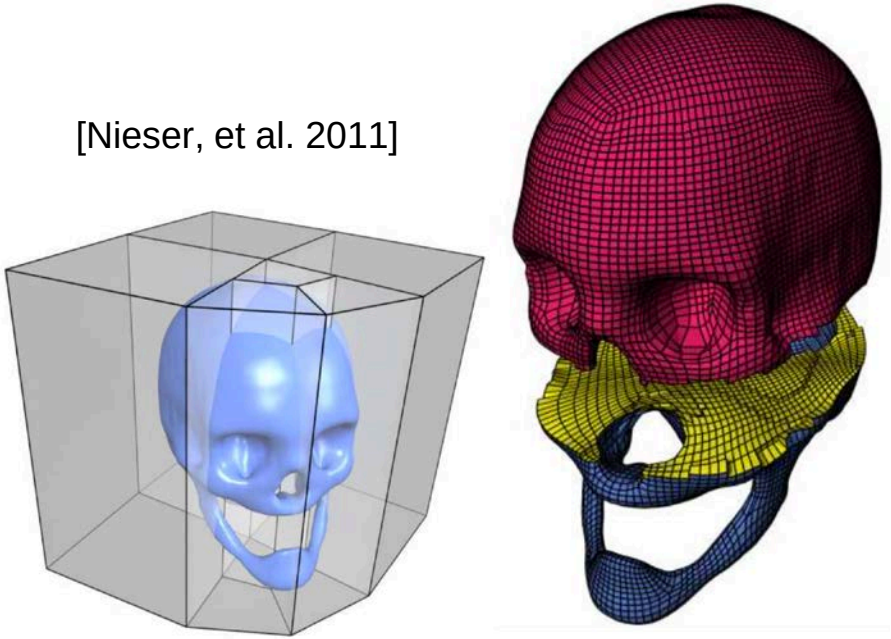
[Gregson, et al. 2011]

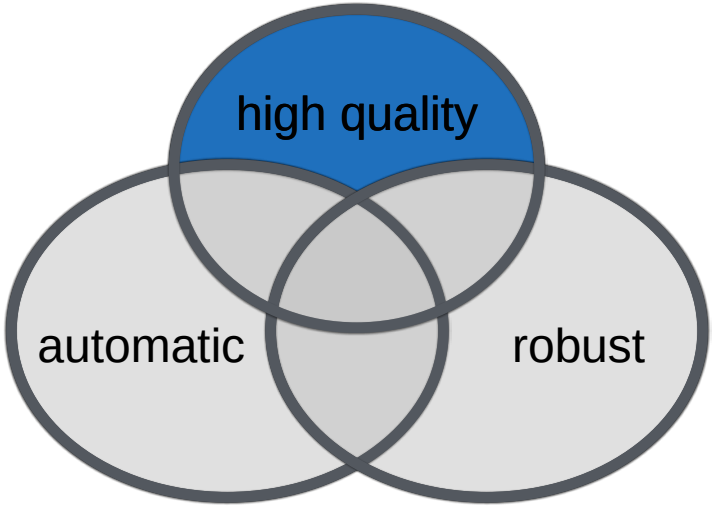


State-of-the-art

- **Hex Meshing**
  - Octahedral field based

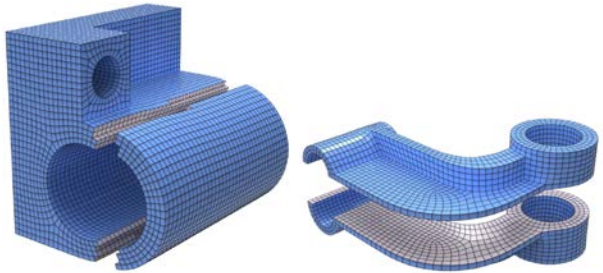
[Nieser, et al. 2011]



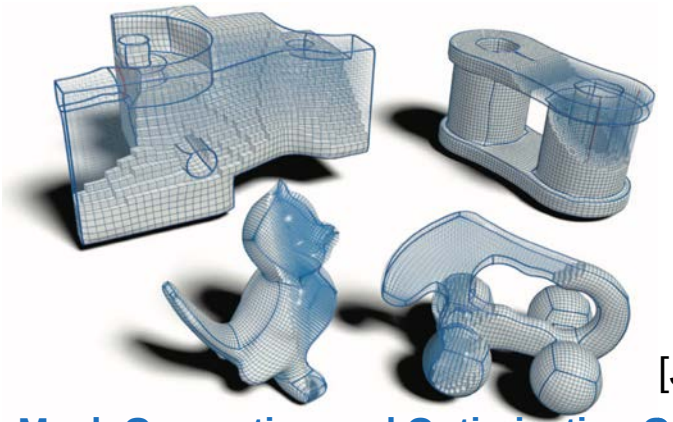


State-of-the-art

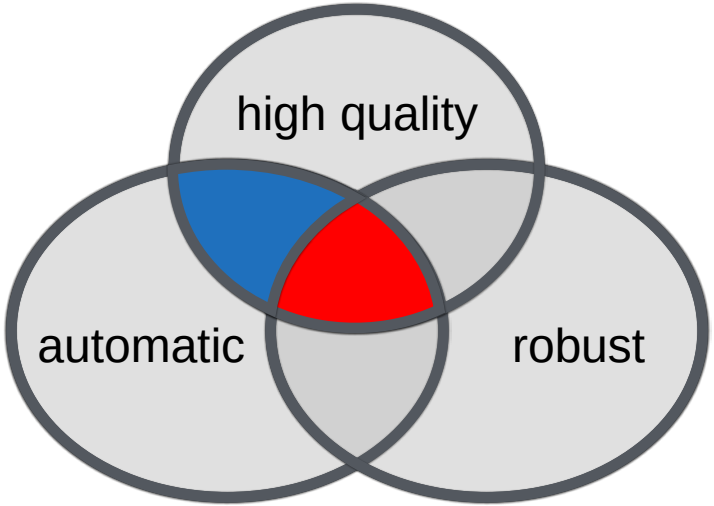
- **Hex Meshing**
  - Octahedral field based



[Li, et al. 2012]

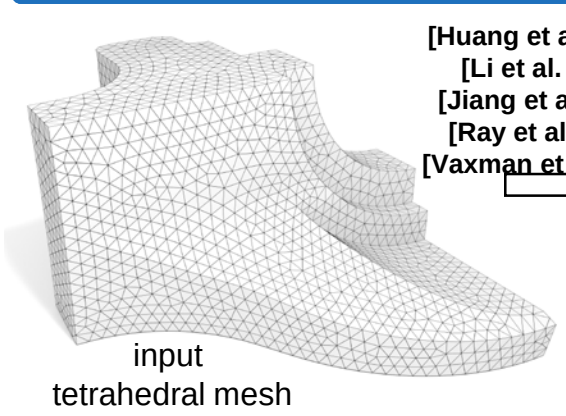


[Jiang, et al. 2013]



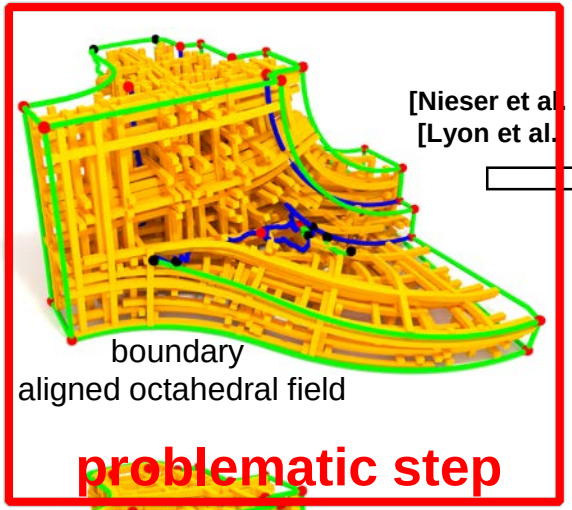
Octahedral Field Based Hex Meshing

Octahedral Field Based Hex Meshing



input
tetrahedral mesh

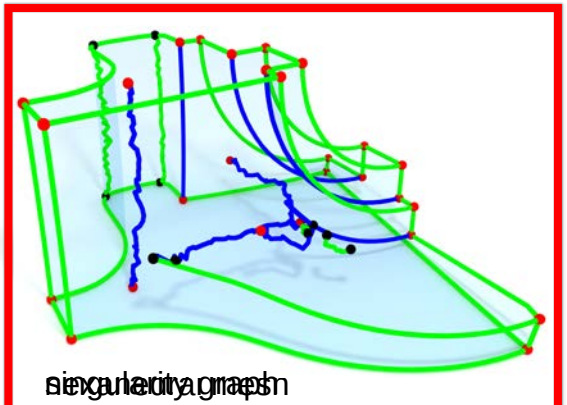
[Huang et al. 2011]
[Li et al. 2012]
[Jiang et al. 2014]
[Ray et al. 2016]
[Vaxman et al. 2017]



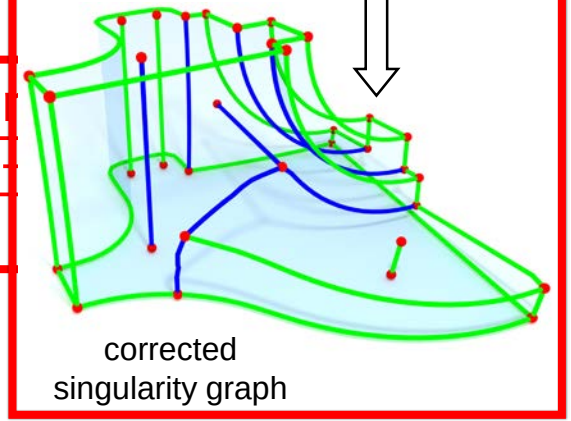
boundary
aligned octahedral field

problematic step

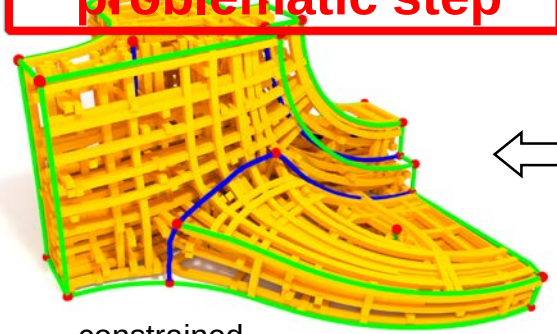
[Nieser et al. 2011]
[Lyon et al. 2016]



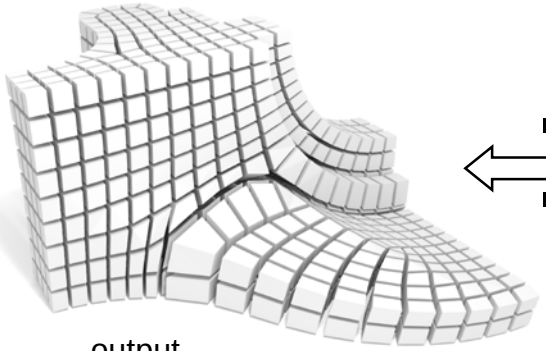
singularity graph



corrected
singularity graph

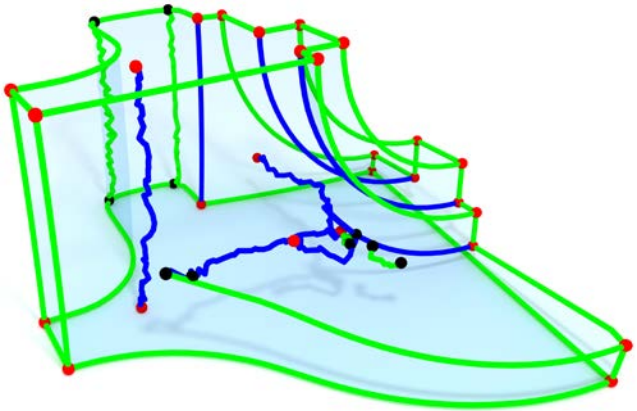


constrained
octahedral field

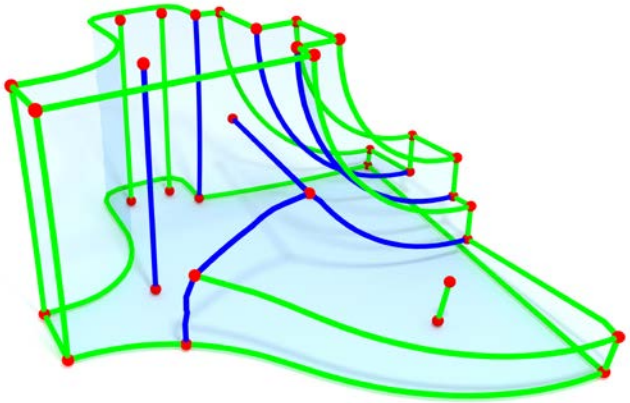


output
hexahedral mesh

Contribution 1



singularity graph



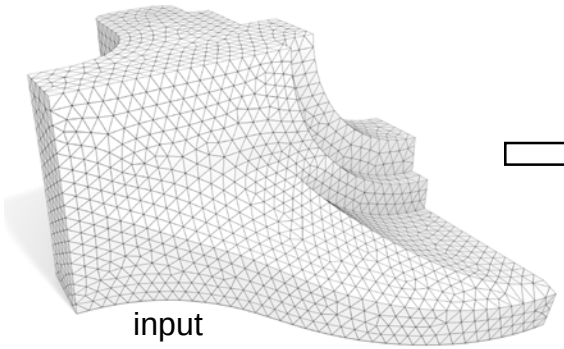
corrected singularity graph

- local configurations
- global necessary condition

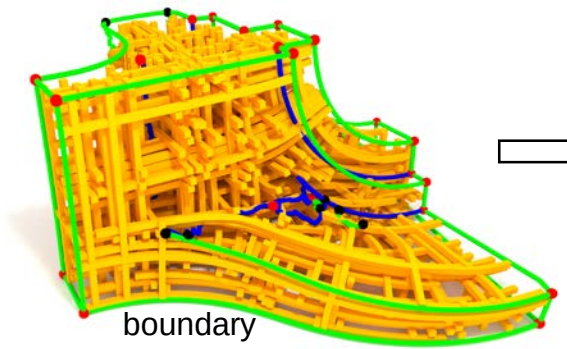
guide

singularity graph correction
current: manual & future: automatic

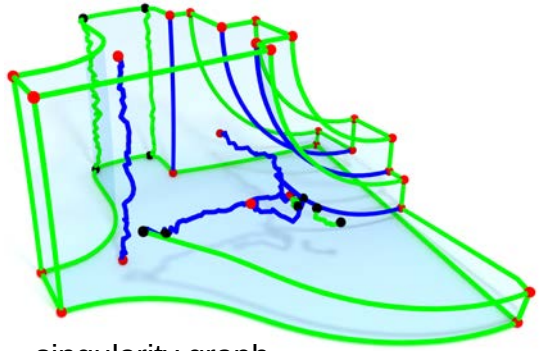
Modified Algorithm



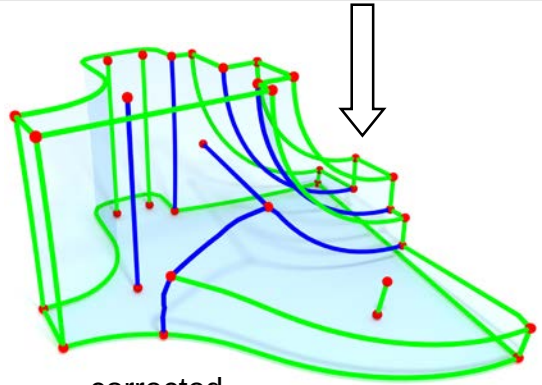
input
tetrahedral mesh



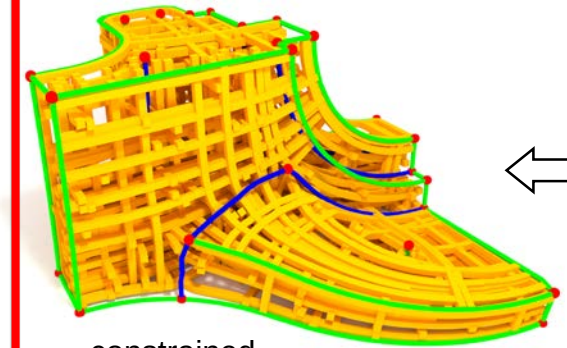
boundary
aligned octahedral field



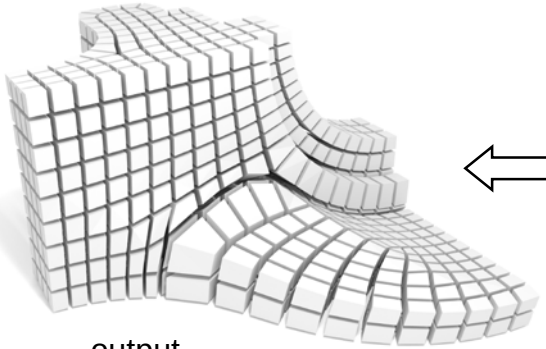
singularity graph



corrected
singularity graph

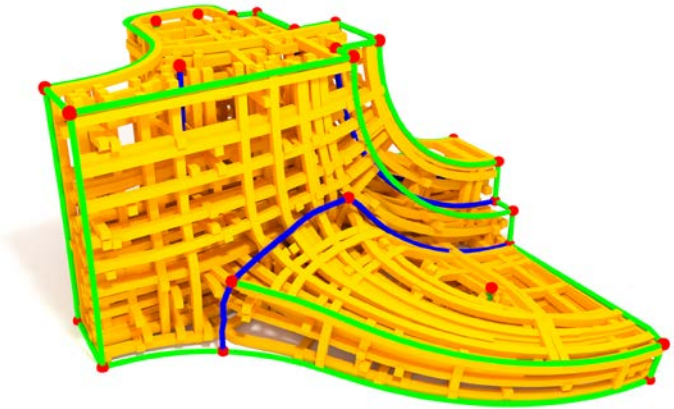


constrained
octahedral field

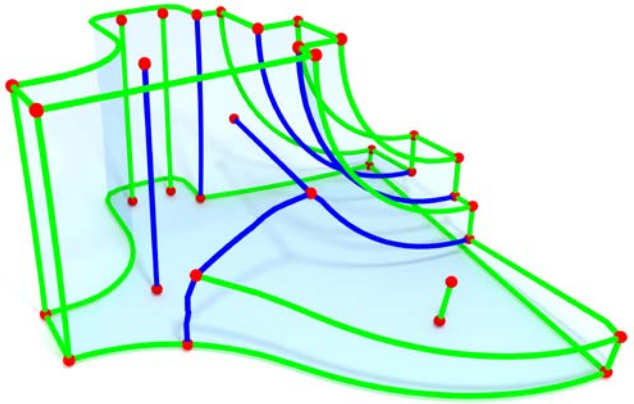


output
hexahedral mesh

Contribution 2



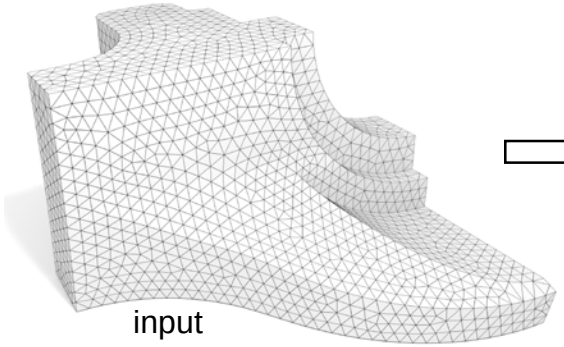
constrained octahedral field



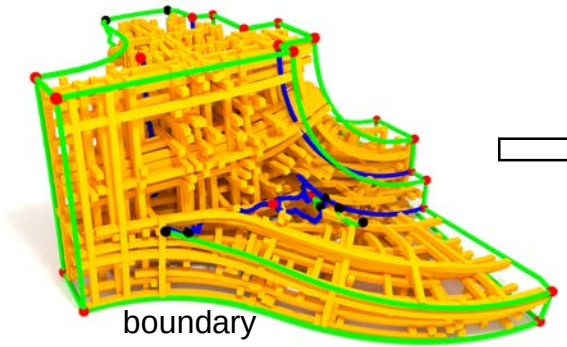
corrected singularity graph

- algorithm to generate octahedral fields with prescribed hex-meshable singularity graphs

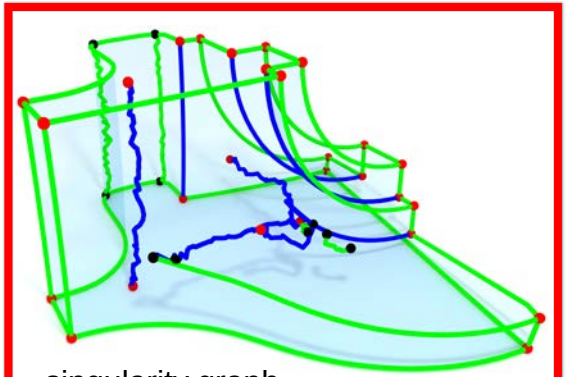
Modified Algorithm



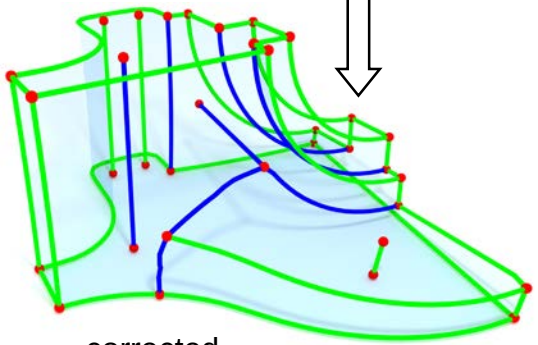
input
tetrahedral mesh



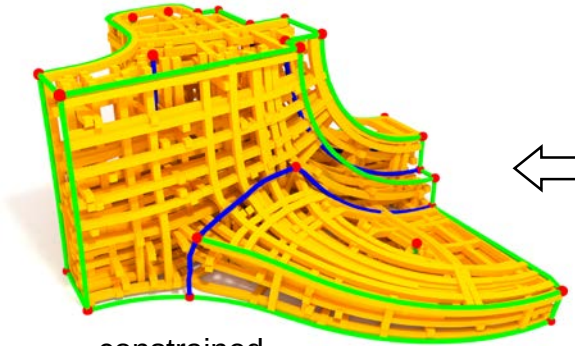
boundary
aligned octahedral-field



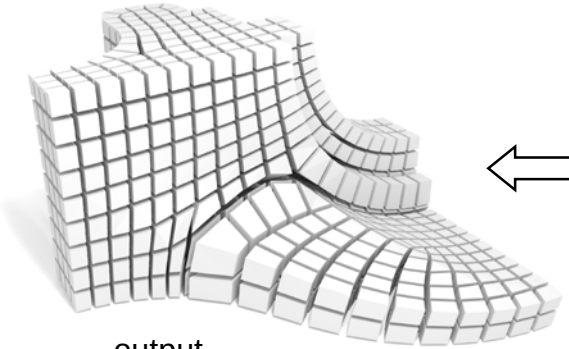
singularity graph



corrected
singularity graph



constrained
octahedral-field

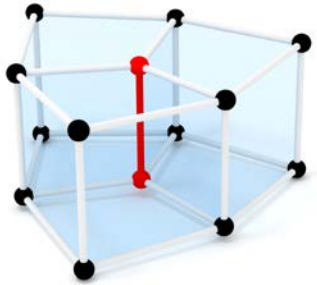


output
hexahedral mesh

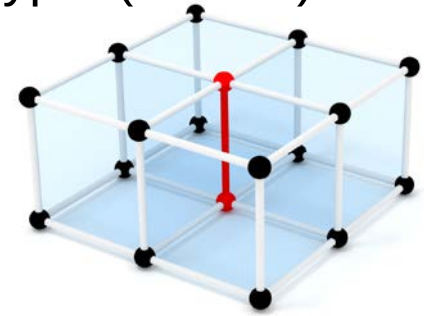
What are hex mesh singularities?

Hex Mesh Singularities

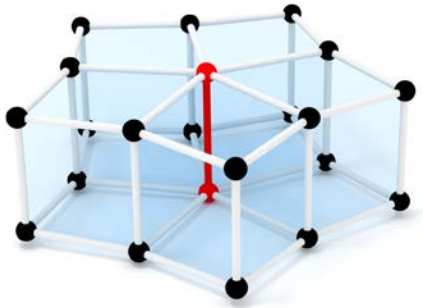
- Local Conditions - Edge types(interior)



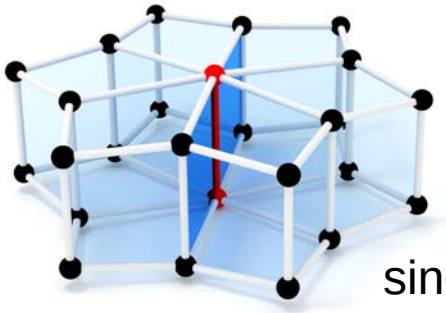
singular
valence 3



regular
valence 4



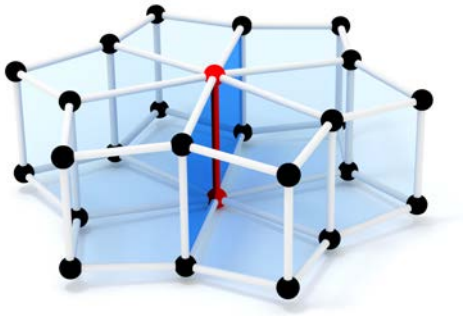
singular
valence 5



singular
valence 6

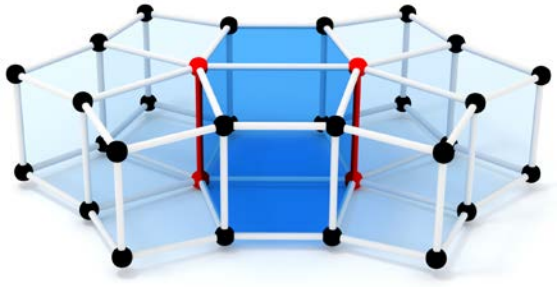
Hex Mesh Singularities

- Local Conditions - Edge types(interior)



singular
valence 6

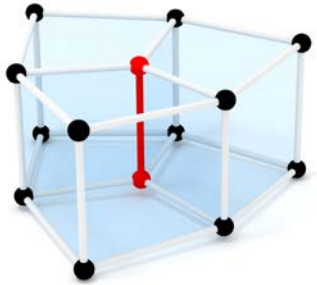
split



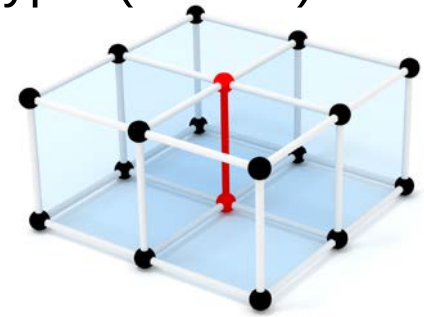
twice valence 5

Hex Mesh Singularities

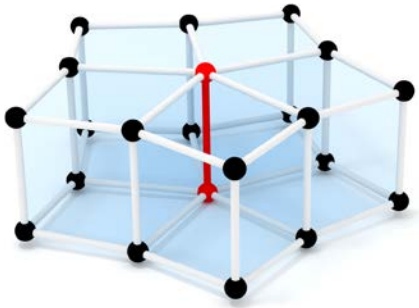
- Local Conditions - Edge types(interior)



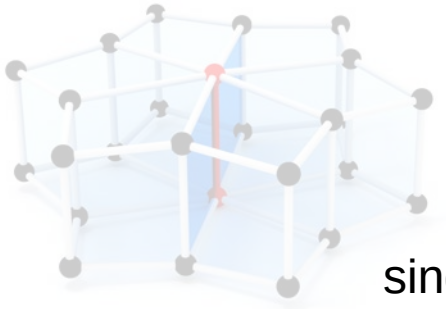
singular
valence 3



regular
valence 4



singular
valence 5

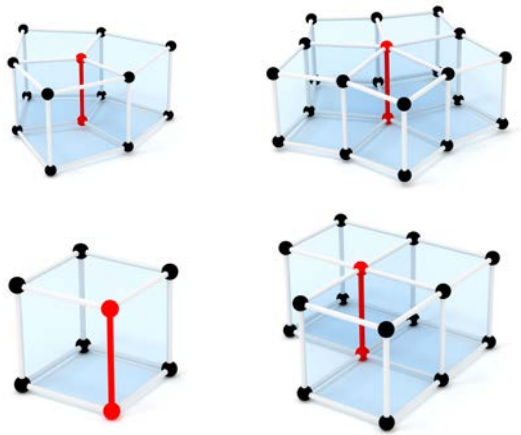


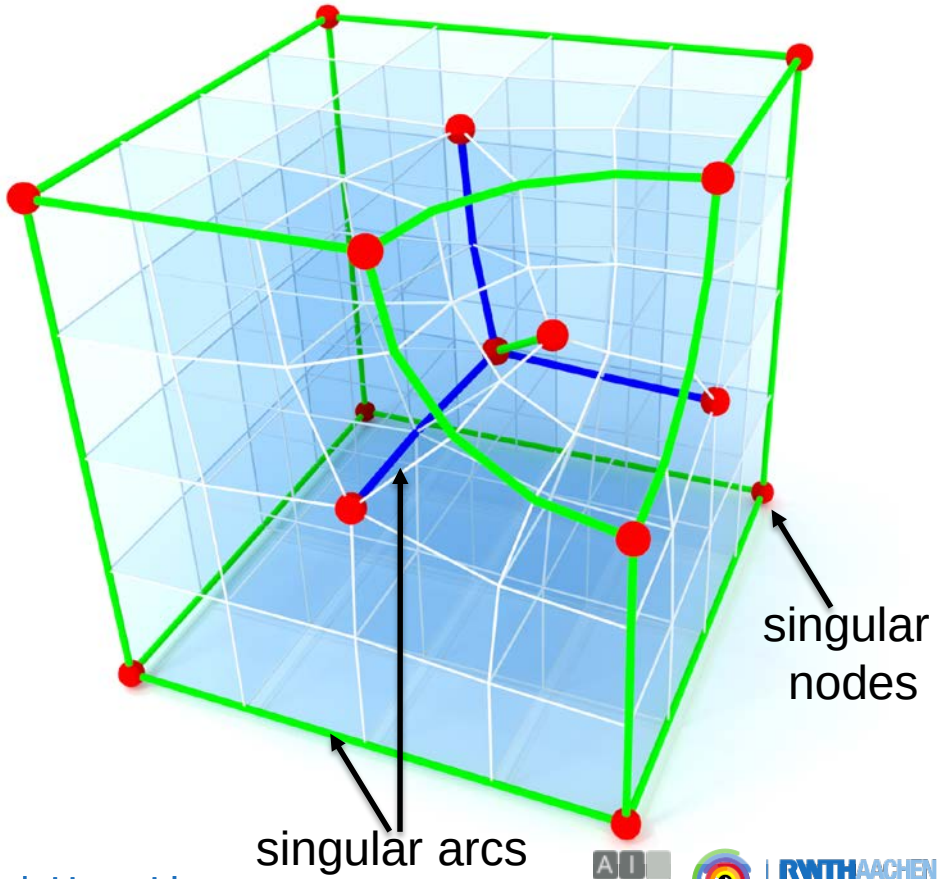
singular
valence 6

Hex Mesh Singularities

- Singularity graph

$$\mathcal{S} = (V_S, E_S)$$



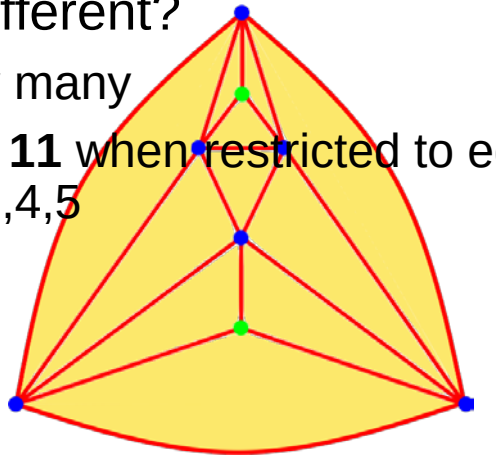


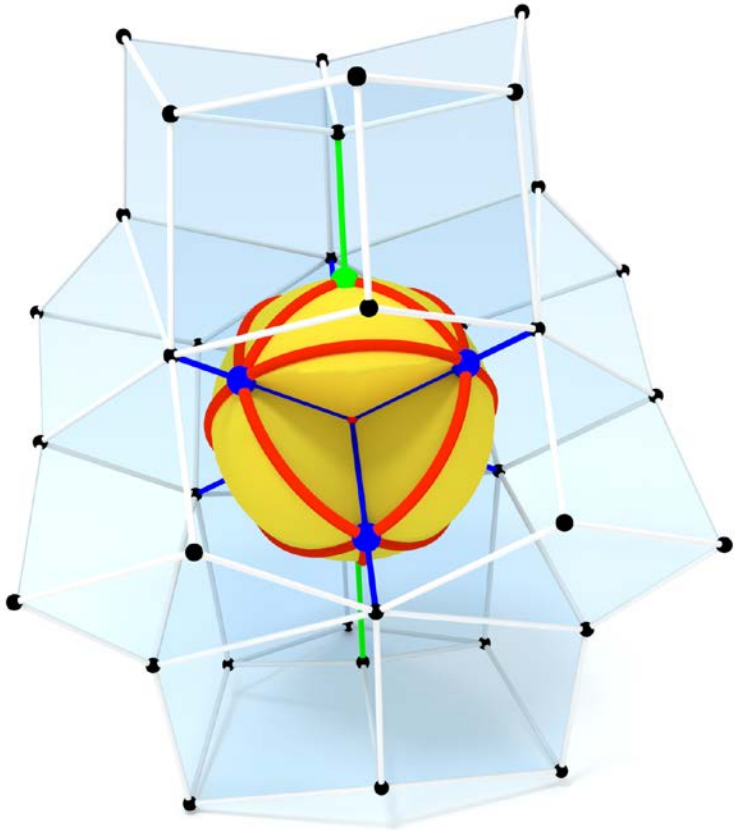
Hex Mesh Singularities

- Vertex types

hex mesh vertices are isomorphic to triangulations of the sphere

- How many different?
  - → infinitely many
  - → but only **11** when restricted to edge valence 3,4,5





Hex Mesh Singularities

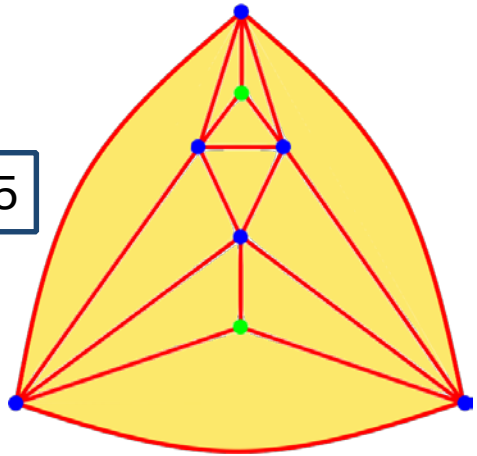
- How many sphere triangulations exists with vertex valences restricted to 3, 4, 5?
  → Answer: **only 11**

- Assume triangulation has #V vertices with signature (i, j, k)

#valence 3

#valence 4

#valence 5



Euler formula
#V - #E + #F = 2

3i + 2j + k = 12
with
i+j+k = #V

consequences:

1. minimal #V = 4 (i=4)
2. maximal #V = 12 (k=12)

Hex Mesh Singularities

$3i + 2j + k = 12$
with
$i+j+k = \#V$

- **#V=4**
  (4,0,0)
- **#V=5**
  (3,1,1), (2,3,0)
- **#V=6**
  (3,0,3), (2,2,2), (1,4,1), (0,6,0)
- **#V=7**
  (0,5,2), (1,3,3), (2,1,4)
- **#V=8**
  (0,4,4), (1,2,5), (2,0,6)
- **#V=9**
  (0,3,6), (1,1,7)
- **#V=10**
  (0,2,8), (1,0,9)
- **#V=11**
  (0,1,10)
- **#V=12**
  (0,0,12)

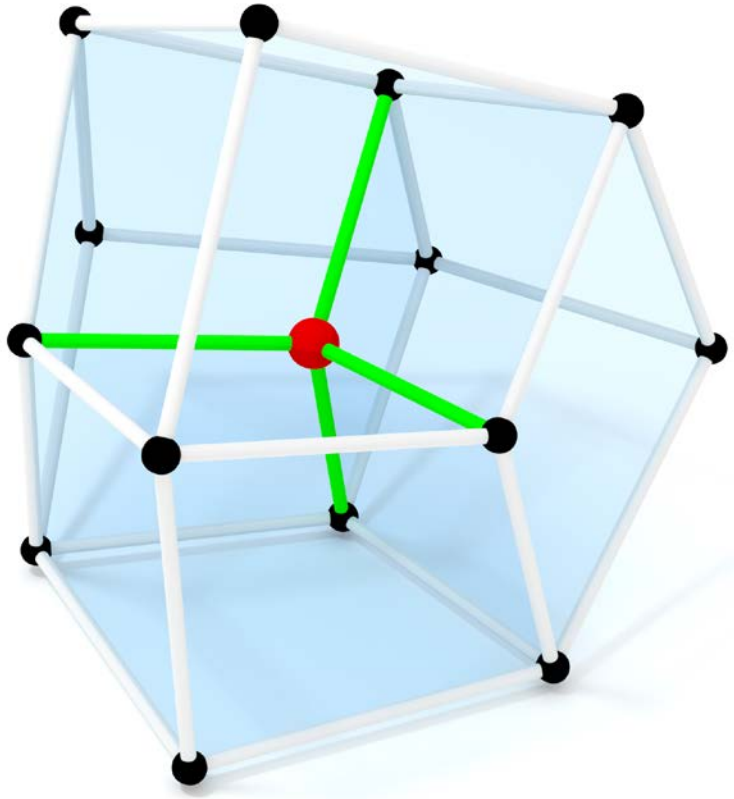
valence 5 with 5 vertices requires self-connection

[Schmeichel and Hakimi 1977]
"On Planar Graphical Degree Sequences"

[Mishra and Sarvate 2007]
"A note on Non-Regular Planar Graphs"

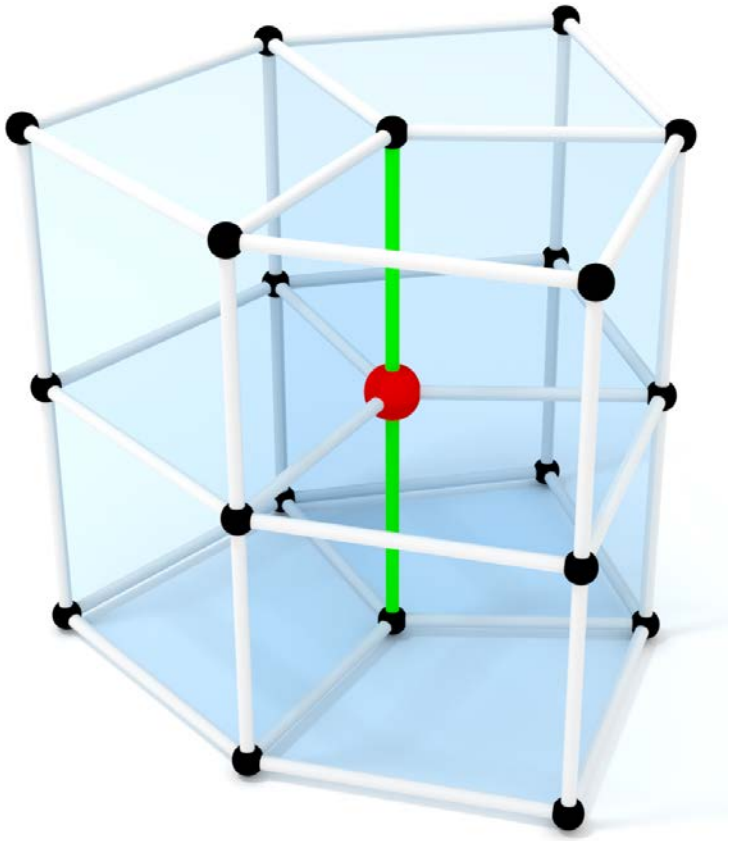
Hex Mesh Singularities

- **#V=4**
  (4,0,0)
- **#V=5**
  (2,3,0)
- **#V=6**
  (2,2,2), (0,6,0)
- **#V=7**
  (0,5,2), (1,3,3)
- **#V=8**
  (0,4,4), (2,0,6)
- **#V=9**
  (0,3,6)
- **#V=10**
  (0,2,8)
- **#V=12**
  (0,0,12)



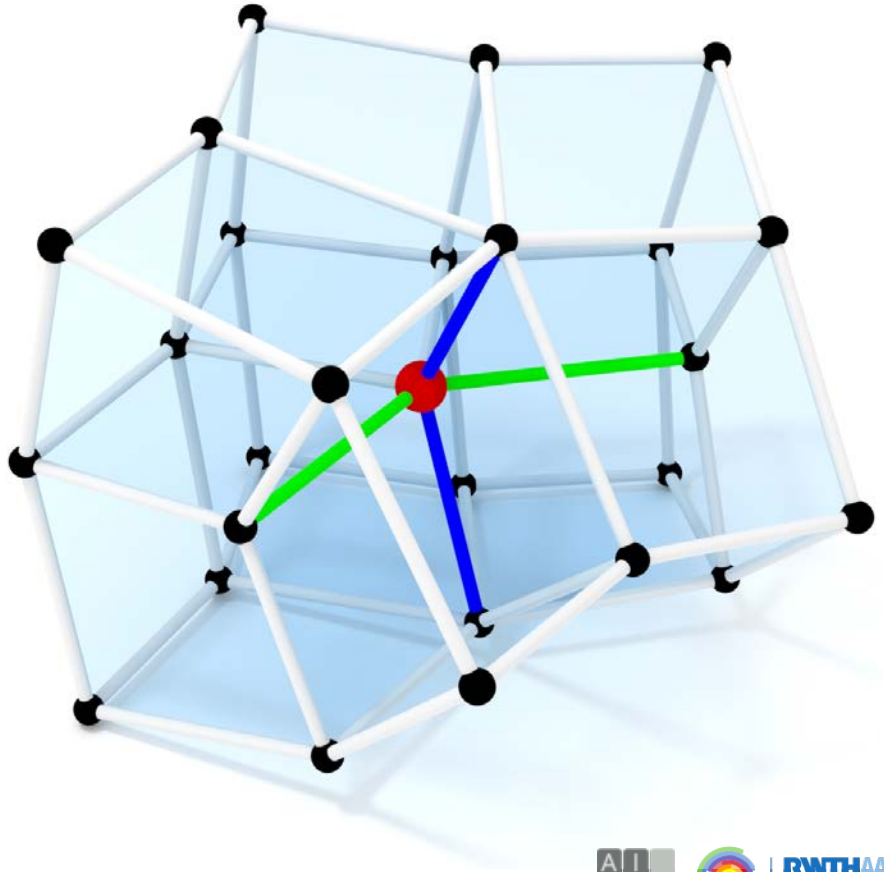
Hex Mesh Singularities

- **#V=4**
  (4,0,0)
- **#V=5**
  (2,3,0)
- **#V=6**
  (2,2,2), (0,6,0)
- **#V=7**
  (0,5,2), (1,3,3)
- **#V=8**
  (0,4,4), (2,0,6)
- **#V=9**
  (0,3,6)
- **#V=10**
  (0,2,8)
- **#V=12**
  (0,0,12)



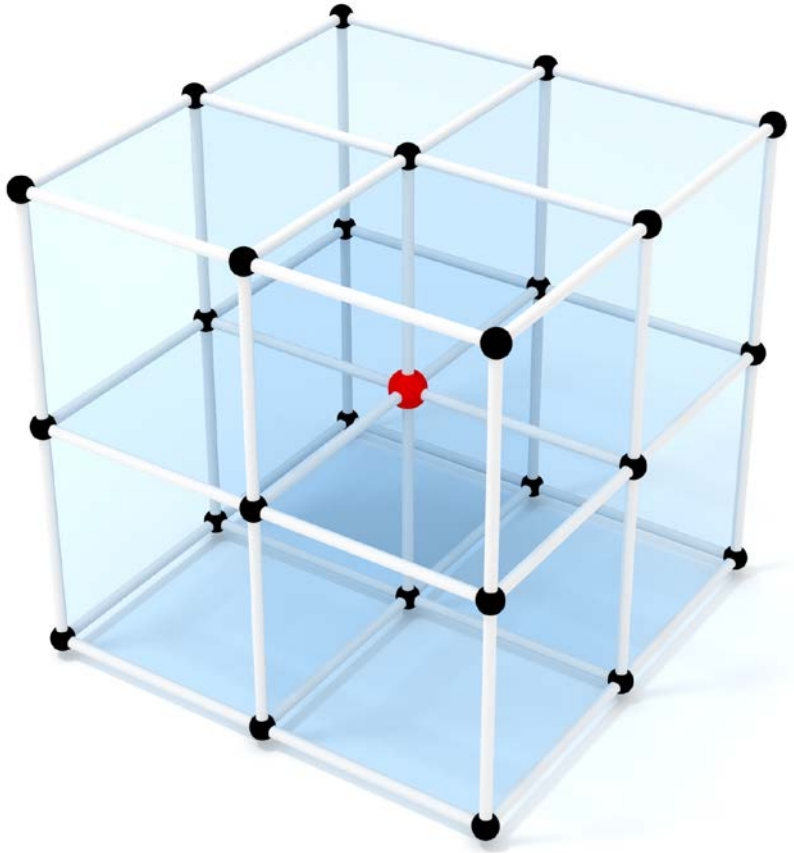
Hex Mesh Singularities

- **#V=4**
  (4,0,0)
- **#V=5**
  (2,3,0)
- **#V=6**
  (2,2,2), (0,6,0)
- **#V=7**
  (0,5,2), (1,3,3)
- **#V=8**
  (0,4,4), (2,0,6)
- **#V=9**
  (0,3,6)
- **#V=10**
  (0,2,8)
- **#V=12**
  (0,0,12)



Hex Mesh Singularities

- **#V=4**
  (4,0,0)
- **#V=5**
  (2,3,0)
- **#V=6**
  (2,2,2), **(0,6,0)**
- **#V=7**
  (0,5,2), (1,3,3)
- **#V=8**
  (0,4,4), (2,0,6)
- **#V=9**
  (0,3,6)
- **#V=10**
  (0,2,8)
- **#V=12**
  (0,0,12)



Hex Mesh Singularities

- **#V=4**
  (4,0,0)
- **#V=5**
  (2,3,0)
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  (2,2,2), (0,6,0)
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  (0,5,2), (1,3,3)
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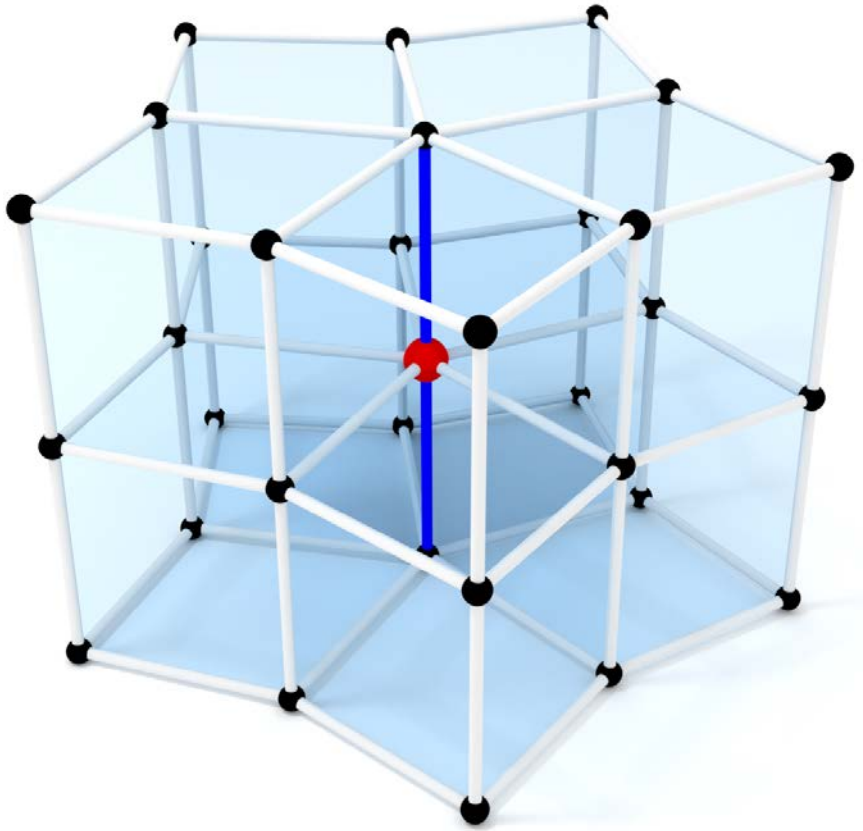
  (0,4,4), (2,0,6)
- **#V=9**
  (0,3,6)
- **#V=10**
  (0,2,8)
- **#V=12**
  (0,0,12)



Hex Mesh Singularities

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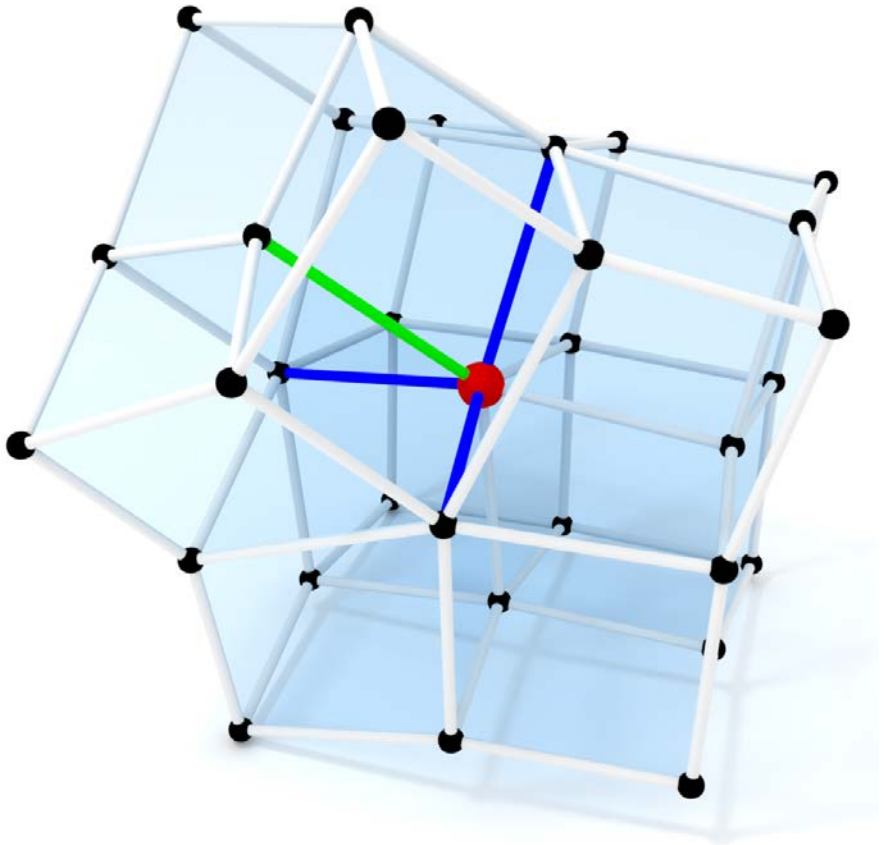
  (4,0,0)
- **#V=5**
  (2,3,0)
- **#V=6**
  (2,2,2), (0,6,0)
- **#V=7**
  (0,5,2), **(1,3,3)**
- **#V=8**
  (0,4,4), (2,0,6)
- **#V=9**
  (0,3,6)
- **#V=10**
  (0,2,8)
- **#V=12**
  (0,0,12)



Hex Mesh Singularities

- **#V=4**
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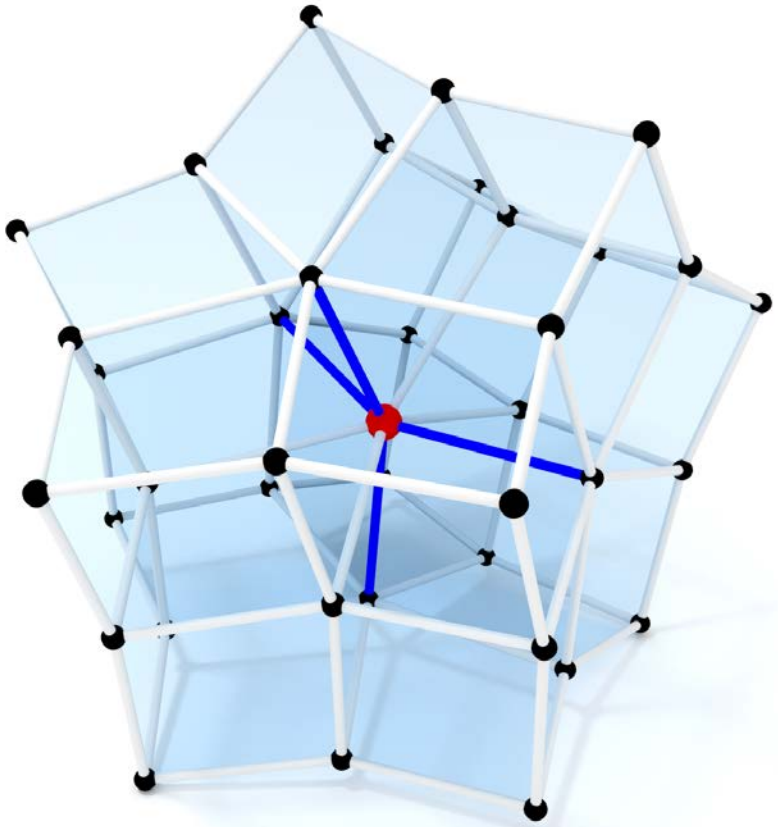
- **#V=5**
  (2,3,0)
- **#V=6**
  (2,2,2), (0,6,0)
- **#V=7**
  (0,5,2), (1,3,3)
- **#V=8**
  (0,4,4), (2,0,6)
- **#V=9**
  (0,3,6)
- **#V=10**
  (0,2,8)
- **#V=12**
  (0,0,12)



Hex Mesh Singularities

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  (2,3,0)
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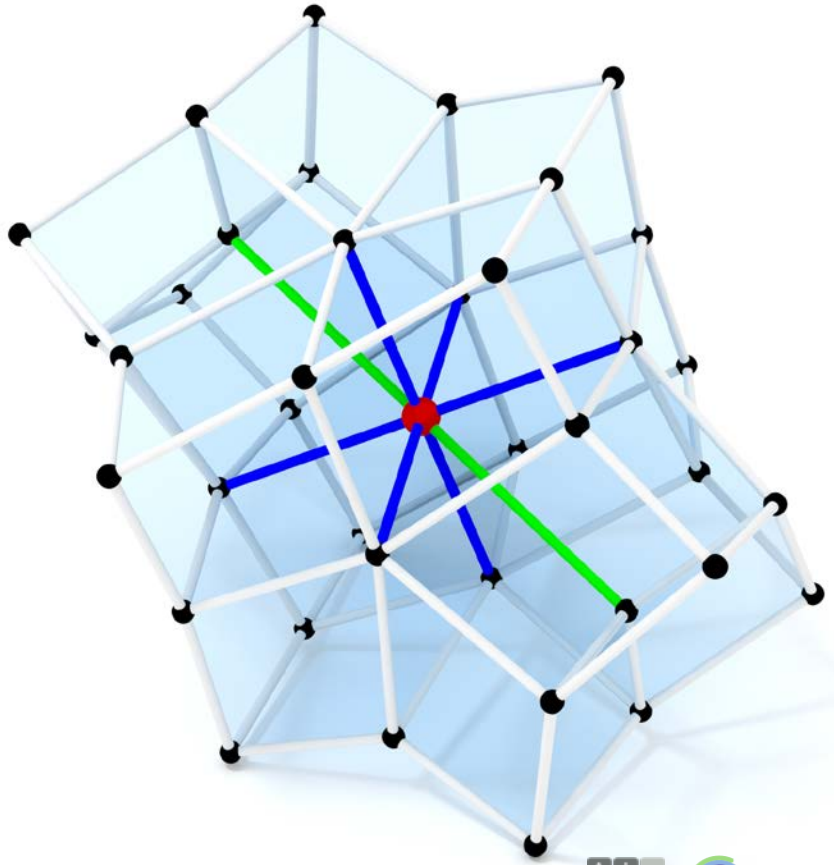
  (0,5,2), (1,3,3)
- **#V=8**
  (0,4,4), **(2,0,6)**
- **#V=9**
  (0,3,6)
- **#V=10**
  (0,2,8)
- **#V=12**
  (0,0,12)



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  (2,2,2), (0,6,0)
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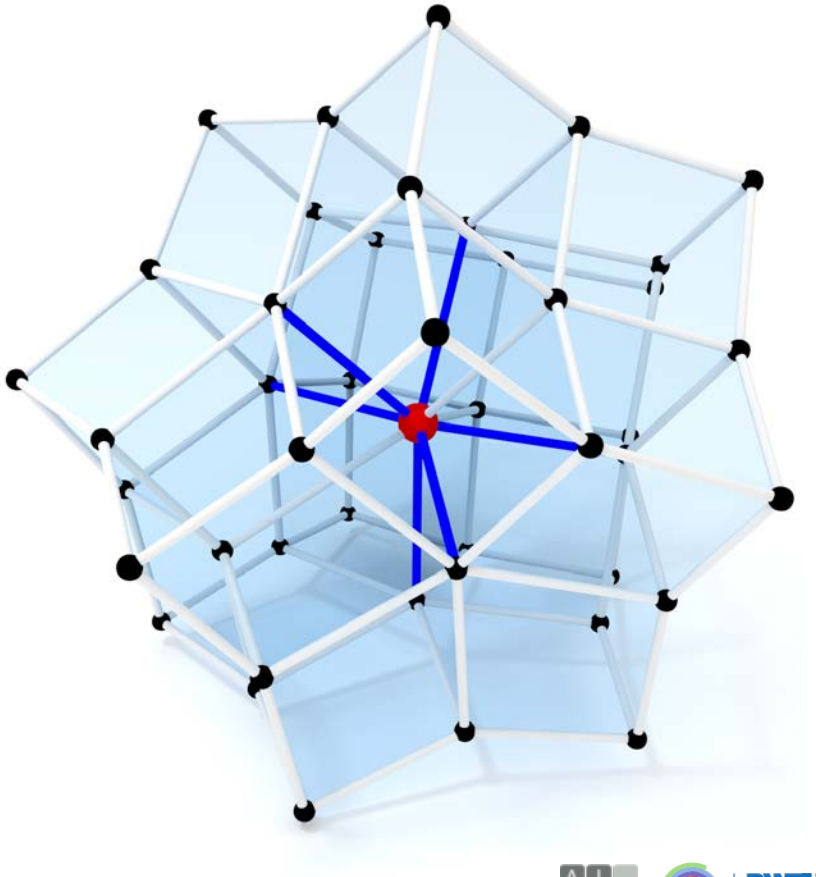
- **#V=12**
  (0,0,12)



Hex Mesh Singularities

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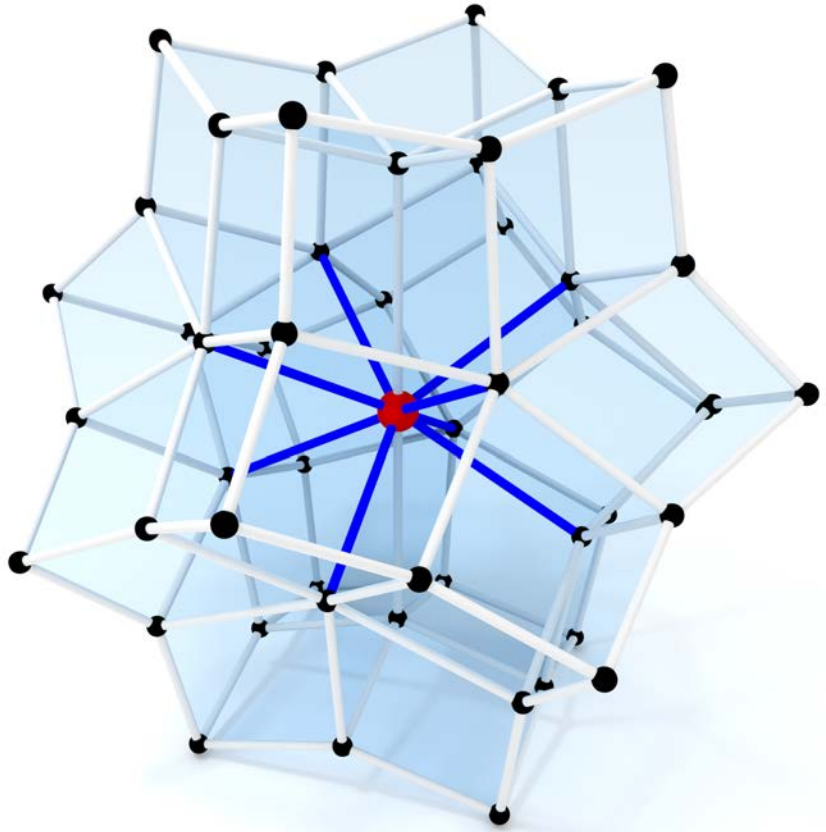
  (0,0,12)



Hex Mesh Singularities

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  (2,2,2), (0,6,0)
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- **#V=12**
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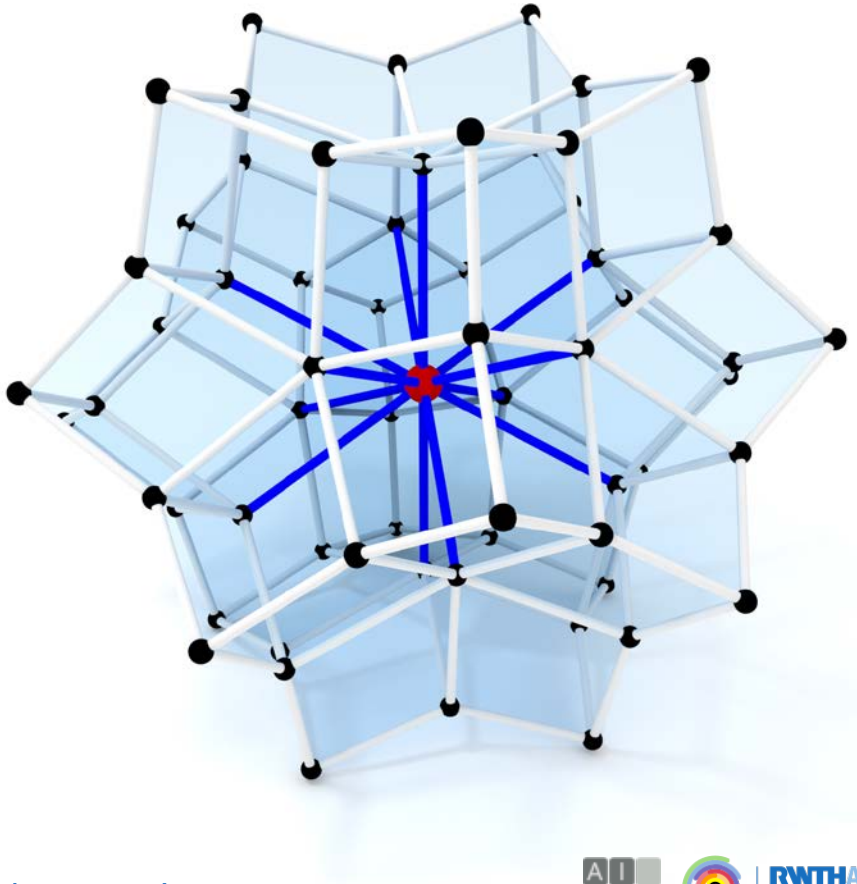
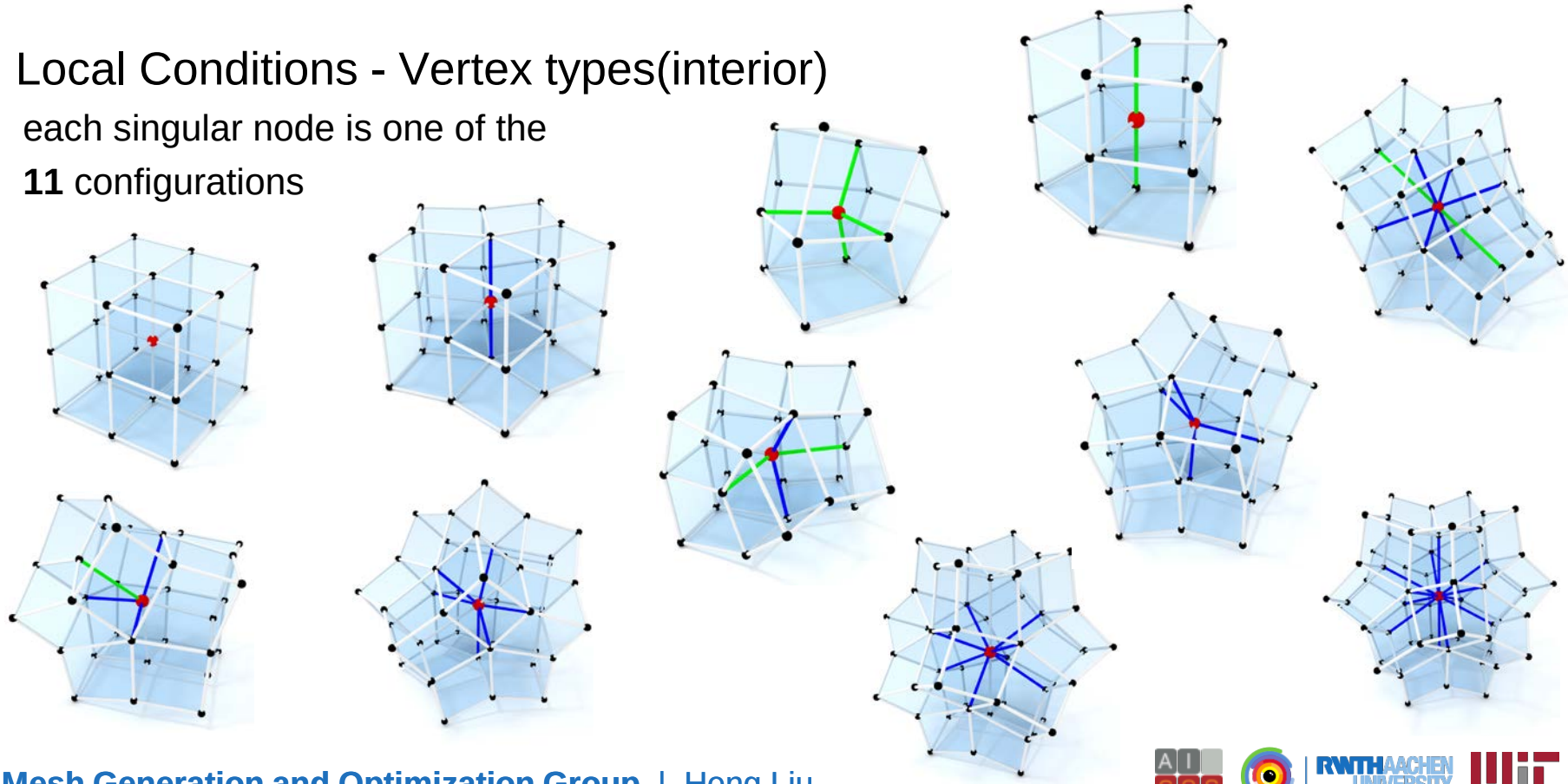


Hex Meshable Singularity Graphs (valence 3/4/5)

- Local Conditions - Vertex types(interior)

  each singular node is one of the

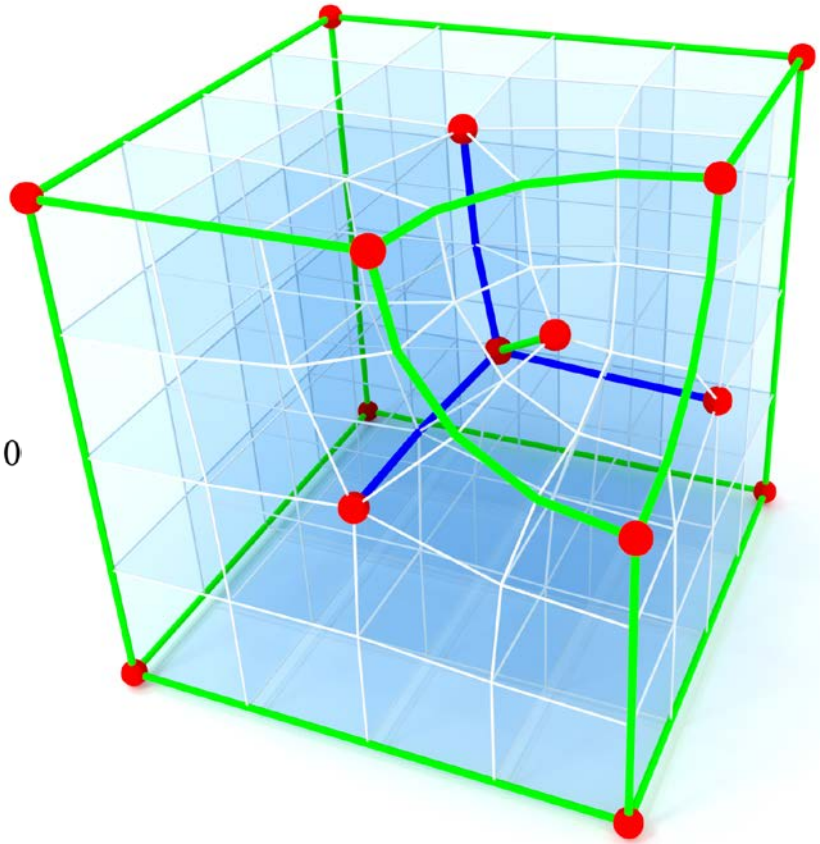
  11 configurations



Hex Meshable Singularity Graphs (valence 3/4/5)

- Global Necessary Condition

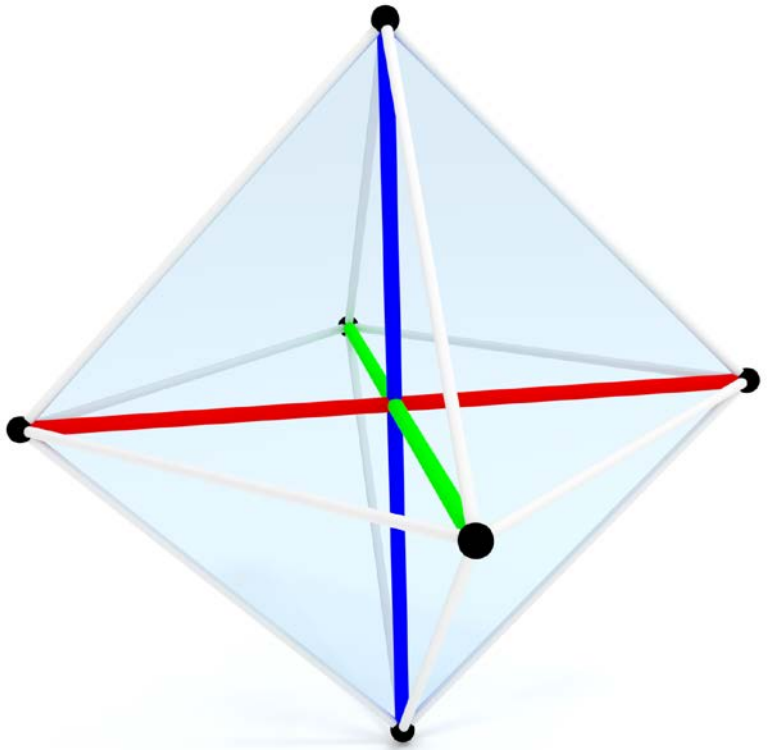
$$\sum_{v \in \partial V_S} \frac{1}{2}\left(1 - \frac{\mathrm{val}_h(v)}{4}\right) - \sum_{e \in \partial E_S^-} \mathrm{idx}(e) + \sum_{v \in \mathring{V}_S} \left(1 - \frac{\mathrm{val}_h(v)}{8}\right) - \sum_{e \in \mathring{E}_S^-} \mathrm{idx}(e) = 0$$



What are Octahedral Field singularities?

Octahedral Field Singularities

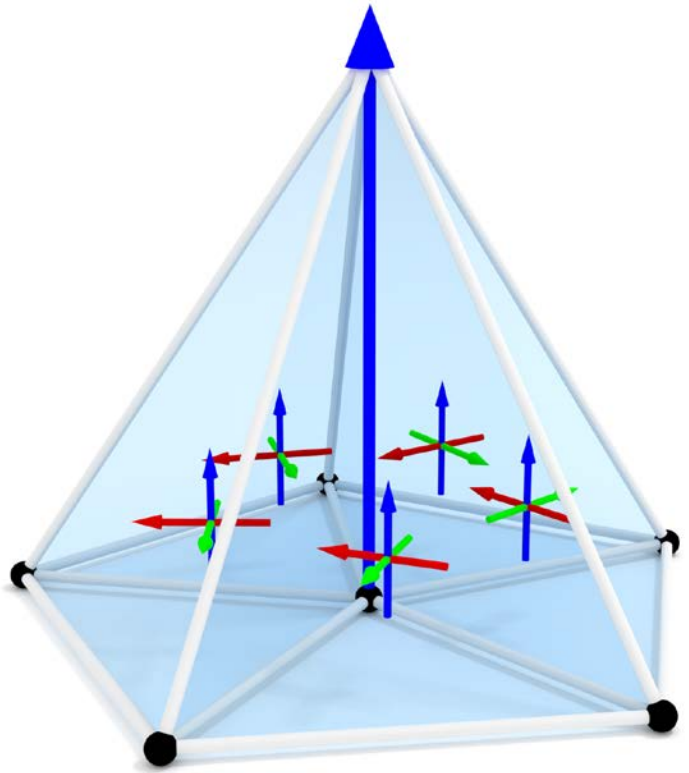
- Octahedral-Fields



Octahedral Field Singularities

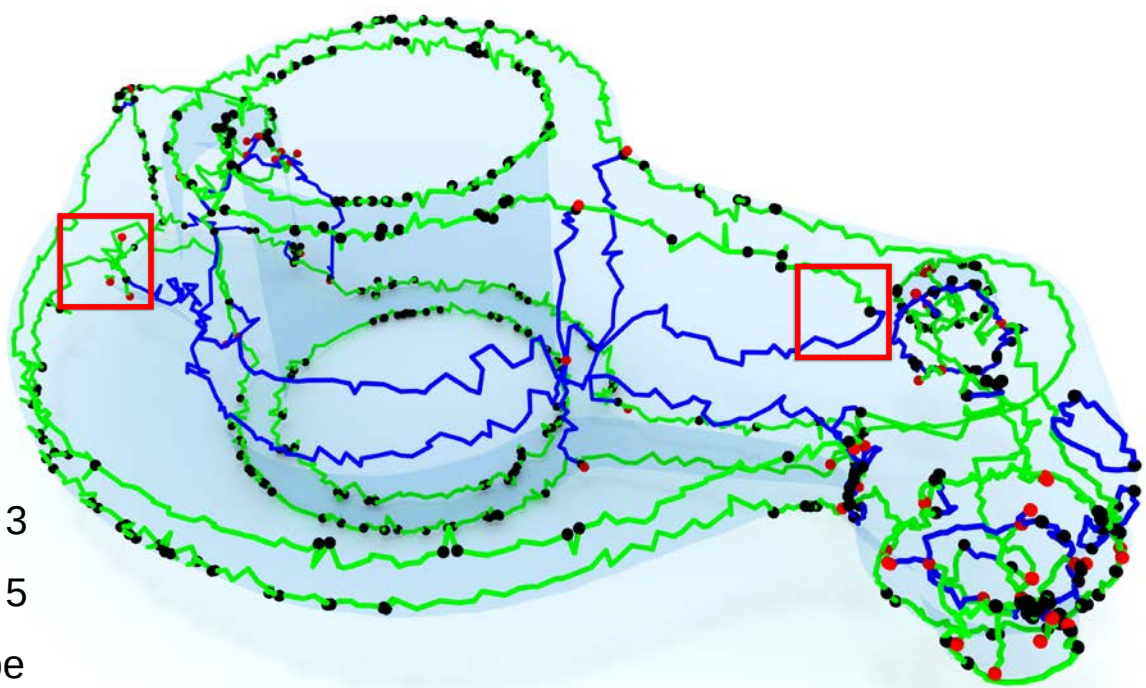
- Octahedral-Fields
  - one frame per tet q_i (unit quaternion)
  - $R \in \mathrm{Oct}$ matching quaternion per face
  - 24 classes of matchings

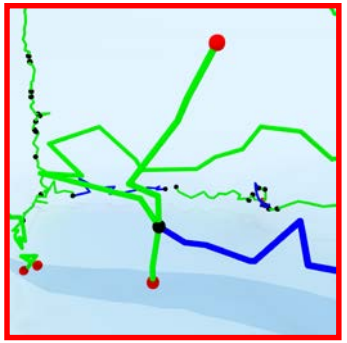
singularities form a graph



Octahedral-Field Singularities

- Defects





: valance 3

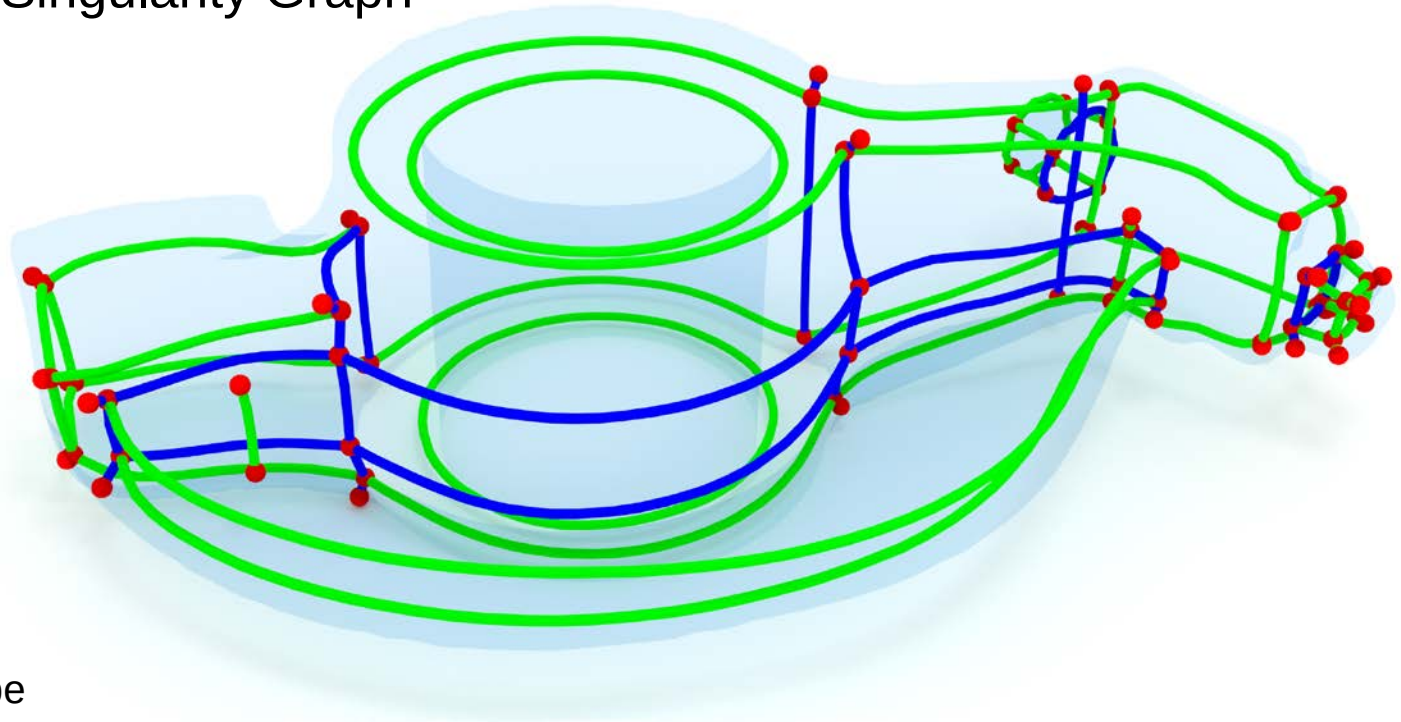
: valance 5

: valid type

: invalid type

Octahedral-Field Singularities

- Corrected Singularity Graph



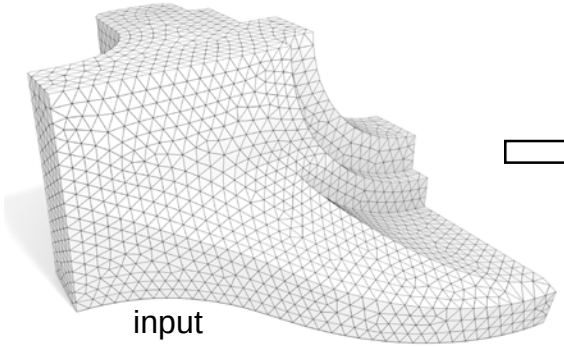
: valance 3

: valance 5

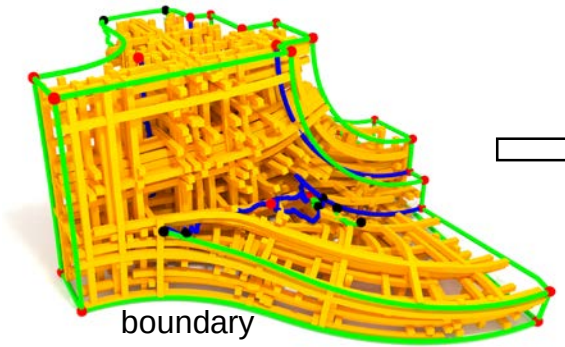
: valid type

: invalid type

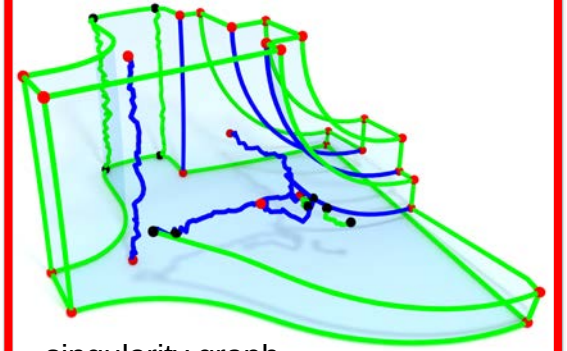
Modified Algorithm



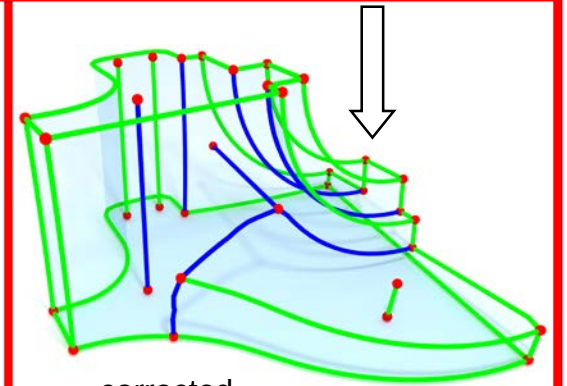
input
tetrahedral mesh



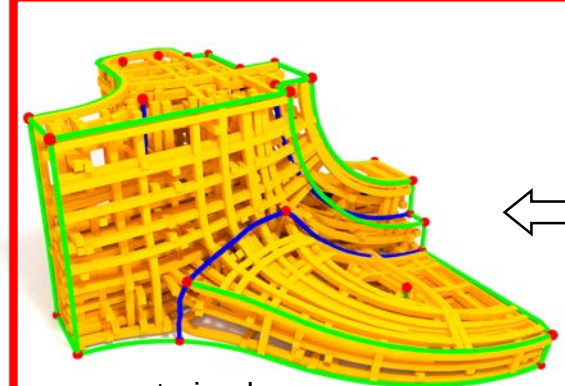
boundary
aligned octahedral-field



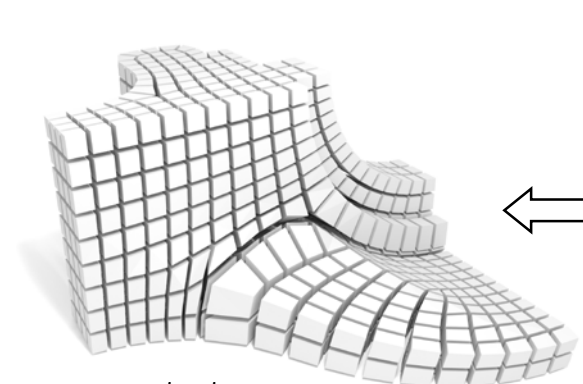
singularity graph



corrected
singularity graph



constrained
octahedral-field

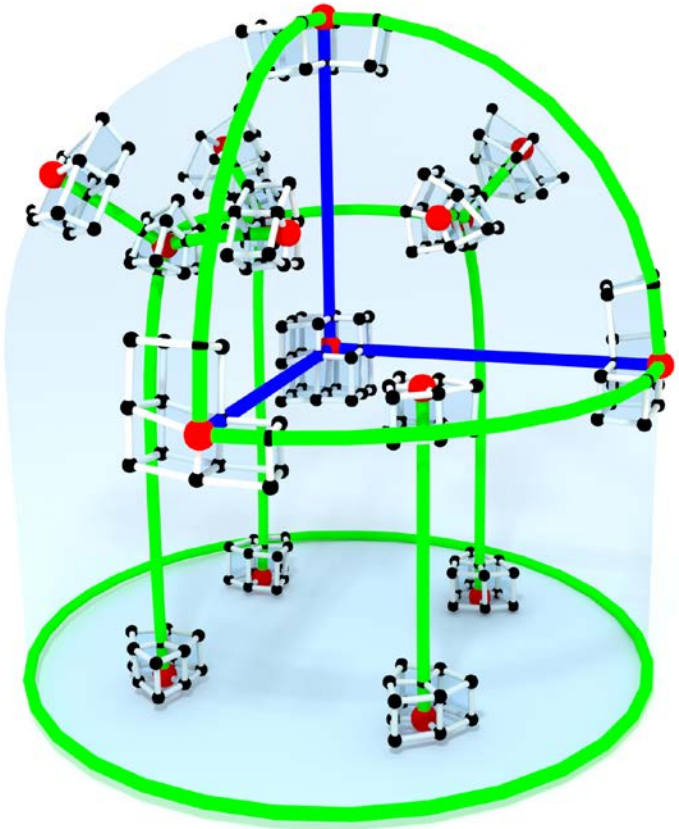


output
hexahedral mesh

Singularity Constrained Hexable Octahedral-Fields

Challenge

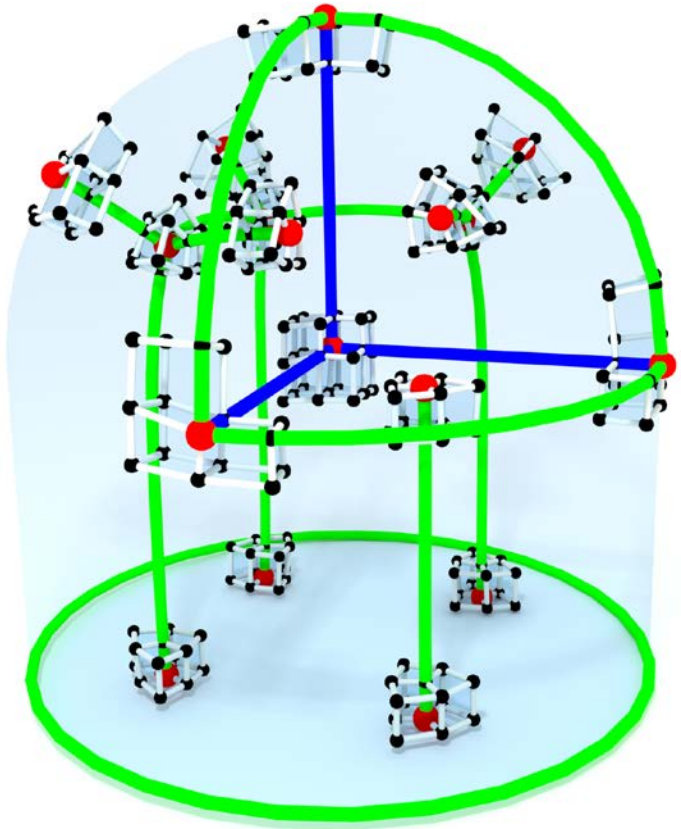
- **Input**:
  - Singularity graph $\mathcal{S} = (V_S, E_S)$, satisfying
  - → global necessary condition
  - → local conditions
- **Output**:
  - an octahedral field O that is boundary-aligned and matches the singularity graph $\mathcal{S}$



Challenge

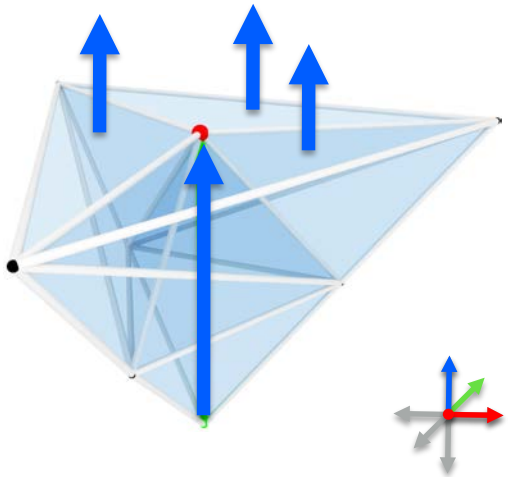
- **Input**:
  - alignment constraints As
  - edge type constraints Es
  - vertex type constraints Vs
- **Output**:
  - topological octahedral field

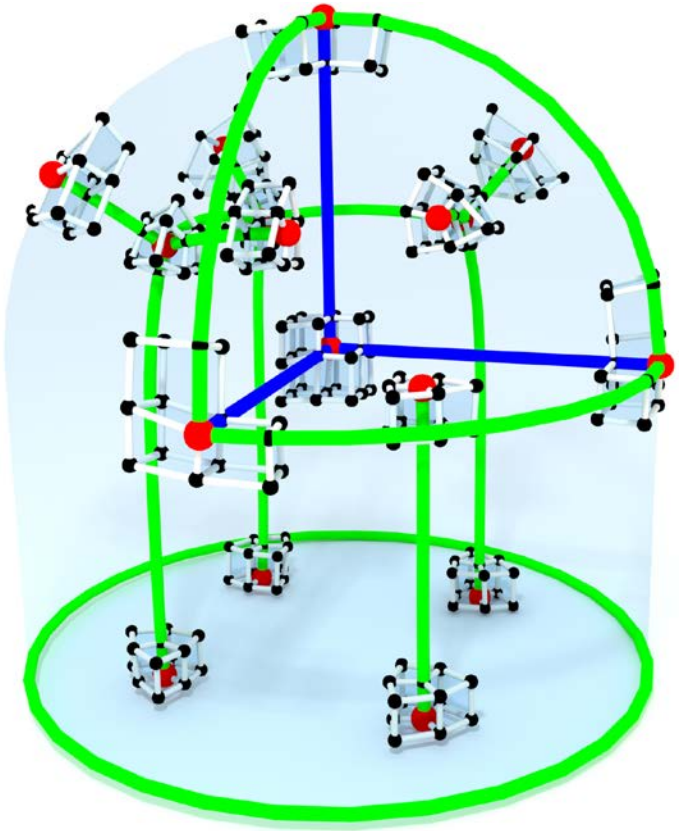
    → matchings and field alignment



Singularity Graph Constraints

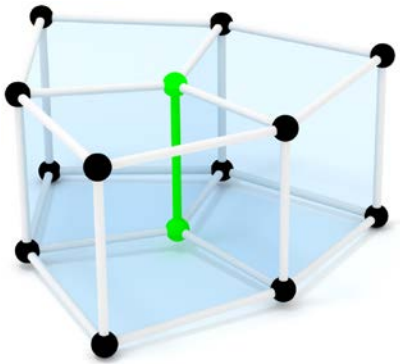
- **Input**:
  - alignment constraints As
  - edge type constraints Es
  - vertex type constraints Vs



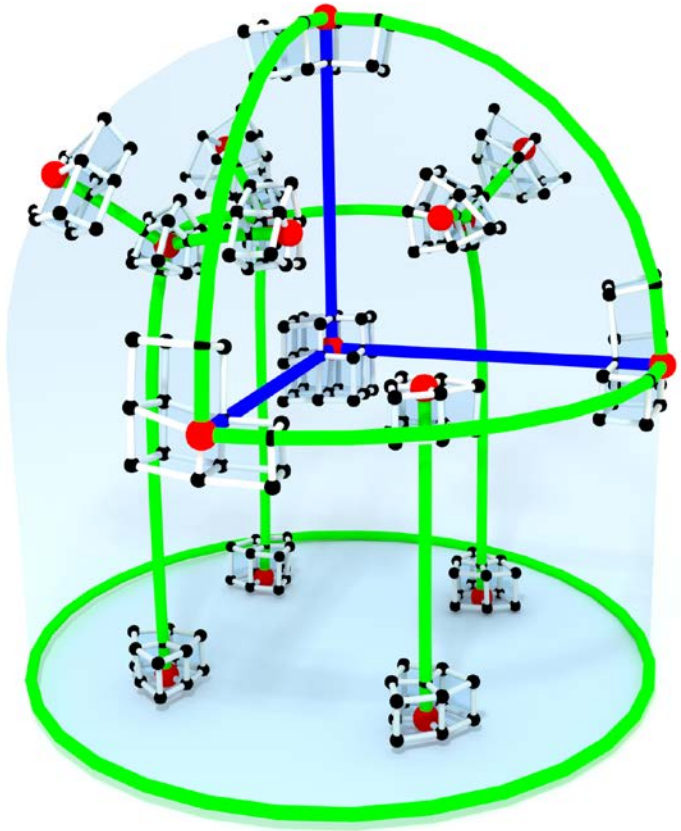


Singularity Graph Constraints

- **Input**:
  - alignment constraints As
  - edge type constraints Es
  - vertex type constraints Vs

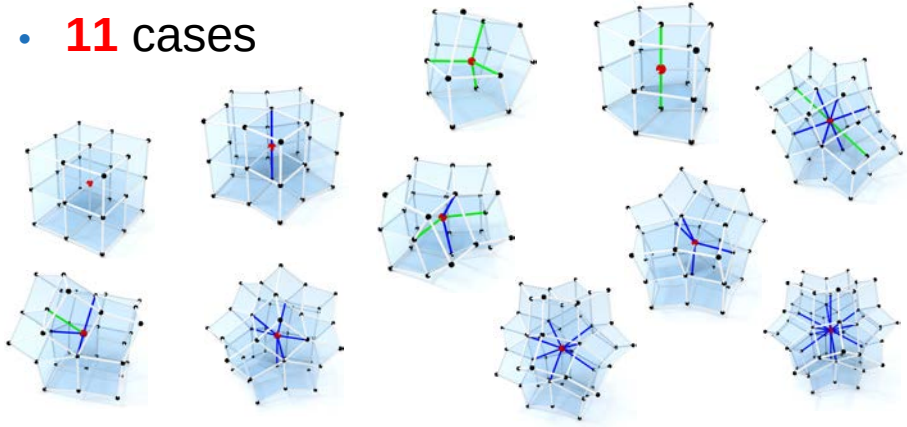


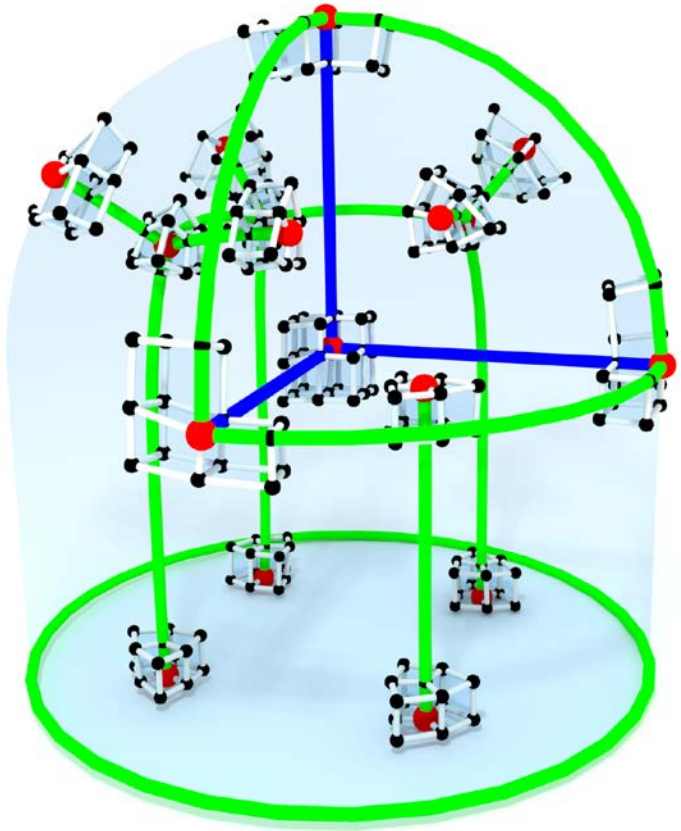
singular
valence 3



Singularity Graph Constraints

- **Input**:
  - alignment constraints As
  - edge type constraints Es
  - vertex type constraints Vs
  - **11** cases

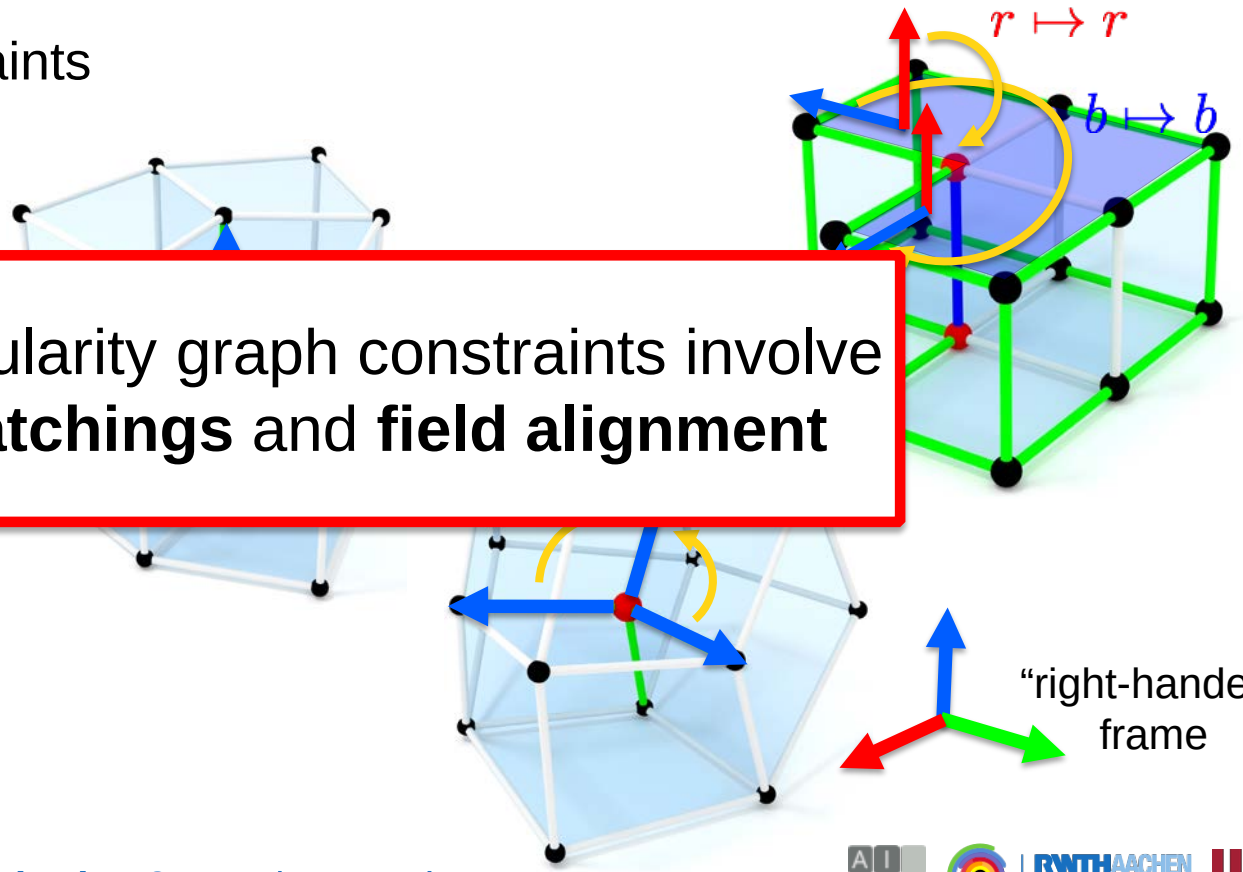




Singularity Graph Constraints

- Vertex type constraints

1. “axis-to-axis”
   → 4 possibilities
2. “3-axes-corne
   → 4 possibilities
3. “2-axes-to-2-a
   → unique



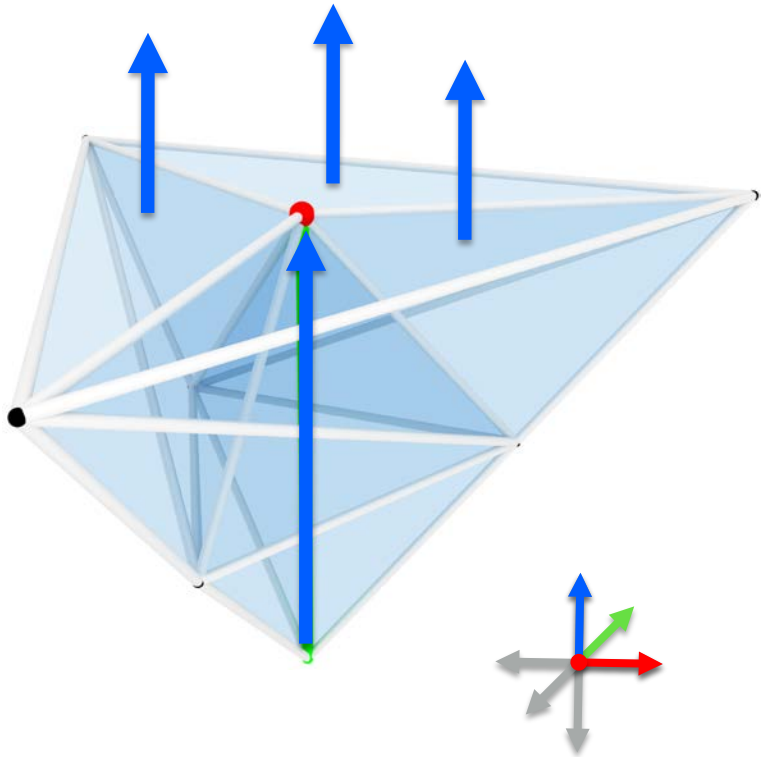
Singularity graph constraints involve **matchings** and **field alignment**

“right-handed” frame

Singularity Graph Constraints

- **Field alignment topologically matters for**
  - singular edges
  - surface normals

- **Idea:**
  - fix field alignment (arbitrarily) and then derive consistent matchings

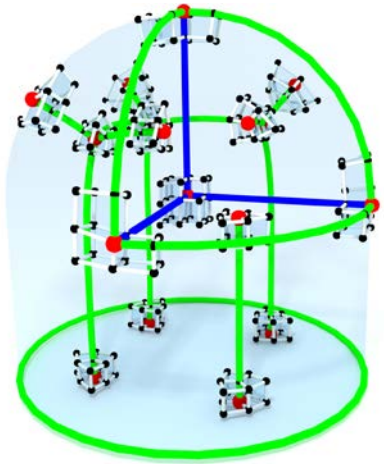


Simplified Algebraic System

- **Input**:
  - axis alignment constraints
  - one edge type constraint for each tetmesh edge
  - vertex type constraints between adjacent singularities

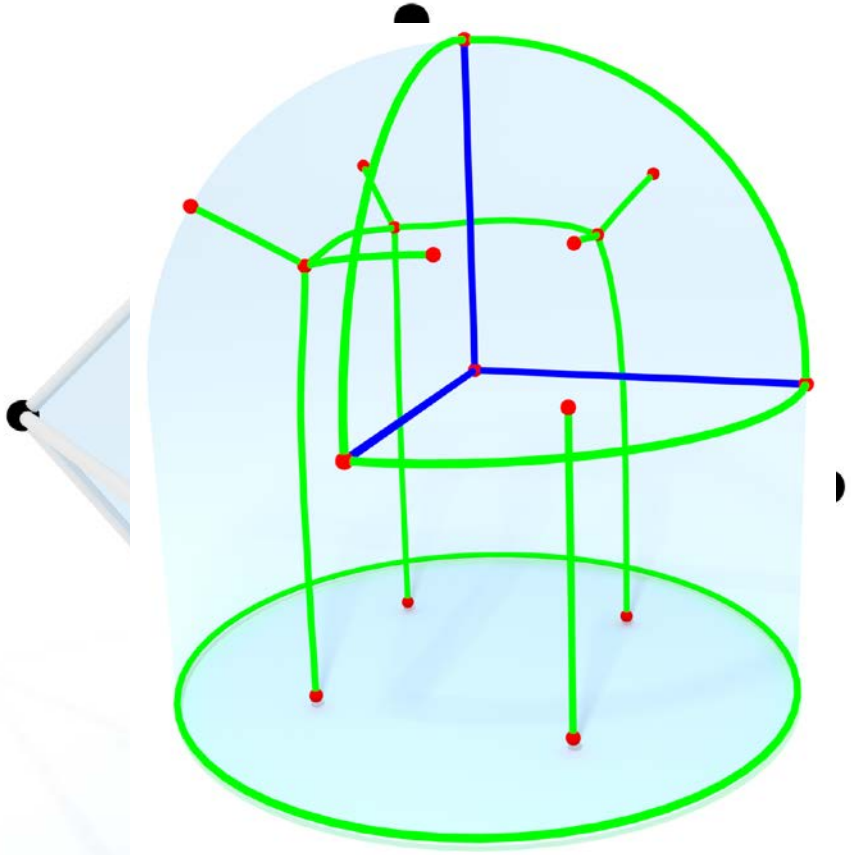
} difficult algebraic system

- **Task**:
  - find consistent matchings
- **Idea**:
  - proceed from where local constraint information is available
  1. locally unique matchings (“2-axes-to-2-axes”)
  2. partially constrained matchings(“axis-to-axis”)



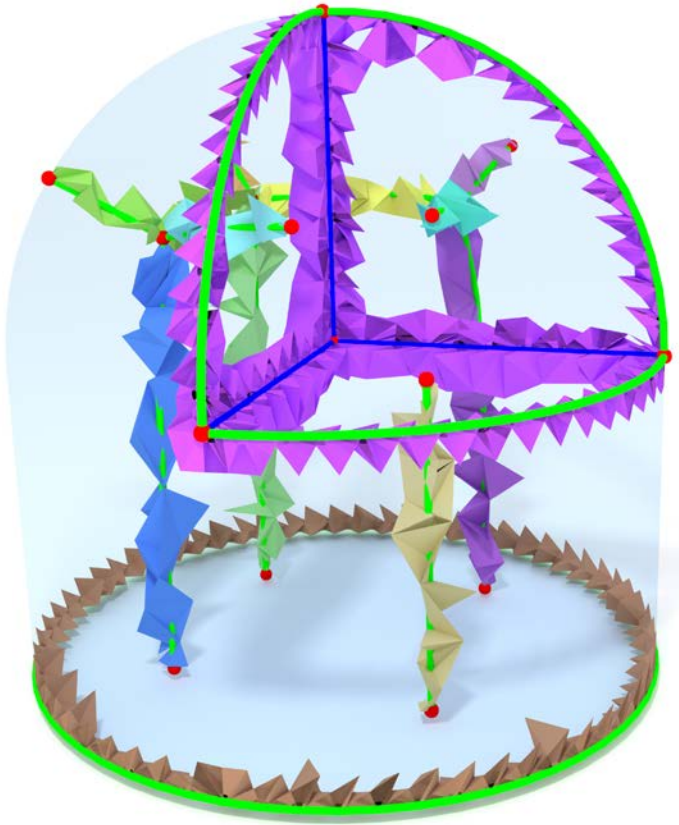
Algorithm: Chart-Merging

- **Algorithm**
  1. MergeSingularArcCharts
  2. MergeBoundaryCharts
  3. MergeVolumeCharts



Algorithm: Chart-Merging

- **Algorithm**
  1. MergeSingularArcCharts
  2. MergeBoundaryCharts
  3. MergeVolumeCharts



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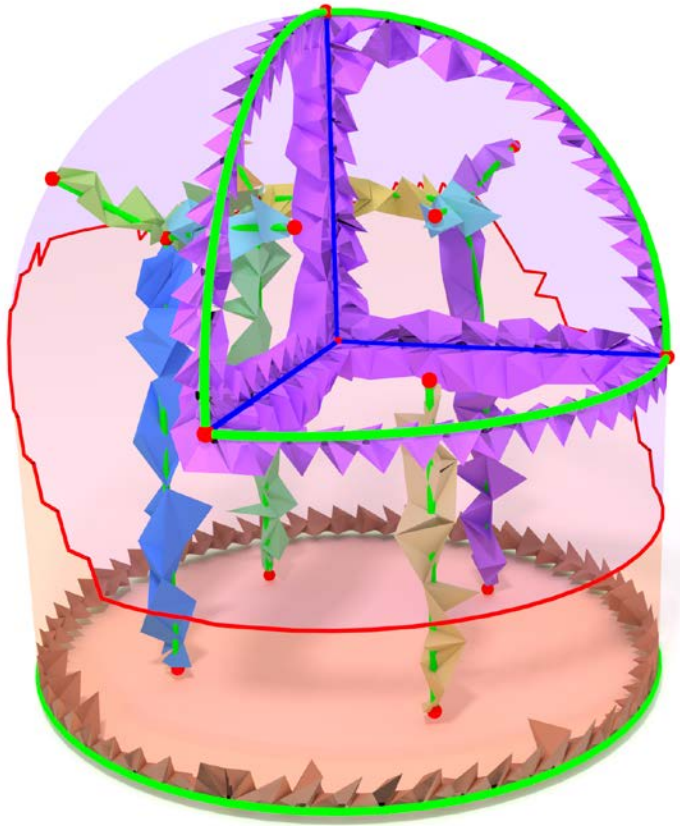

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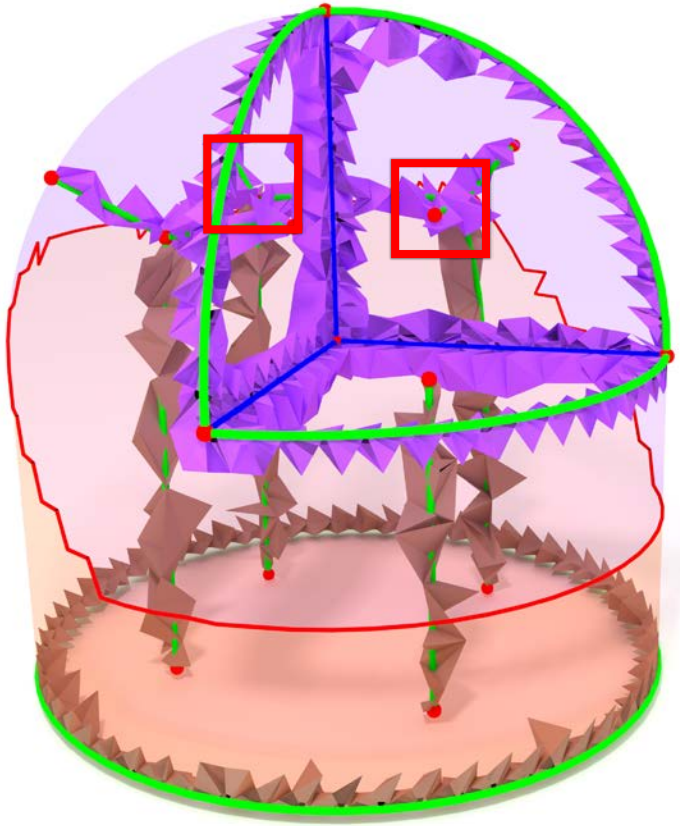

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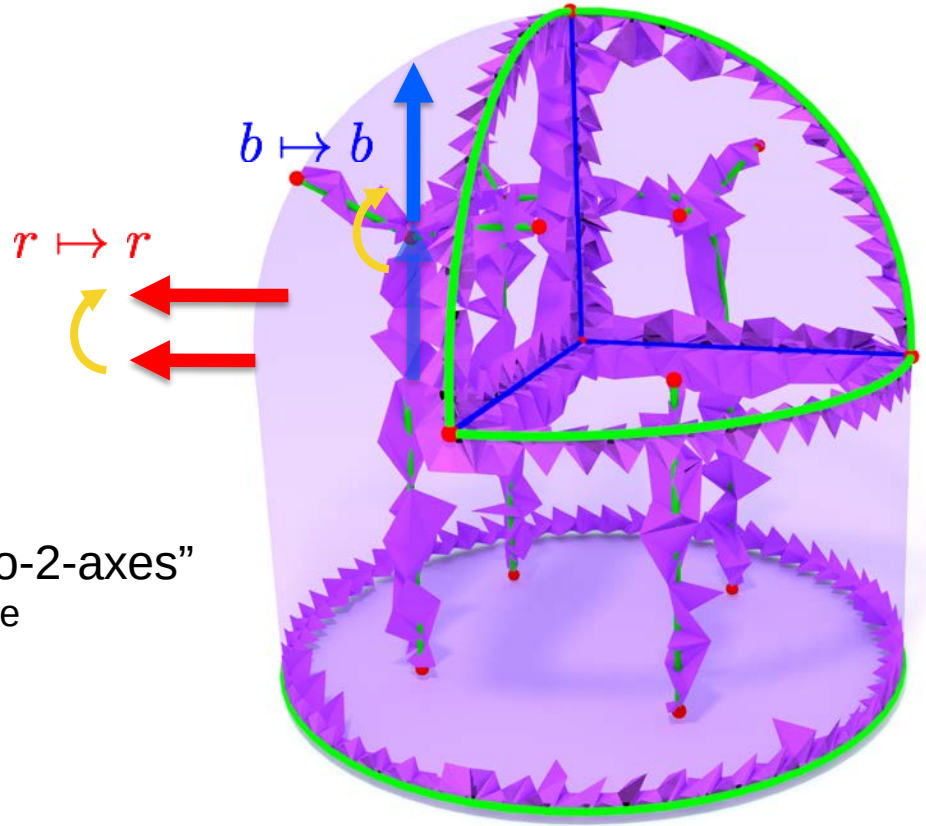
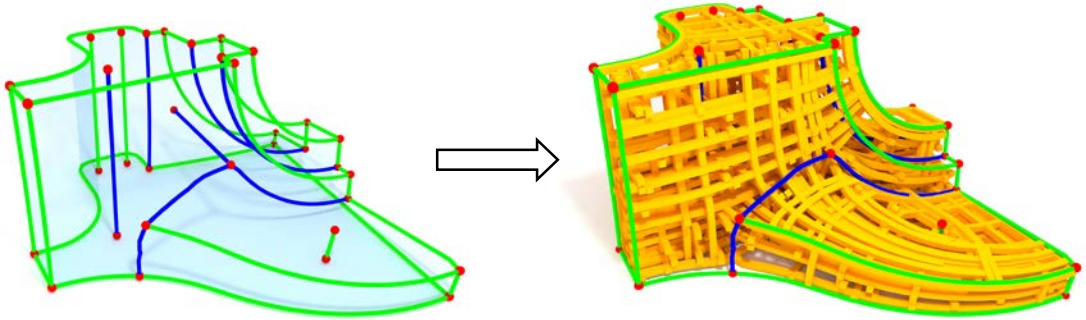


“2-axes-to-2-axes”
→ unique

Algorithm: Topological Field to Geometrical Field

- **Input:**
  - Axes alignment
  - Matchings



topological field　　geometrical field

minimize $\int_{\Omega} \| \nabla q \|^2 dV$

subject to: $A_i q_i = 0, q_i \in B$

$\| q_i \|^2 = 1, i = 1, \ldots, n$

Relaxation

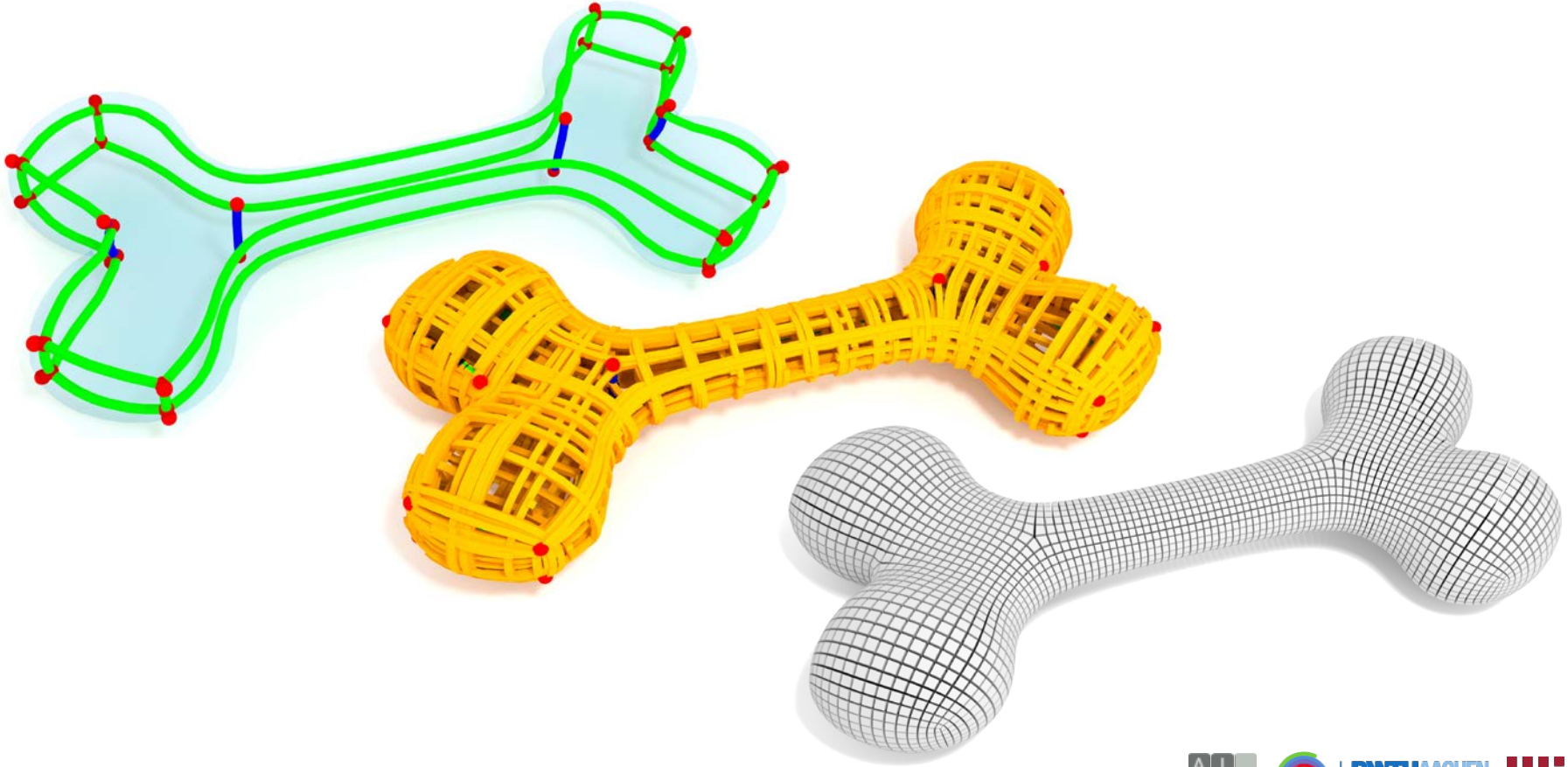
minimize $q^T Q q$

subject to: $\sum_i \| q_i \|^2 = n, i = 1, \ldots, n$

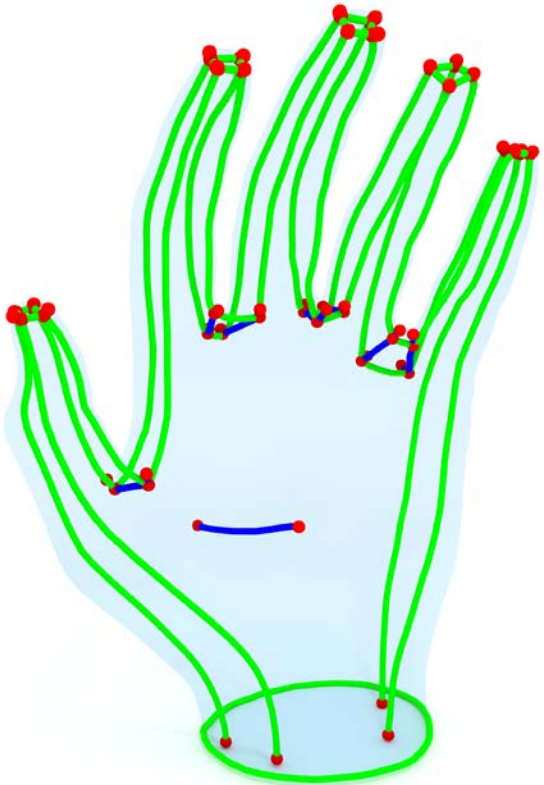
Eigenvalue Problem

Results

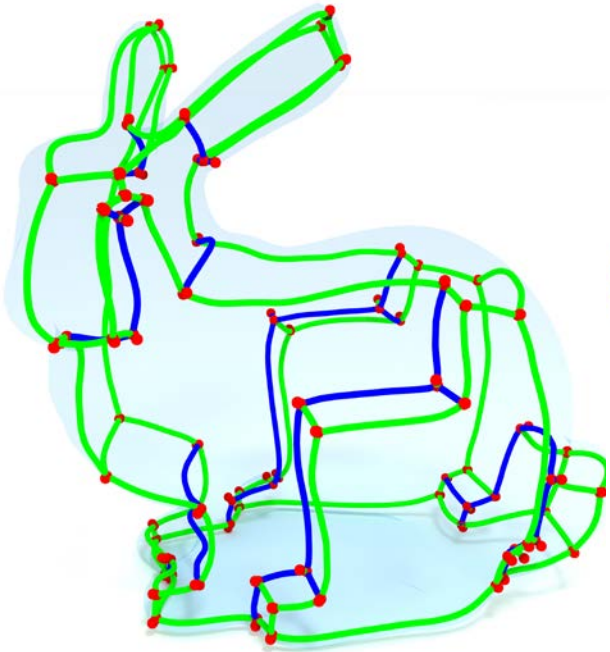
Results-Bone[Tets: 71k Time: 0.9s/54.0s]

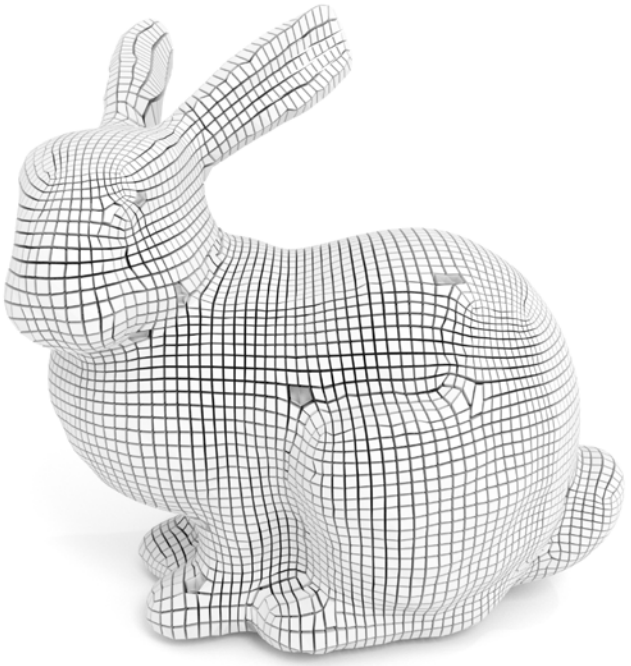


Results-Hand[Tets: 113k Time: 1.8s/34.9s]

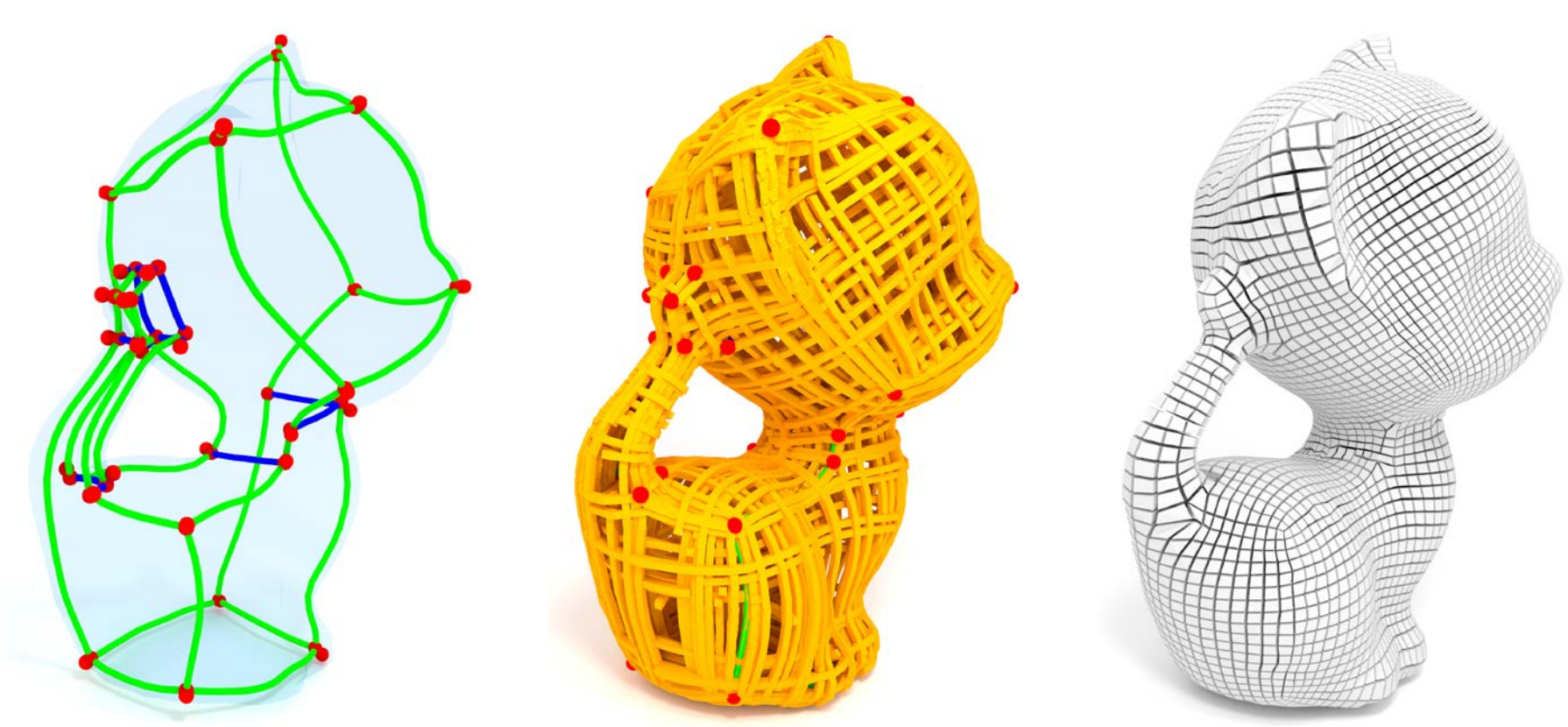


Results-Bunny[Tets: 130k Time: 2.7s/53.2s]

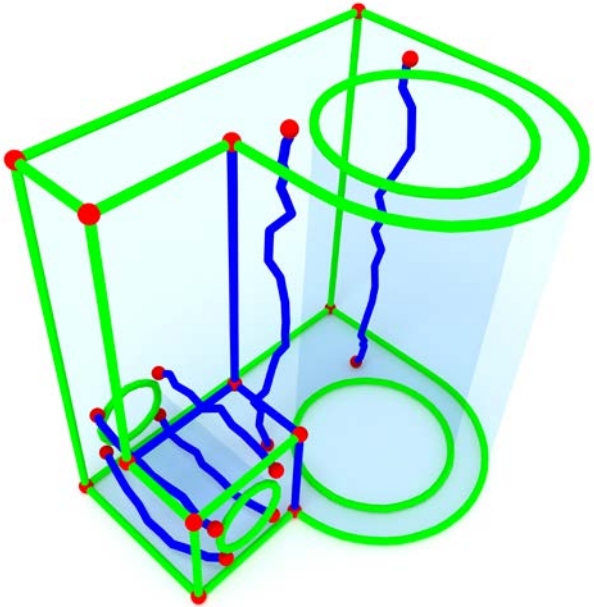


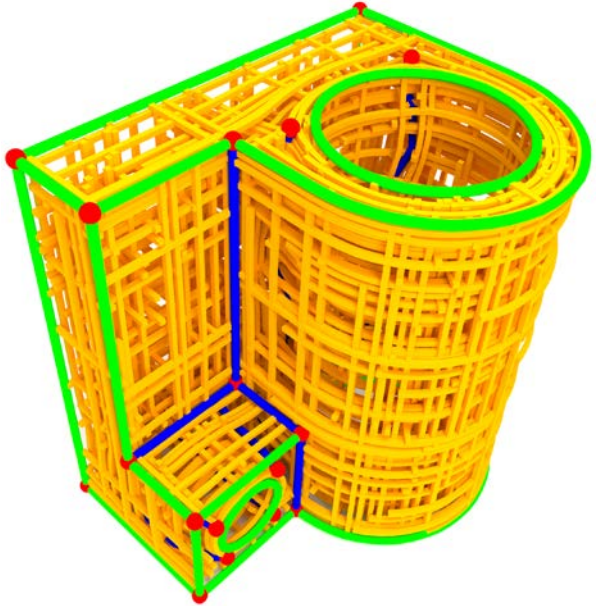


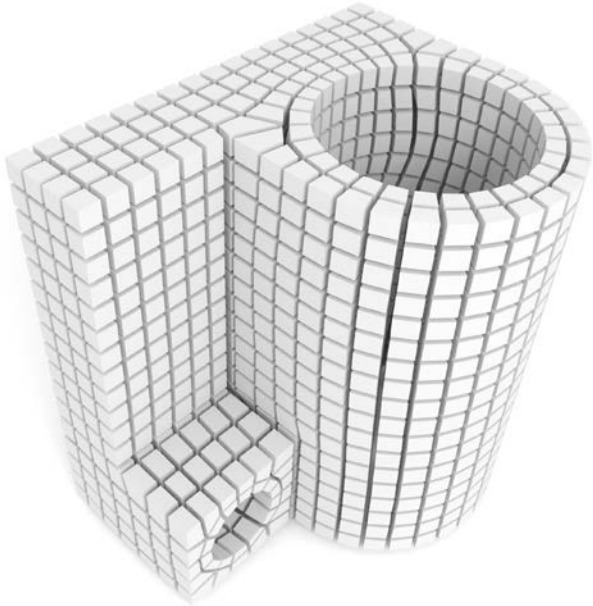
Results-Kitten[Tets: 55k Time: 1.8s/28.3s]



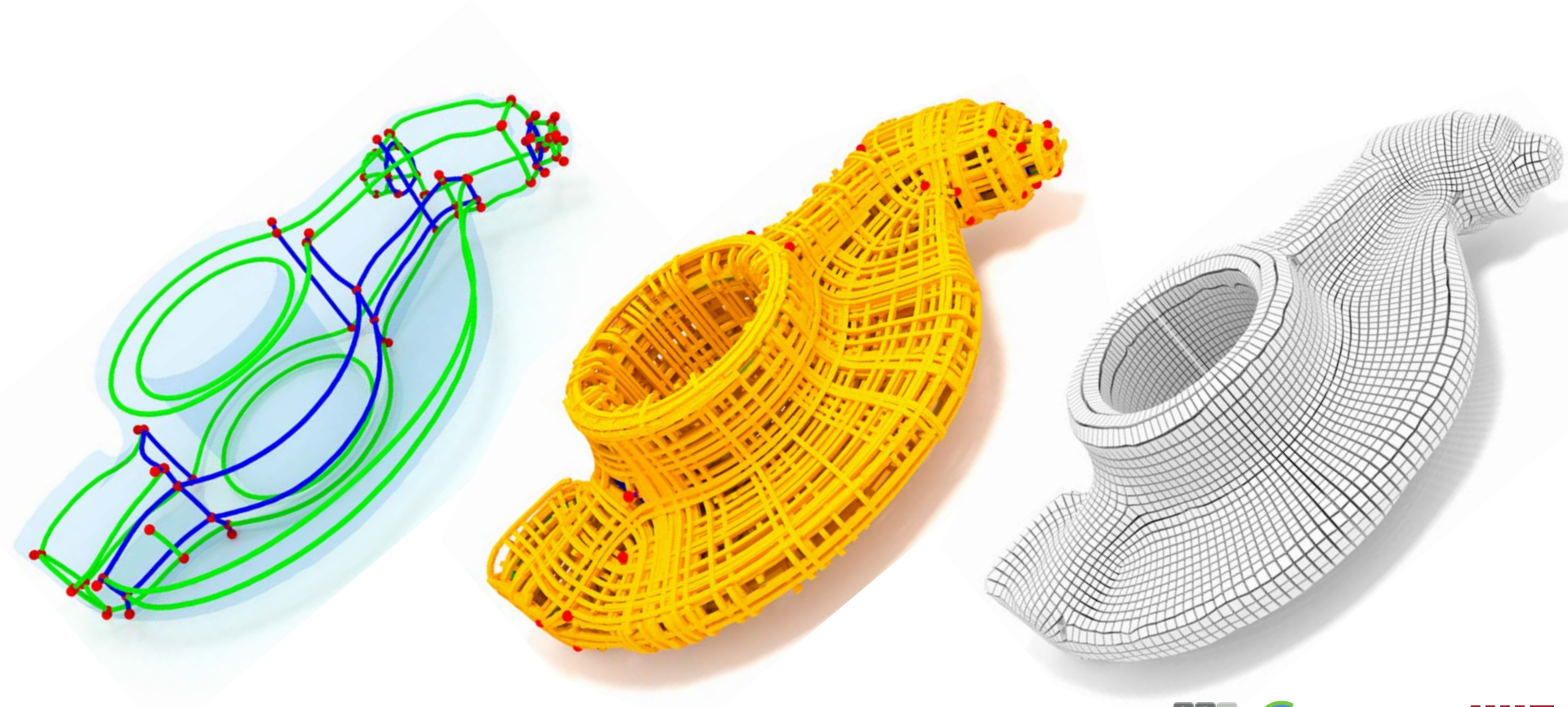
Results-Joint[Tets: 44k Time: 0.5s/13.6s]



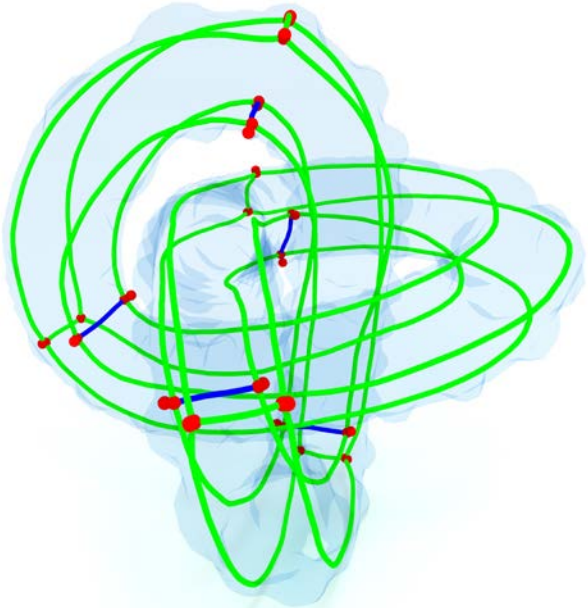


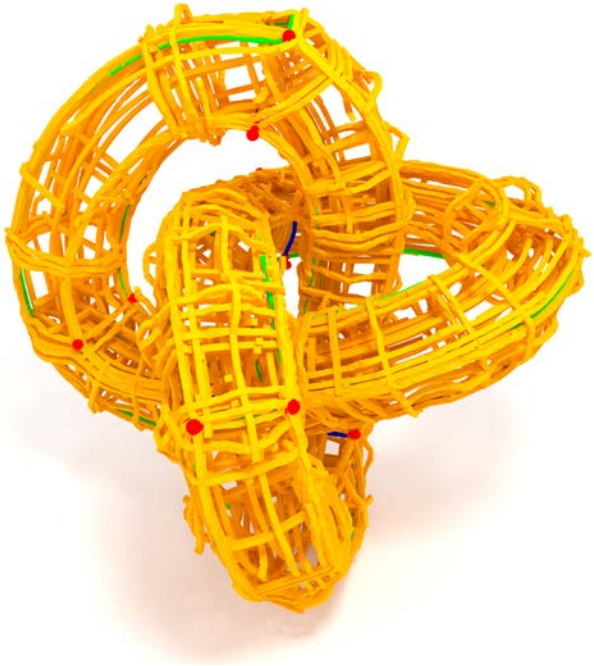


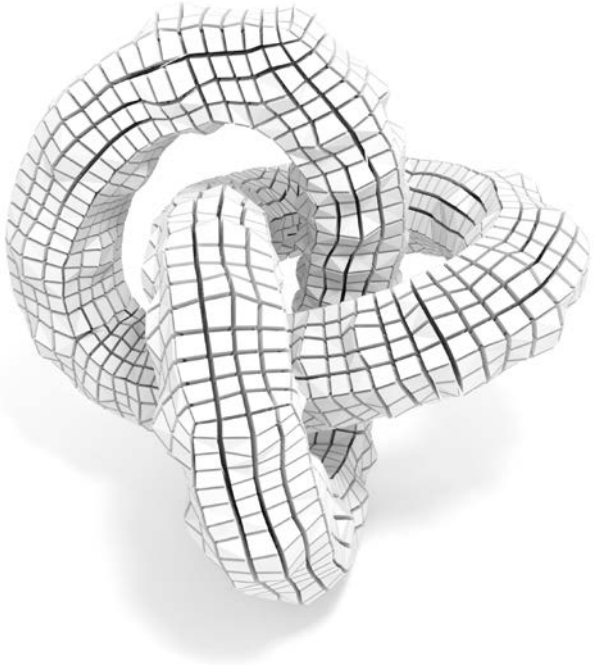
Results-Rockerarm[Tets: 122k Time: 3.1s/67.5s]



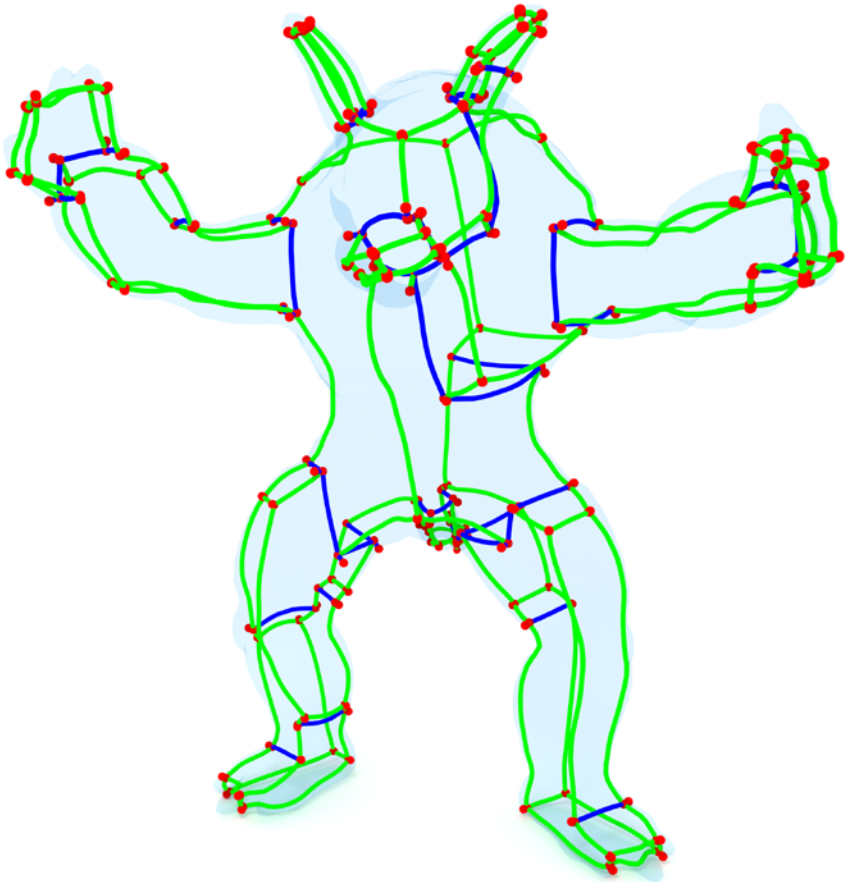
Results-Knot[Tets: 131k Time: 1.8s/47.3s]

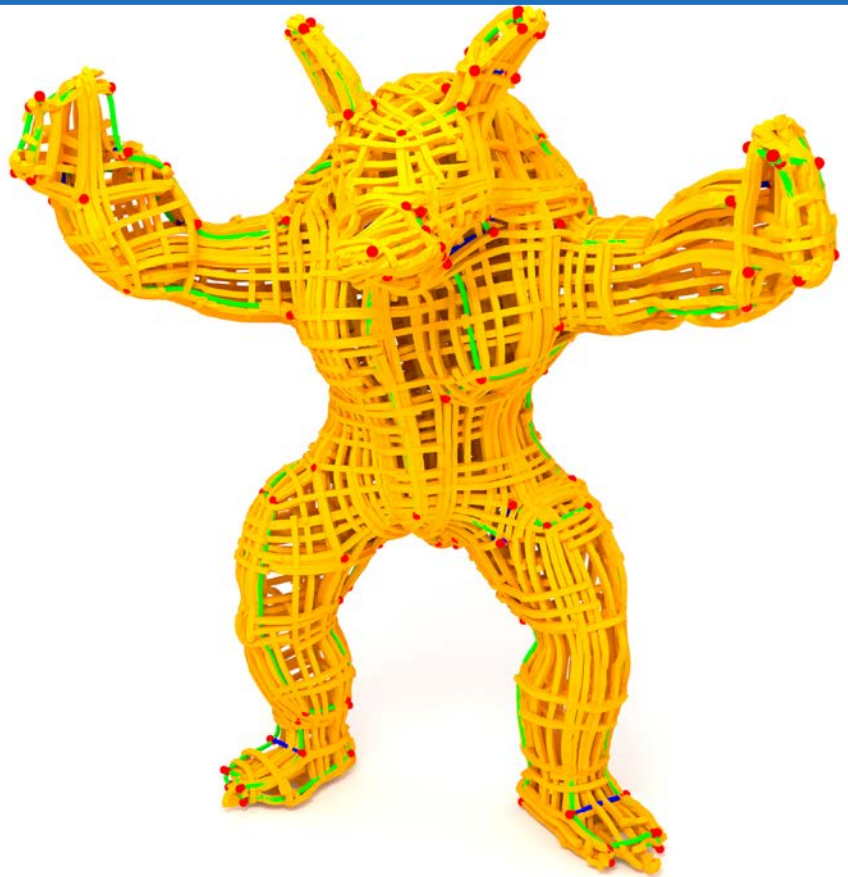




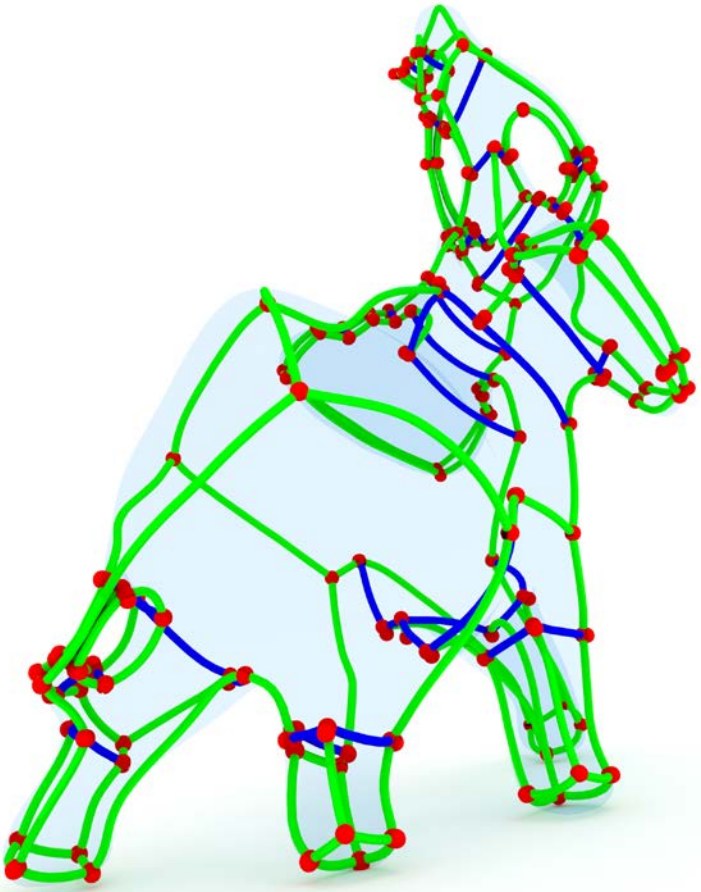


Results-Armadillo[Tets: 206k Time: 6.0s/83.4s]





Results-Elephant[Tets: 301k Time: 9.9s/88s]



Summary & Outlook

Summary & Outlook

- **Contribution**
  - Necessary conditions: hex mesh singularities with valence 3, 4, 5 (and some more)
  - Algorithm: octahedral-fields with fixed topology

- **Future Challenges**
  - Sufficient conditions: local consistency & global consistency
  - Automatic correction of invalid singularity graph
  - Robust integer-quantization & mapping ⇨ automatic high-quality hex meshing

Thank You!