Computing a High-Dimensional Euclidean Embedding from an Arbitrary Smooth Riemannian Metric

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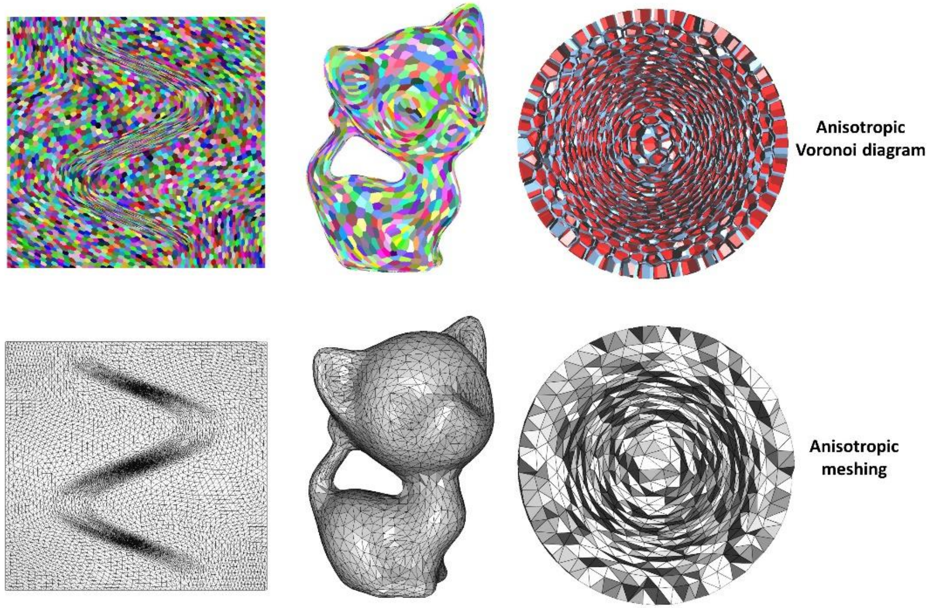

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Short Bio

- **Assistant Professor,** Department of Computer Science, *Wayne State University*, USA, 2015 – Now
- **Postdoc,** Department of Radiation Oncology, *UT Southwestern Medical Center*, USA, 2014 – 2015
- **Ph.D.,** Computer Science, *UTDallas*, USA, 2009 – 2014 (Advisor: Dr. Xiaohu Guo)
- **B.S. and M.S.,** Computer Science, *UESTC*, China**,** 2002 – 2006, 2006 – 2009

Research interests: Computer Graphics, Geometric Modeling (specifically Surface and Volume Mesh Generations), Medical Imaging Processing (specifically Deformable Image Registration, 3D/4D Image Reconstruction), Visualization, and GPU Algorithms

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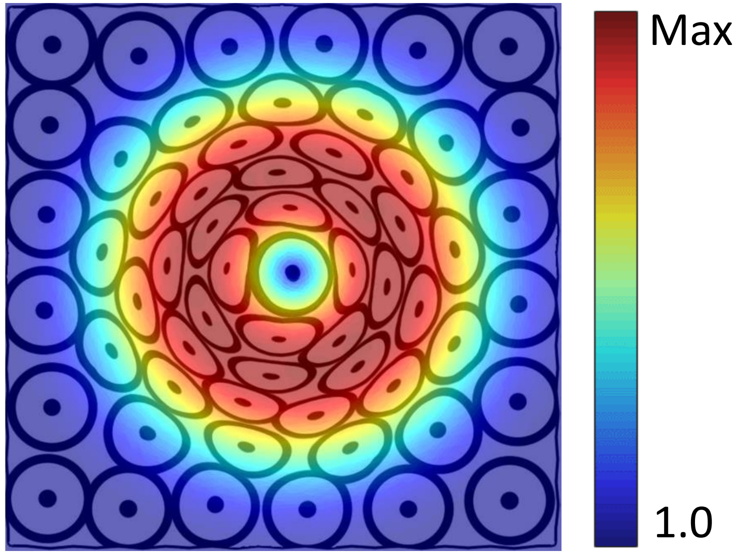
Introduction

Introduction

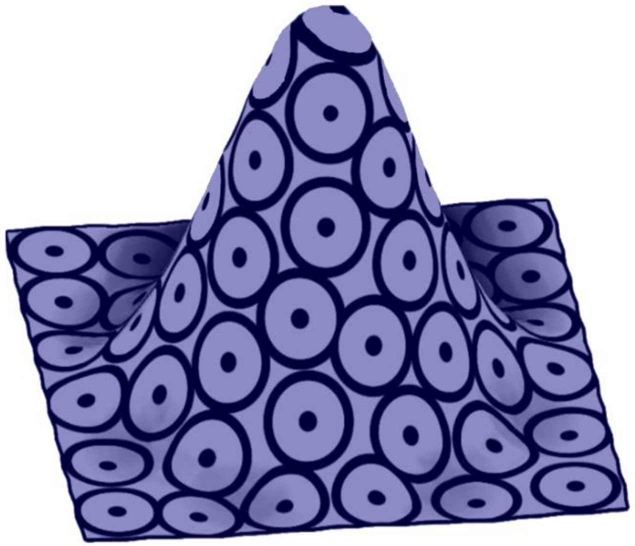
Nash embedding theorem [Nash 1954; Kuiper 1955]:



Ω

Original Riemannian manifold, i.e., 2D

is mapped to



$\bar{\Omega}$

Higher dimensional Euclidean embedded manifold, i.e., 3D

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Existing work on embeddings

- **Mathematical community (theoretical work):** isometric embedding for Riemannian manifolds

  [Gromov and Rokhlin 1970; Hong 1993; Han and Hong 2006; Borrelli et al. 2012; Gromov 2017] …

- **Computer graphics community:** surface meshing by embedding

  [Cañas and Gortler 2006; Boissonnat et al. 2008; Kovacs et al. 2010] …

- **Recent closest related work:**
  - Implicit embedding: *Particle-based anisotropic meshing [Zhong et al. 2013]*
  - Explicit embedding:
    - *3D embedding / immersion [Panozzo et al. 2014; Chern et al. 2018]*
    - *5D or 6D embedding [Dassi et al. 2014, 2015; Lévy and Bonneel 2012]*

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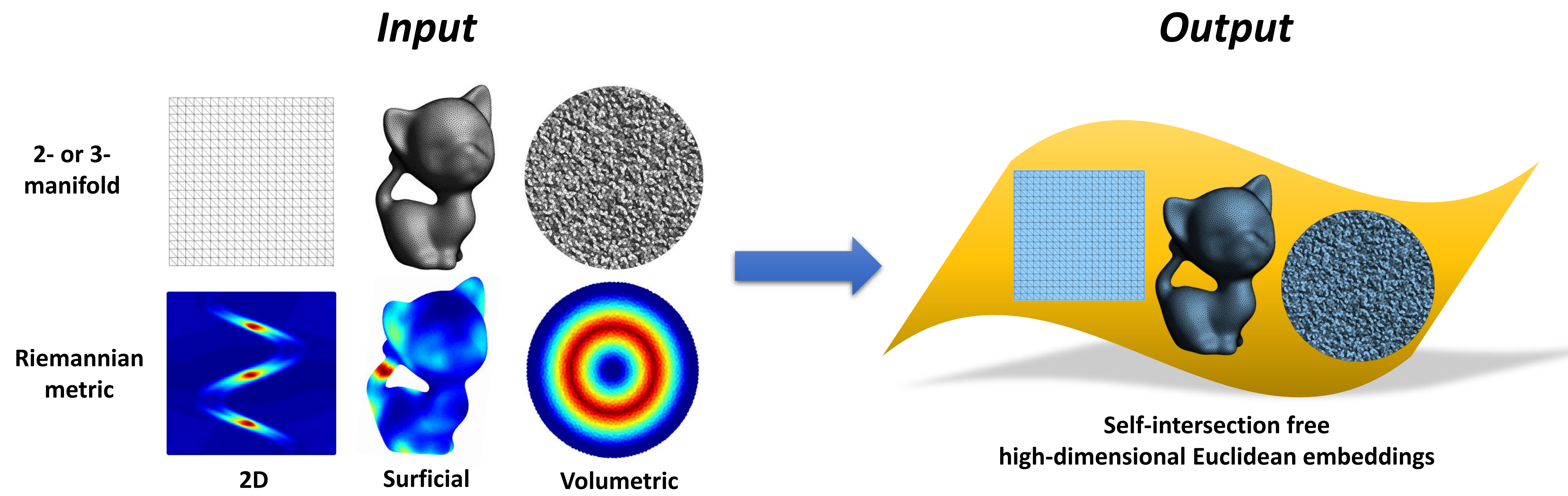
Motivations



Contributions

The proposed Euclidean embedding formulation: to minimize the deviation between the given metric and the deformation gradient of a map from the original surface / volume to the high-d embedded one

Contributions:

- A general *high-d* embedding framework for *arbitrary smooth Riemannian metric*
- An effective computational algorithm for *arbitrary topological* surface and *volume* manifolds
- The new computational strategies for *anisotropic meshing algorithms* in high-d Euclidean space

Our Method

Anisotropic Metric

- Anisotropy represents how distances and angles are distorted, which can be measured by the dot product in geometry:

$$<\mathbf{a}, \mathbf{b}>_{\mathbf{M}(\mathbf{x})} = \mathbf{a}^T \mathbf{M}(\mathbf{x}) \mathbf{b}$$

- A symmetric $m \times m$ matrix $\mathbf{M}(\mathbf{x})$ represents the metric:

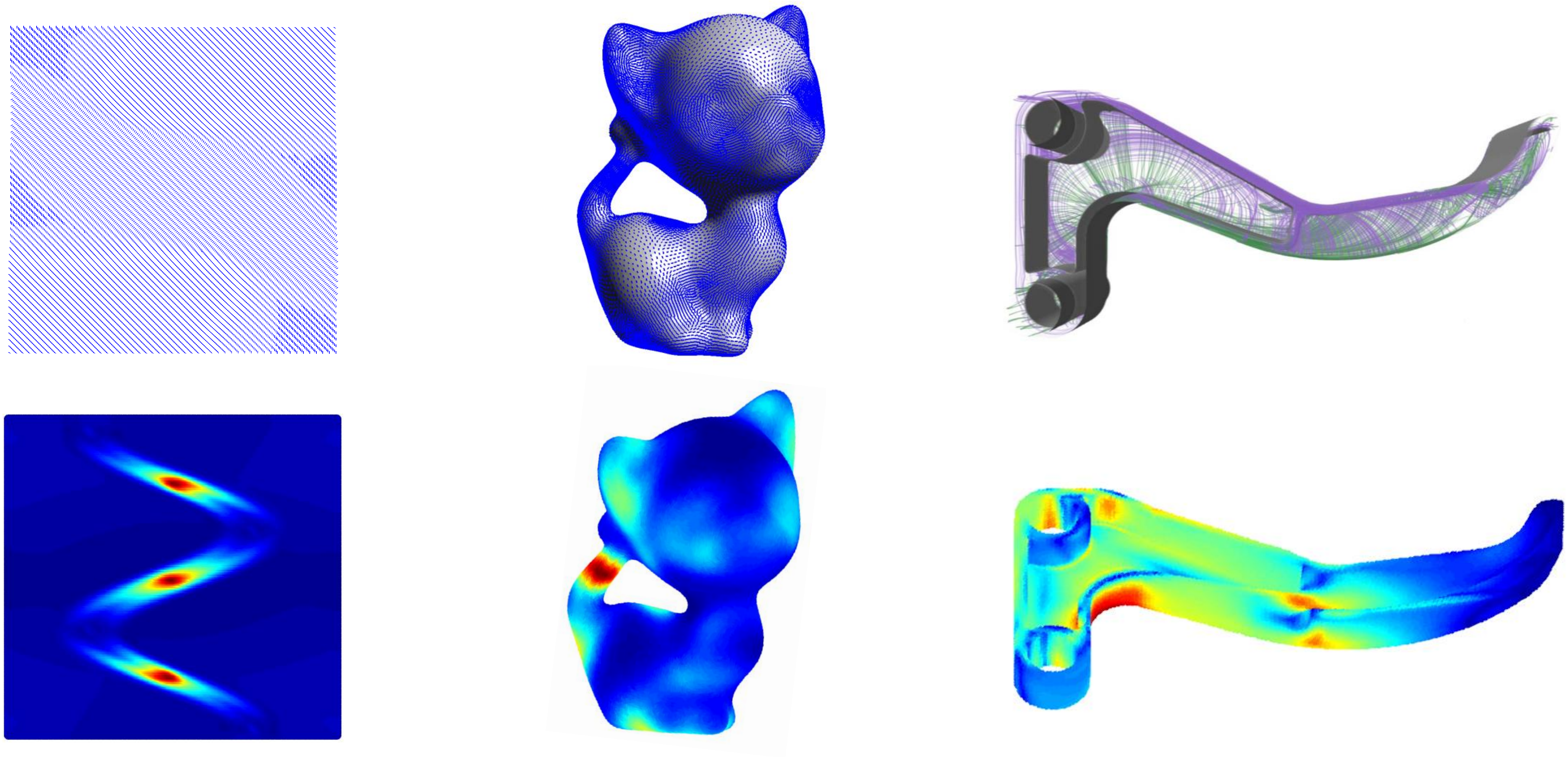
$$\mathbf{M}(\mathbf{x}) = \mathbf{R}(\mathbf{x})^T \mathbf{S}(\mathbf{x})^2 \mathbf{R}(\mathbf{x})$$

where the diagonal matrix $\mathbf{S}(\mathbf{x})$ is a scaling field, and the orthogonal matrix $\mathbf{R}(\mathbf{x})$ is a rotation field

Input Metrics

Directions
$\mathbf{R}(\mathbf{x})$

Scalings
$\mathbf{S}(\mathbf{x})$



2D analytic tensor

3D surface curvature tensor

3D stress tensor

Metric through High-D Embedding

For an arbitrary metric field **M**(**x**) defined on the surface or volume $\Omega \subset \mathbb{R}^m$(i.e., Riemannian 2- or 3-manifold):

Nash theorem [Nash 1954] states that there exists a high-d space $\mathbb{R}^{\bar{m}}$ (i.e., $m \leq \bar{m}$), in which the surface or volume can be embedded with Euclidean metric as $\bar{\Omega} \subset \mathbb{R}^{\bar{m}}$

Importance of High-D Embedding

(1) **More degrees of freedom**: to deform and embed the given surface or volume -> to obtain better embedding quality

(2) **Avoid self-intersections:** of the embedded surface or volume, instead of embedding them in the original space (e.g., 2D or 3D)

(3) **Simplify several Riemannian geometric applications**: such as computing high-quality anisotropic RVD and meshing on surface and volume by using only isotropic Euclidean computations

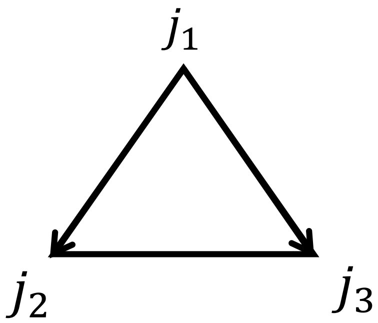
High-D Embedding Transformation

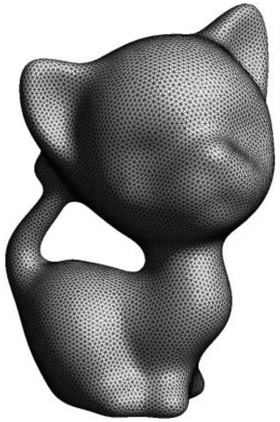
- For a triangle / tetrahedron *j* on the original surface / volume Ω , and high-d embedded $\bar{\Omega}$, the matrices of corresponding edge vectors are:

Surface:

$$\mathbf{W}_j = [\mathbf{v}_{j_2} - \mathbf{v}_{j_1}, \mathbf{v}_{j_3} - \mathbf{v}_{j_1}]$$

$$\bar{\mathbf{W}}_j = [\bar{\mathbf{v}}_{j_2} - \bar{\mathbf{v}}_{j_1}, \bar{\mathbf{v}}_{j_3} - \bar{\mathbf{v}}_{j_1}]$$

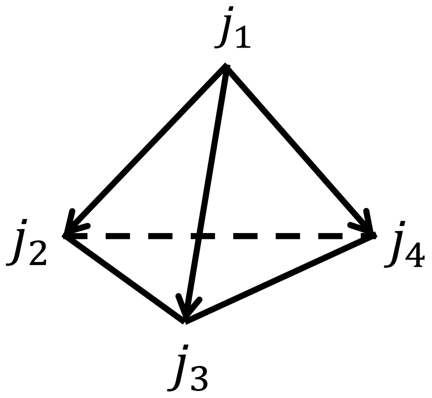


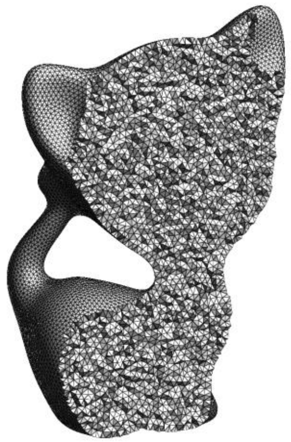


Volume:

$$\mathbf{W}_j = [\mathbf{v}_{j_2} - \mathbf{v}_{j_1}, \mathbf{v}_{j_3} - \mathbf{v}_{j_1}, \mathbf{v}_{j_4} - \mathbf{v}_{j_1}]$$

$$\bar{\mathbf{W}}_j = [\bar{\mathbf{v}}_{j_2} - \bar{\mathbf{v}}_{j_1}, \bar{\mathbf{v}}_{j_3} - \bar{\mathbf{v}}_{j_1}, \bar{\mathbf{v}}_{j_4} - \bar{\mathbf{v}}_{j_1}]$$



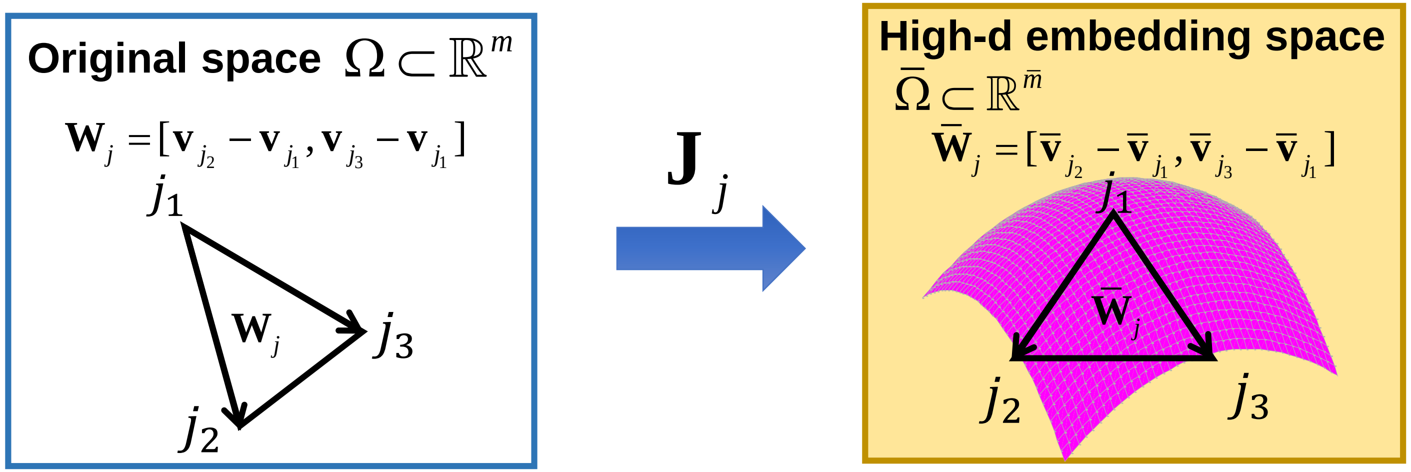


High-D Embedding Transformation

- Their relationship can be represented as:

$$\bar{\mathbf{W}}_j = \mathbf{J}_j \mathbf{W}_j$$

where $\mathbf{J}_j$ is the *Jacobian transformation* matrix for triangle or tetrahedron *j*, and $\mathbf{J}_j^T \mathbf{J}_j = \mathbf{M}_j$

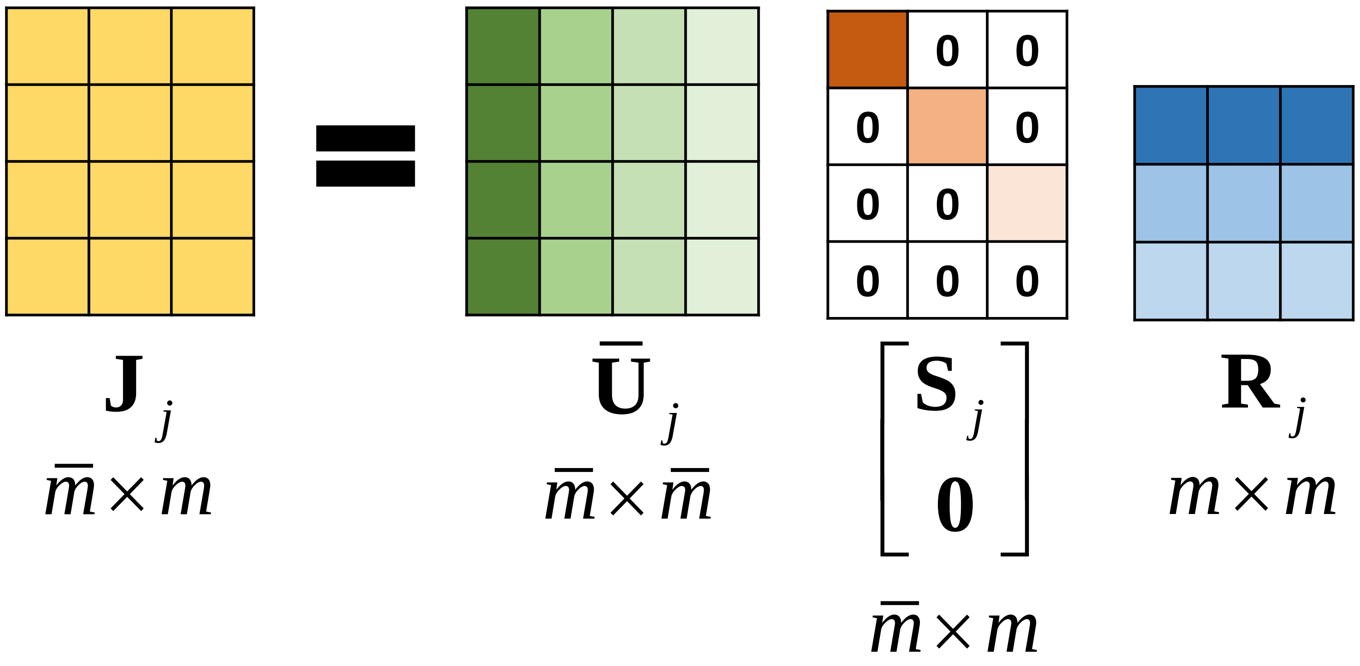


High-D Embedding Transformation

- $\mathbf{J}_j$ is an $\bar{m} \times m$ matrix, and is represented as the product of a rotation in the high-d embedding space, and a scaling and rotation in the original space:

$$\mathbf{J}_j = \bar{\mathbf{U}}_j \begin{bmatrix} \mathbf{S}_j \mathbf{R}_j \\ \mathbf{0} \end{bmatrix} = \bar{\mathbf{U}}_j \mathbf{Q}_j$$

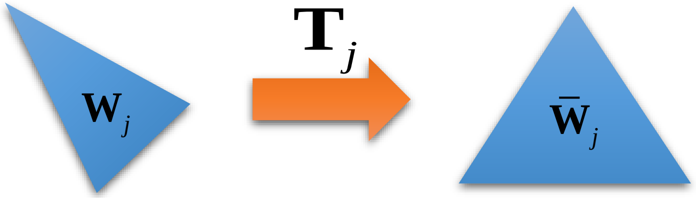
where $\mathbf{Q}_j = \begin{bmatrix} \mathbf{S}_j \mathbf{R}_j \\ \mathbf{0} \end{bmatrix}$



High-D Deformation Gradient

- Intuitively, the transformation between the original surface / volume and its high-d embedded one can be considered as the deformation. It is represented by the field of the *deformation gradient* over the surface / volume:

$$\mathbf{T}_j \mathbf{W}_j = \bar{\mathbf{W}}_j$$



where it is intuitively shown by Sumner and Popović [2004]

Surface: $\mathbf{T}_j = \bar{\mathbf{W}}_j \mathbf{W}_j^{+}$ ($\mathbf{W}_j^{+}$: pseudoinverse of $\mathbf{W}_j$)

Volume: $\mathbf{T}_j = \bar{\mathbf{W}}_j \mathbf{W}_j^{-1}$

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Embedding Optimization

- In essence, the high-d embedding transformation $\mathbf{J}_j$ and the high-d deformation gradient $\mathbf{T}_j$ are the same

- We can formulate an expression to minimize the function E_{em}:

$$E_{em}(\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_{n_v}) = \min \sum_{j=1}^{n_{ele}} \left\| \mathbf{T}_j - \mathbf{J}_j \right\|_F^2$$

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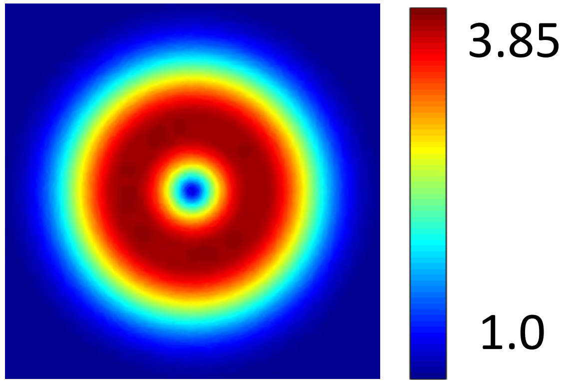
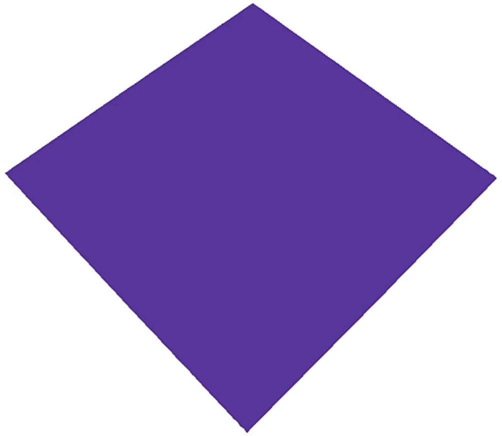
Embedding Optimization

Surface: $$E_{em}(\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_{n_v}) = \min \sum_{j=1}^{n_{ele}} \left\| \bar{\mathbf{W}}_j \mathbf{W}_j^{+} - \bar{\mathbf{U}}_j \mathbf{Q}_j \right\|_F^2$$

Volume: $$E_{em}(\bar{\mathbf{v}}_1, \ldots, \bar{\mathbf{v}}_{n_v}) = \min \sum_{j=1}^{n_{ele}} \left\| \bar{\mathbf{W}}_j \mathbf{W}_j^{-1} - \bar{\mathbf{U}}_j \mathbf{Q}_j \right\|_F^2$$



A 2D domain with anisotropic metric



A 3D embedding

Regularization Term

- The regularity term E_{reg} is a summation of the square of graph Laplacian operations over every vertex in the embedding space:

$$E_{reg}(\overline{\mathbf{v}}_1,\ldots,\overline{\mathbf{v}}_{n_v}) = \sum_{i=1}^{n_{v-b}} \sum_{d=3,4}^{\overline{m}} \left(\frac{\sum_{k \in N(i)} (\overline{v}_k^d - \overline{v}_i^d)}{|N(i)|} \right)^2$$

  where n_{v-b} is the total number of vertices excluding those on the boundaries. *N (i)* is the set of one-ring neighbors of vertex *i*

- To force the embedding to be C^2 smoothness

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Regularized Objective Function

- The embedding optimization includes: the *similarity* between two transformations and the *regularity* used to achieve smoothness of the embedding:

$$E_{total} = E_{em} + \mu E_{reg}$$

where μ is a weighting factor to balance the similarity and regularity terms during optimization. The order of magnitude of μ is 2 in our experiments.

Avoiding Intersections

- According to Nash embedding theorem, using the mapping $\Omega \rightarrow \overline{\Omega}$:

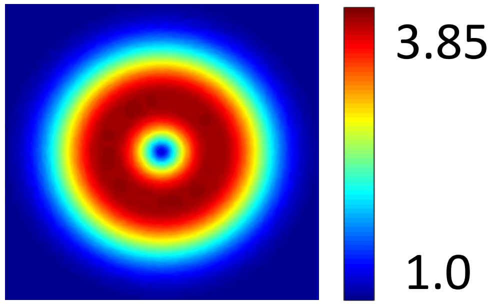
$$\mathbf{v}^{1:m} \rightarrow (\mathbf{v}^{1:m}, \overline{\mathbf{v}}^{m+1}, \dots, \overline{\mathbf{v}}^{\bar{m}})$$

- We keep the original 2D / 3D coordinates to automatically avoid self-intersections in the high-d embedding

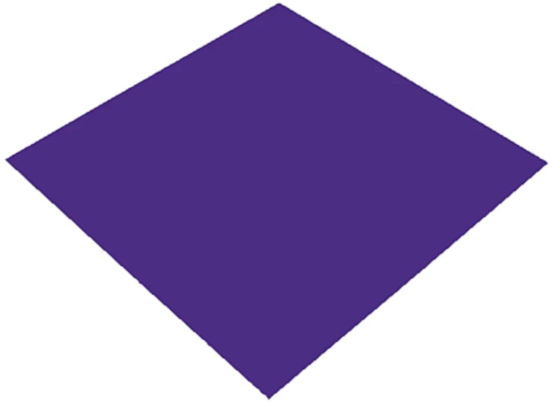
- Note: the Euclidean distance in the high-d space will be at least longer than or equal to the original mesh: we multiply the target Riemannian metric **M** by a suitable global constant (scaling), if any stretching factors are less than one

Numerical Solution Mechanism

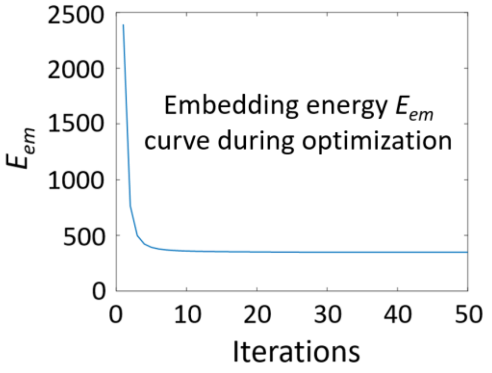
- A non-linear problem with two unknown parameters: $\overline{\mathbf{W}}_j$ and $\overline{\mathbf{U}}_j$
- An iterative method is used to compute the optimal solution



A 2D domain with anisotropic metric



A smooth 3D embedding



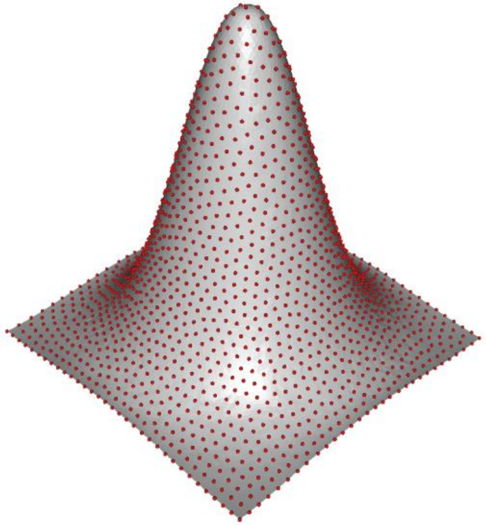
Anisotropic Computations: *High-D Particle Optimization*

- Extend 3D inter-particle energy formulation [Zhong et al. 2013] to high-d case:

$$\bar{E}^{pq} = e^{-\frac{\left\|\bar{\mathbf{x}}_p - \bar{\mathbf{x}}_q\right\|^2}{4\sigma^2}}$$

where $\sigma = 0.3\sqrt[d]{|\bar{\Omega}| / n}$, n is the number of particles, d = 2 in surface and d = 3 in volume, $|\bar{\Omega}|$ denotes the area or the volume of the embedded manifold

- *Advantages:*
  - ➢ *compute the isotropic particle distribution efficiently (inter-particle formulation)*
  - ➢ *search neighboring particles with efficient K-NN (Euclidean embedding space)*



Uniform particle distribution on a high-d embedded surface

Anisotropic Computations: *High-D Restricted Voronoi Diagram*

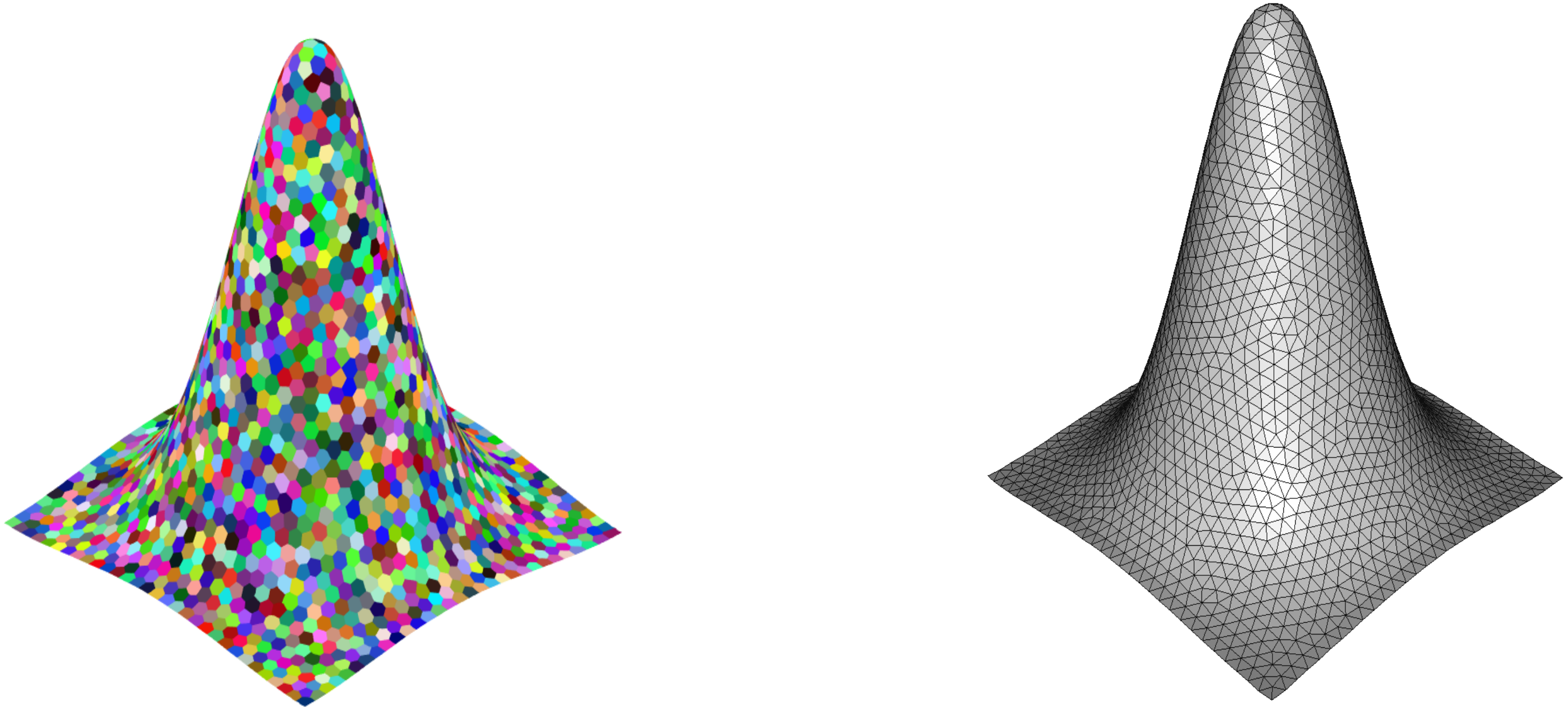
- To identify the high-d Voronoi cells that overlap each triangle / tetrahedron of the embedded surface / volume and compute their intersections

- The RVD computation is based on Lévy and Bonneel's method [2012] with the exact geometric predicates from [Lévy 2016], and then extended in high-d space

- All these computations are done under the Euclidean metric, which is easy and efficient

Anisotropic Computations:
High-D Meshing

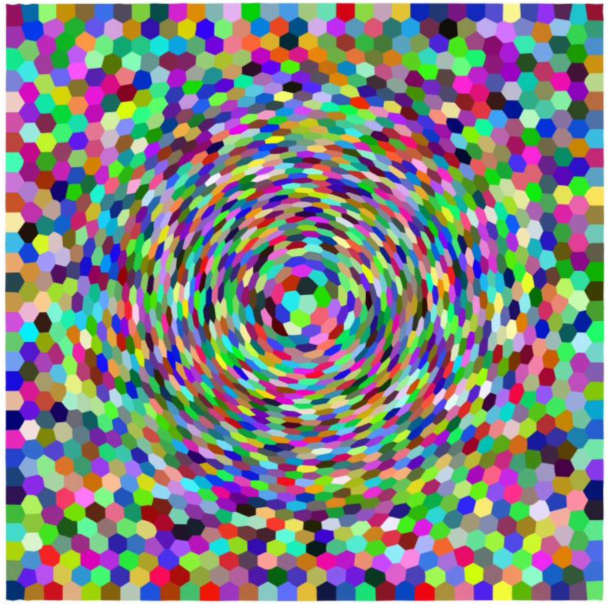
- Once the RVD is obtained, we can easily compute its dual graph, i.e., Restricted Delaunay Triangulation (RDT)

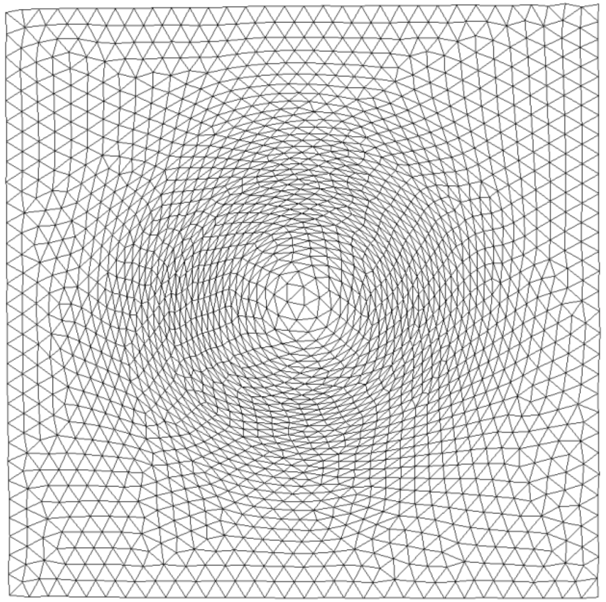


RVD and its dual mesh on a high-d embedded surface

Anisotropic Computations: *Anisotropic RVD and Mesh*

- To generate the final anisotropic RVD and mesh: using the barycentric coordinates of each output site or vertex, we can back-project the RVD and RDT from the high-d embedding space onto the original space





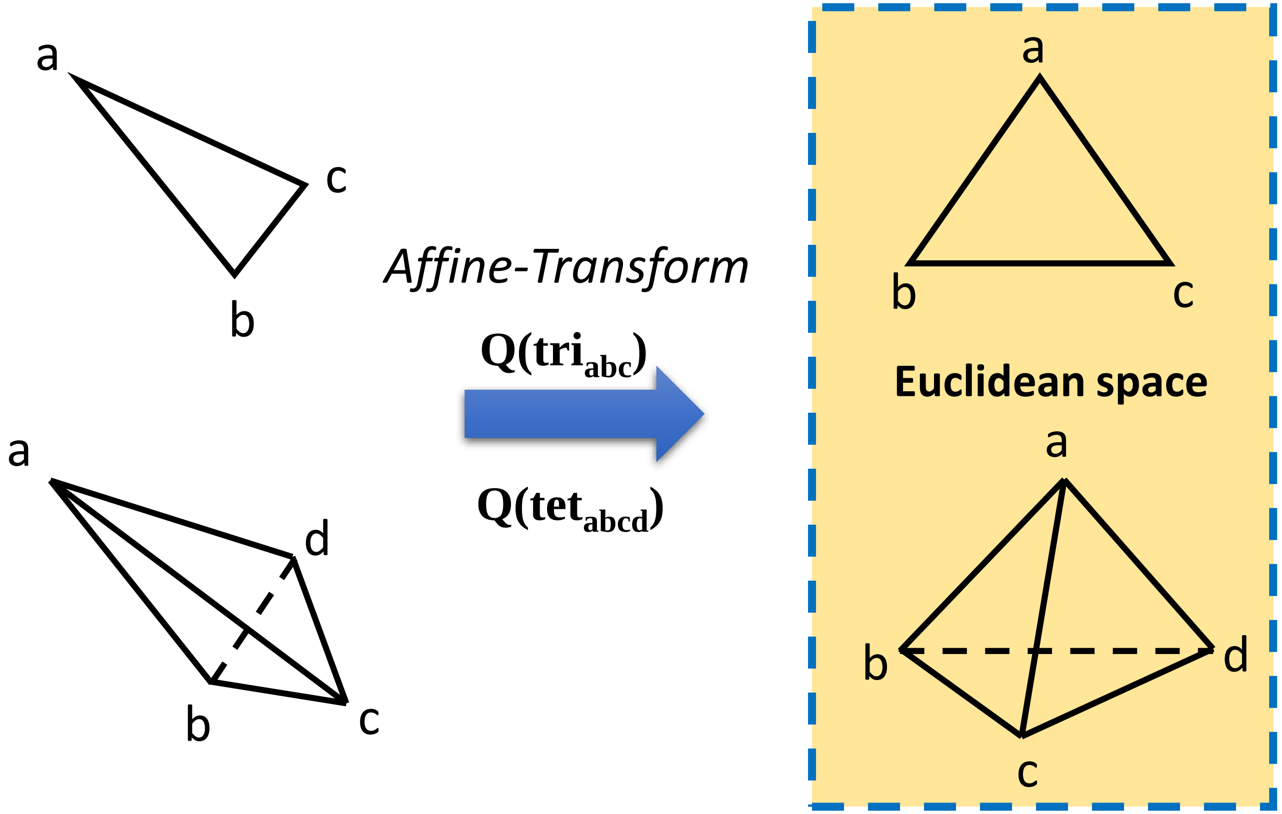
Mapping the RVD and its dual mesh to the original anisotropic metric domain

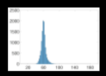
Evaluations: Embedding Quality

- *The relative edge length error* is the percentage of the absolute difference between the ground truth and the edge length of computed embedding with respect to the ground truth

- L_{max}^{rel} and L_{avg}^{rel}: the maximal and average values of relative edge length errors of all embedded triangles / tetrahedrons are evaluated

Evaluations: Anisotropic Mesh Quality



G_{min}	**Minimal quality** of triangles / tetrahedrons
G_{avg}	**Average quality** of triangles / tetrahedrons
θ_{min}	**Smallest value** of the minimal (dihedral) angles
θ_{avg}	**Average value** of the minimal (dihedral) angles
$\%_{<30\,/\,15^\circ}$	**Percentage** of triangles / tetrahedrons with their minimal (dihedral) angles smaller than 30° (triangles) / 15° (tetrahedrons)
	**Angle histogram:** distribution of all (dihedral) angles

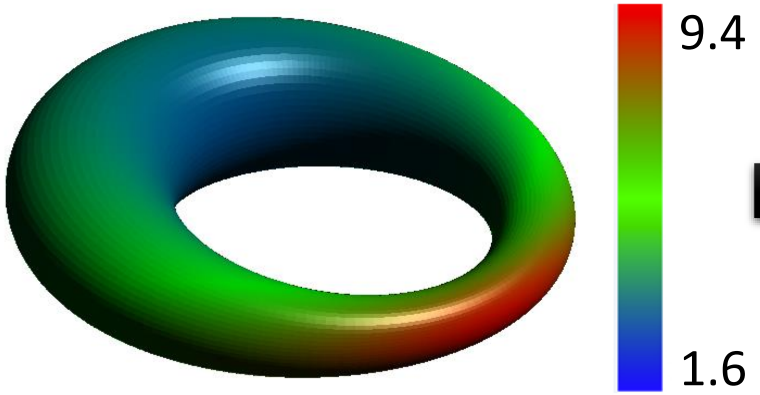
The **quality of a triangle**: $G = 2\sqrt{3}S/(ph)$ where S is the triangle area, p is its half-perimeter, and h is the length of its longest edge

The **quality of a tetrahedron**: $G = 12\sqrt[3]{9V^2}/\sum l_{i,j}^2$ where V is the tetrahedron volume, $l_{i,j}$ is the length of the edge

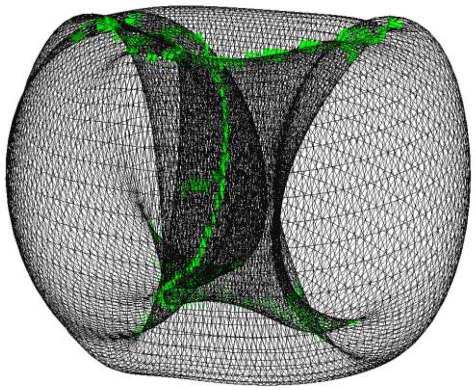
Results

Importance of Higher Dimensions

Stretching ratio



3D surface with anisotropic metric



3D embedding
(Green faces are self-intersecting)

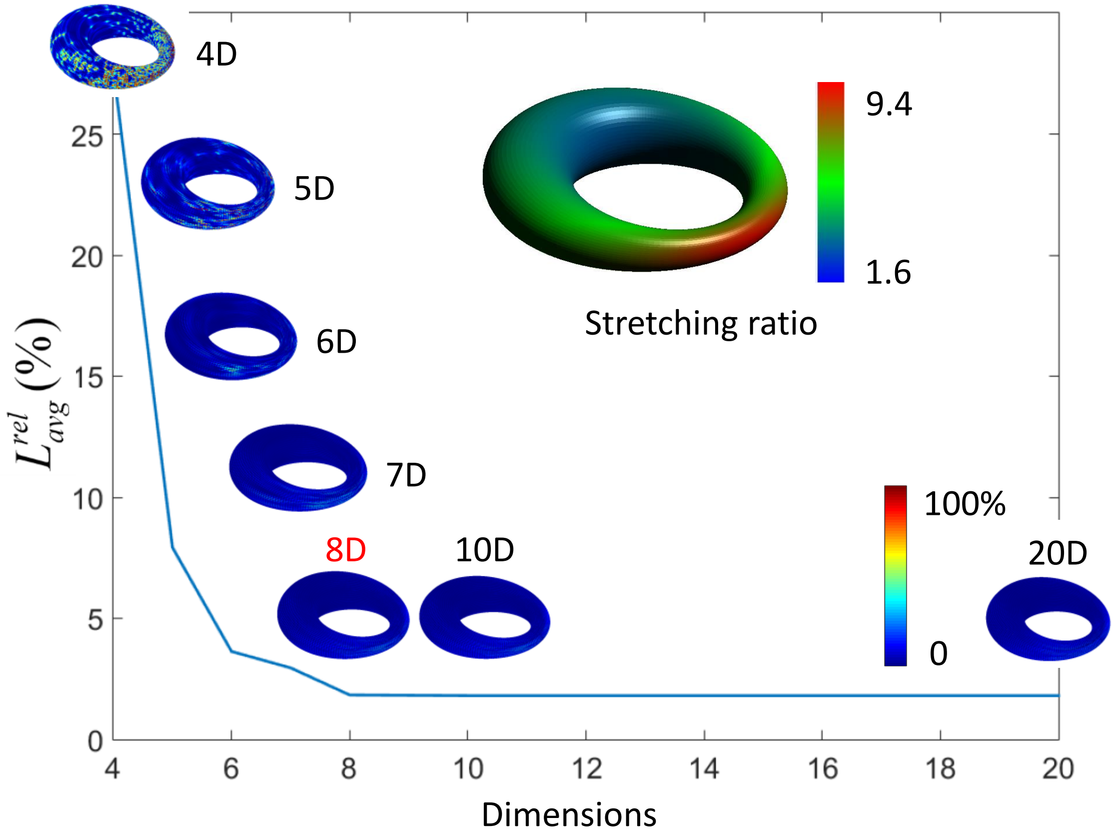
3D embedding result [Panozzo et al. 2014] of a Cyclide surface. There are 1146 self-intersecting faces out of total 21,600 faces as shown in green color

Table 1. Statistics (i.e., numbers and percentages) of self-intersecting faces for embeddings in 3D and high-d spaces on different surfaces.

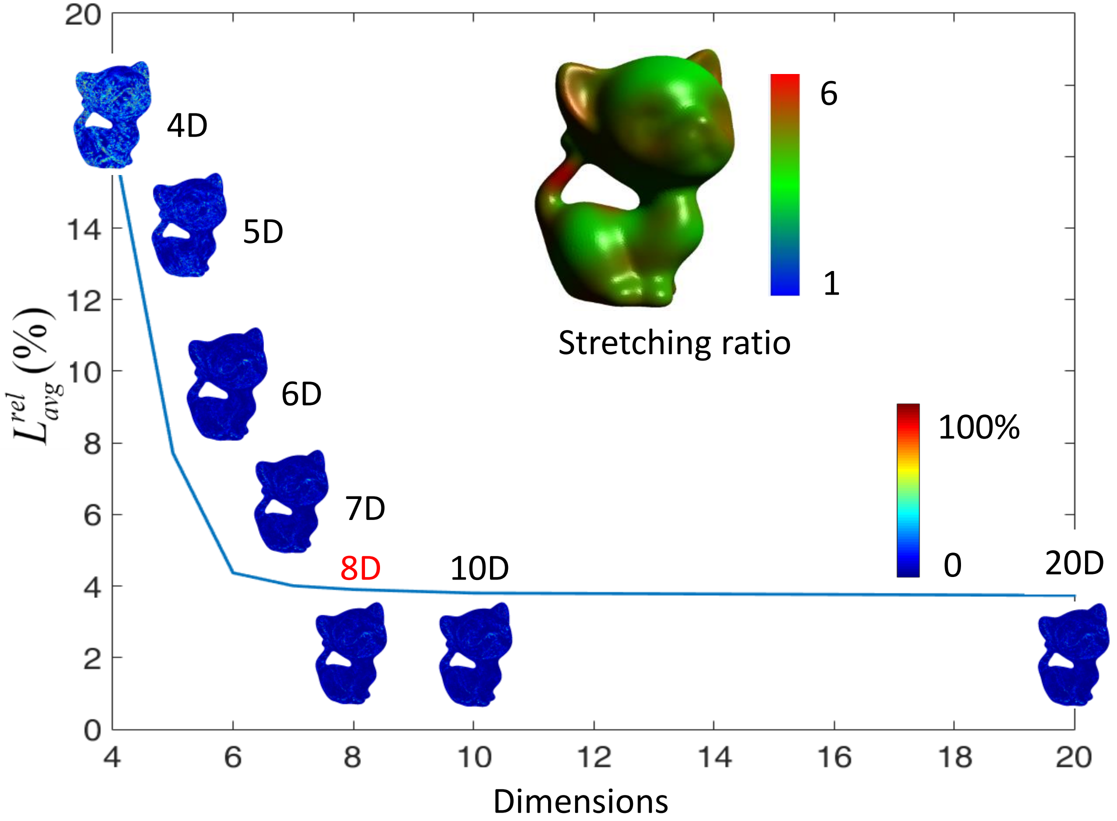
Model	Cyclide1	Cyclide2	Kitten	Gargo	Upright	Nefertiti
3D	1146 5.31%	1751 3.38%	1001 2.50%	2405 2.40%	3584 2.38%	1385 5.61%
High-D	0	0	0	0	0	0

Choosing the Dimension of the Embedding

Surface Examples



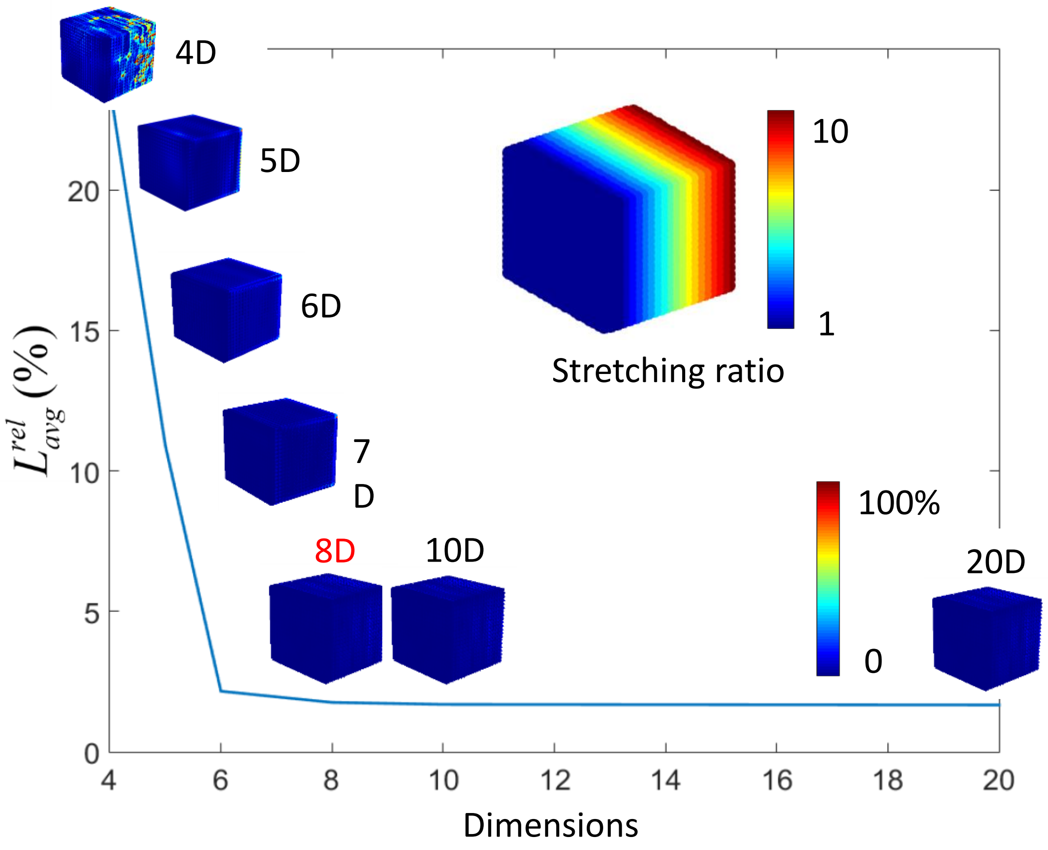
8D: $L_{avg}^{rel} = 1.83\%, L_{max}^{rel} = 25.81\%$

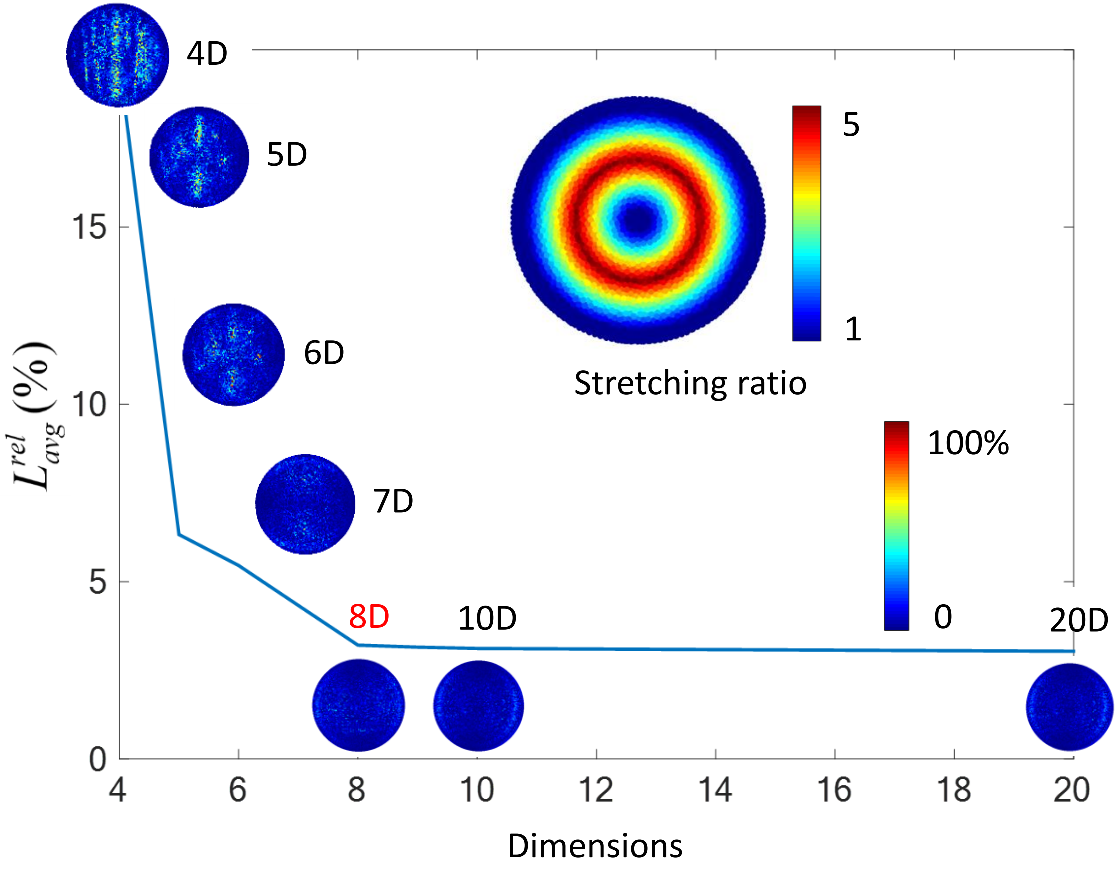


8D: $L_{avg}^{rel} = 3.82\%, L_{max}^{rel} = 139.21\%$

Choosing the Dimension of the Embedding

Volume Examples

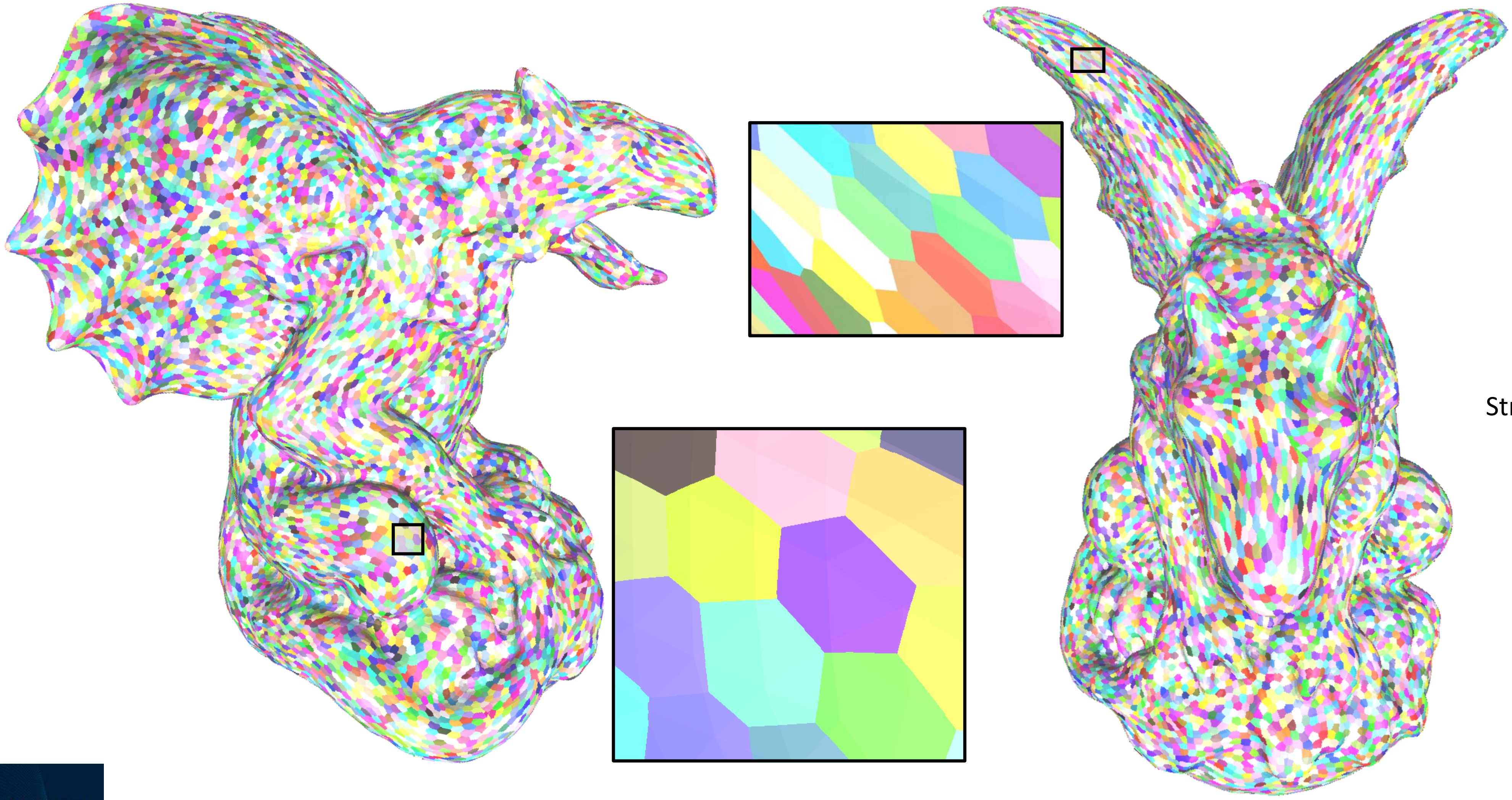




8D: $L_{avg}^{rel} = 1.65\%, L_{max}^{rel} = 17.54\%$

8D: $L_{avg}^{rel} = 3.21\%, L_{max}^{rel} = 65.01\%$

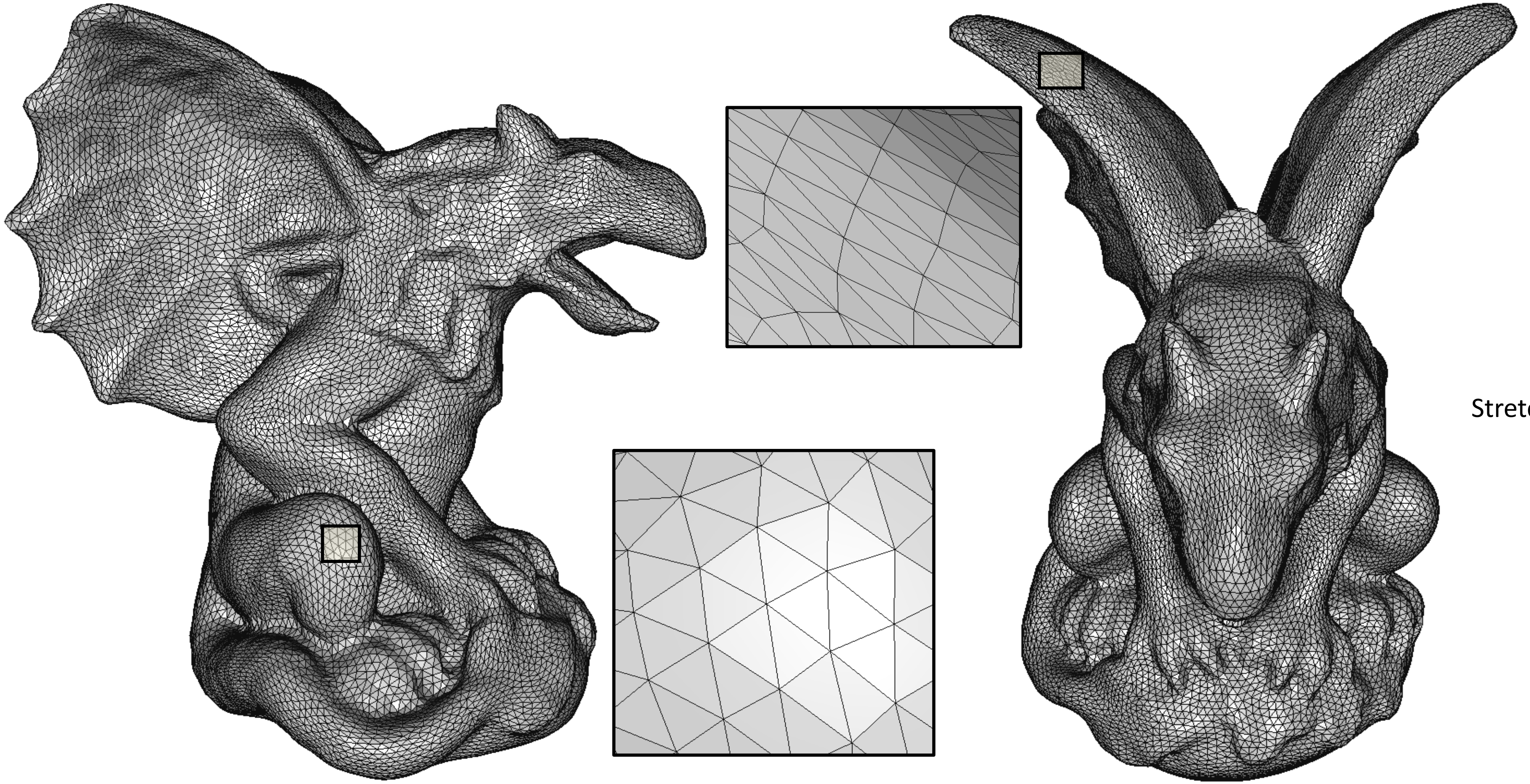
3D Surface RVD and Meshing: Curvature Metrics



Stretching ratio ∈ [1, 7]

Gargo: Anisotropic 3D surface RVD

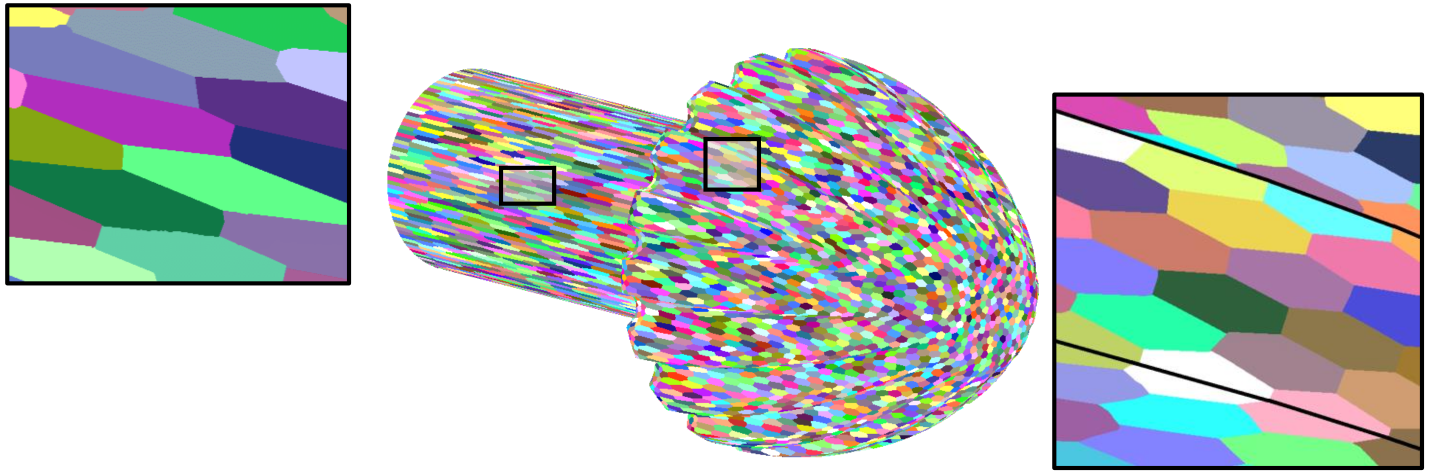
3D Surface RVD and Meshing: Curvature Metrics



Stretching ratio ∈ [1, 7]

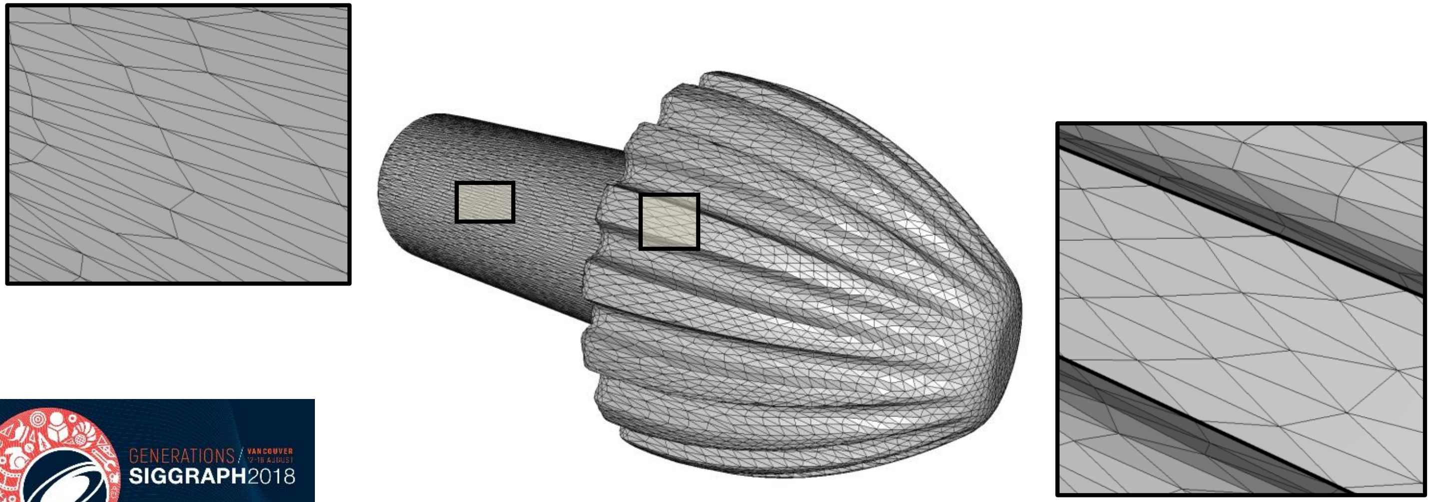
Gargo: Anisotropic 3D surface mesh

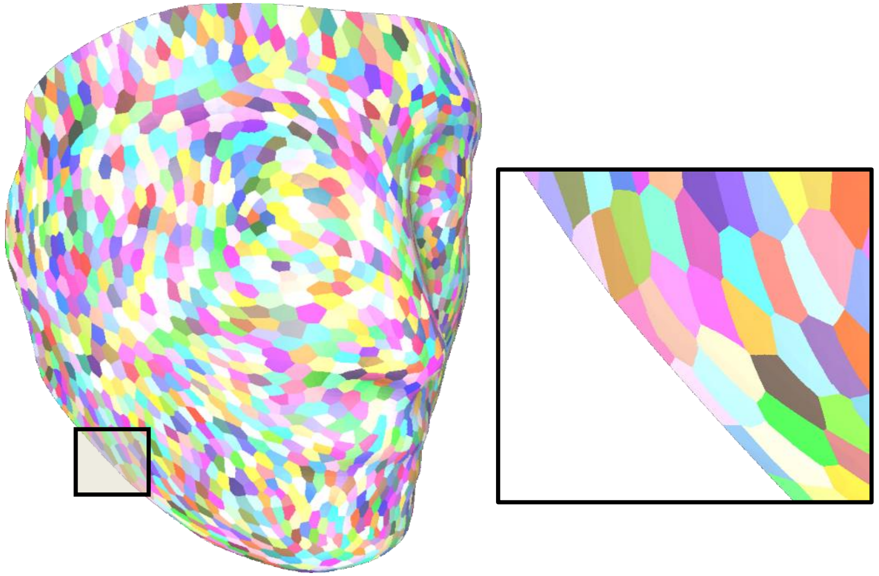
Sharp Feature and Boundary Models



Stretching ratio ∈ [1, 10]

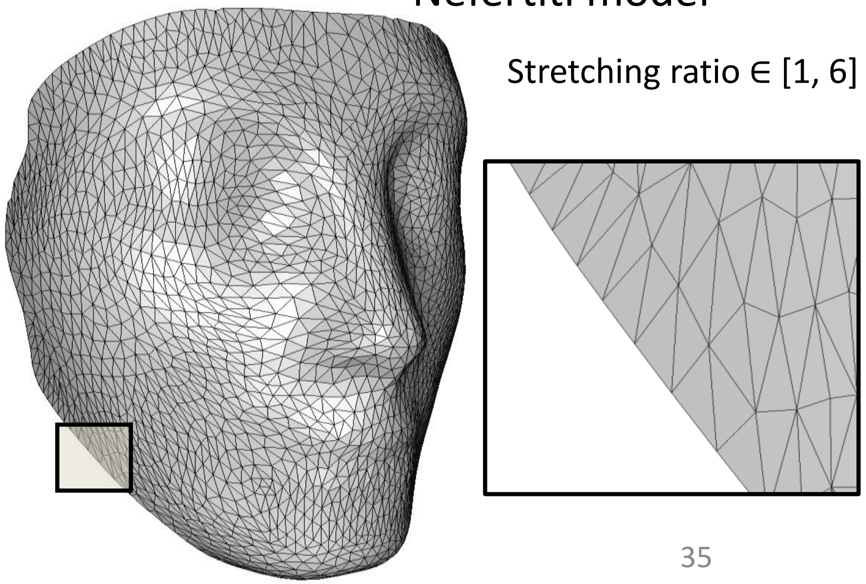
Upright CAD model





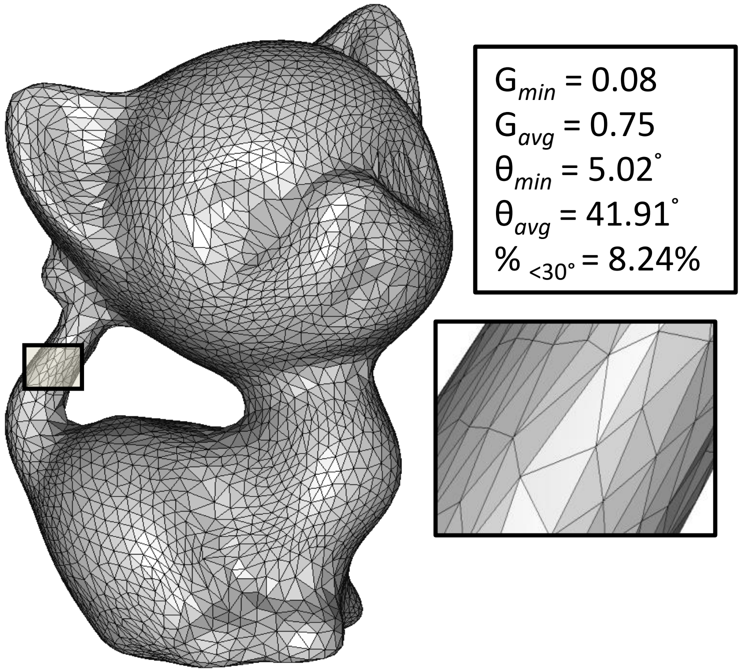
Nefertiti model

Stretching ratio ∈ [1, 6]



Comparison with Other Embedding Meshings

Stretching ratio $\in [1, 6]$



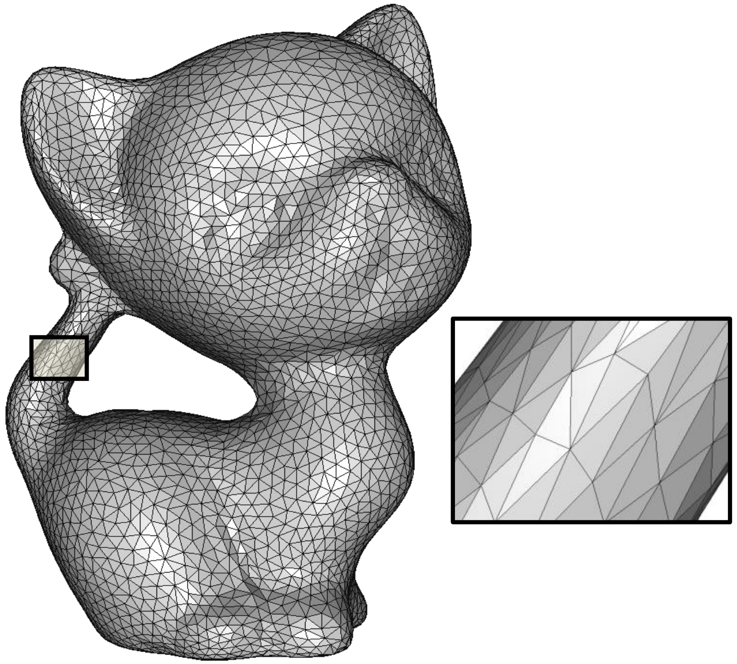
$G_{min} = 0.08$
$G_{avg} = 0.75$
$\theta_{min} = 5.02^{\circ}$
$\theta_{avg} = 41.91^{\circ}$
$\%_{<30^{\circ}} = 8.24\%$

2D conformal embedding

[Zhong et al. 2014]

Hausdorff dist. = 0.013415%

With the curvature tensors

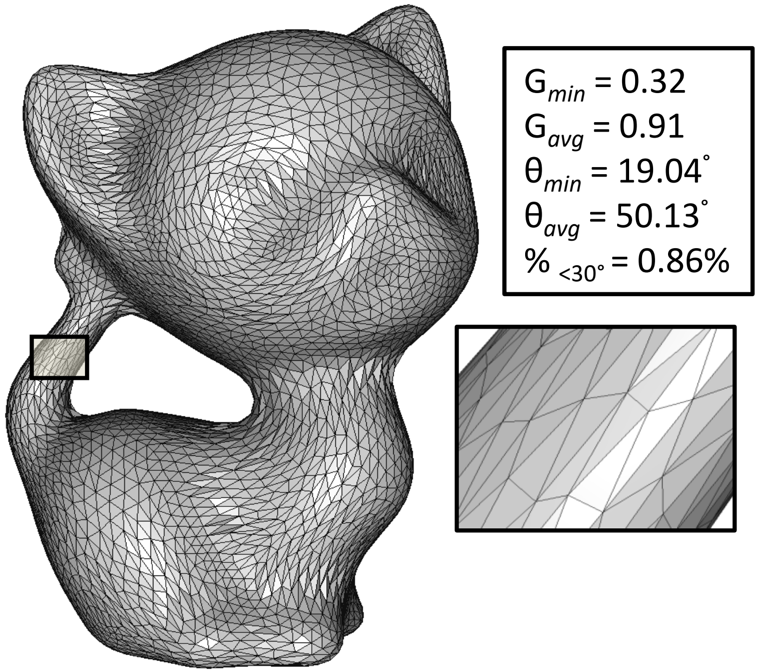


6D embedding

[Lévy and Bonneel 2012]

Hausdorff dist. = 0.005110%

Without the curvature tensors



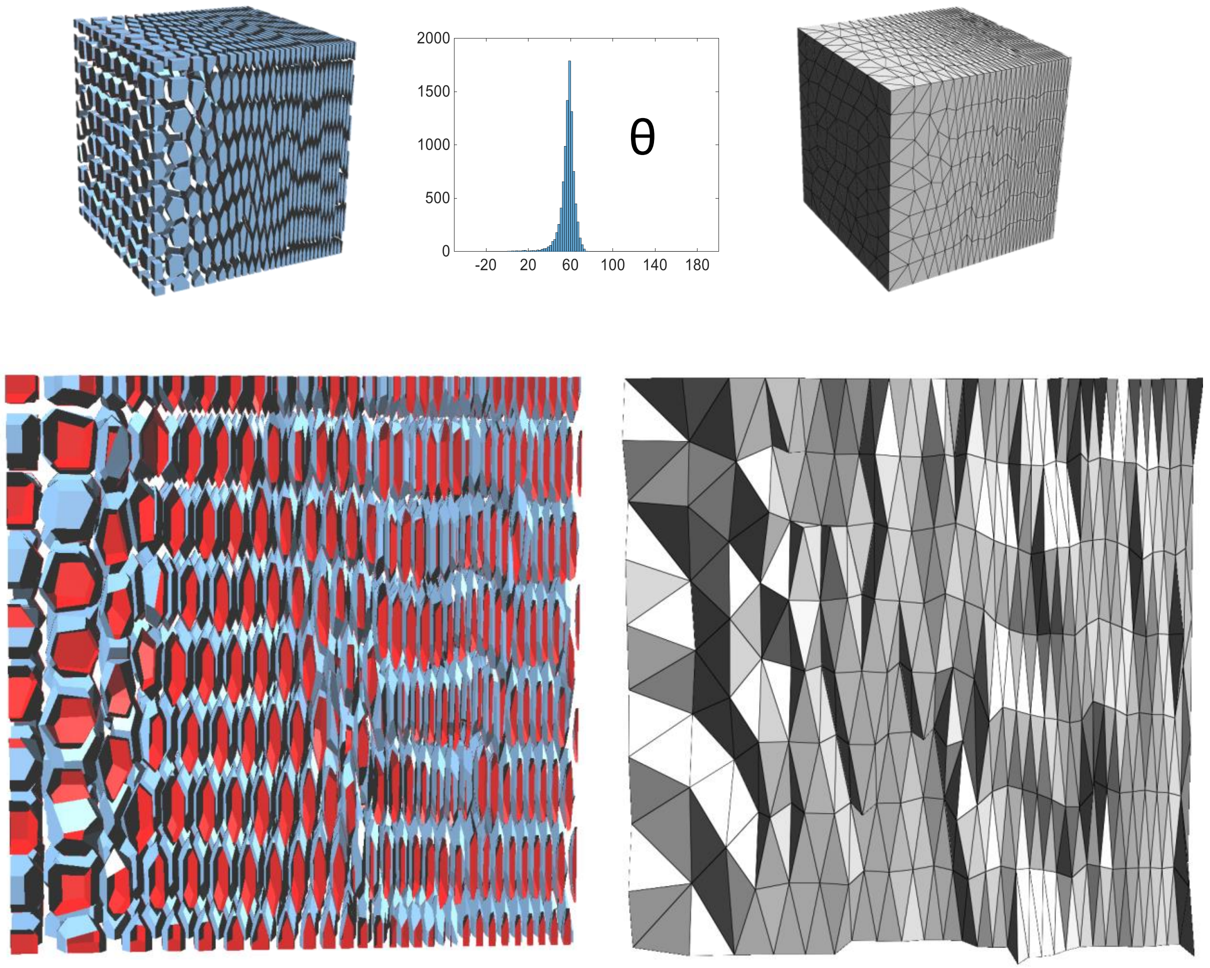
$G_{min} = 0.32$
$G_{avg} = 0.91$
$\theta_{min} = 19.04^{\circ}$
$\theta_{avg} = 50.13^{\circ}$
$\%_{<30^{\circ}} = 0.86\%$

Our embedding

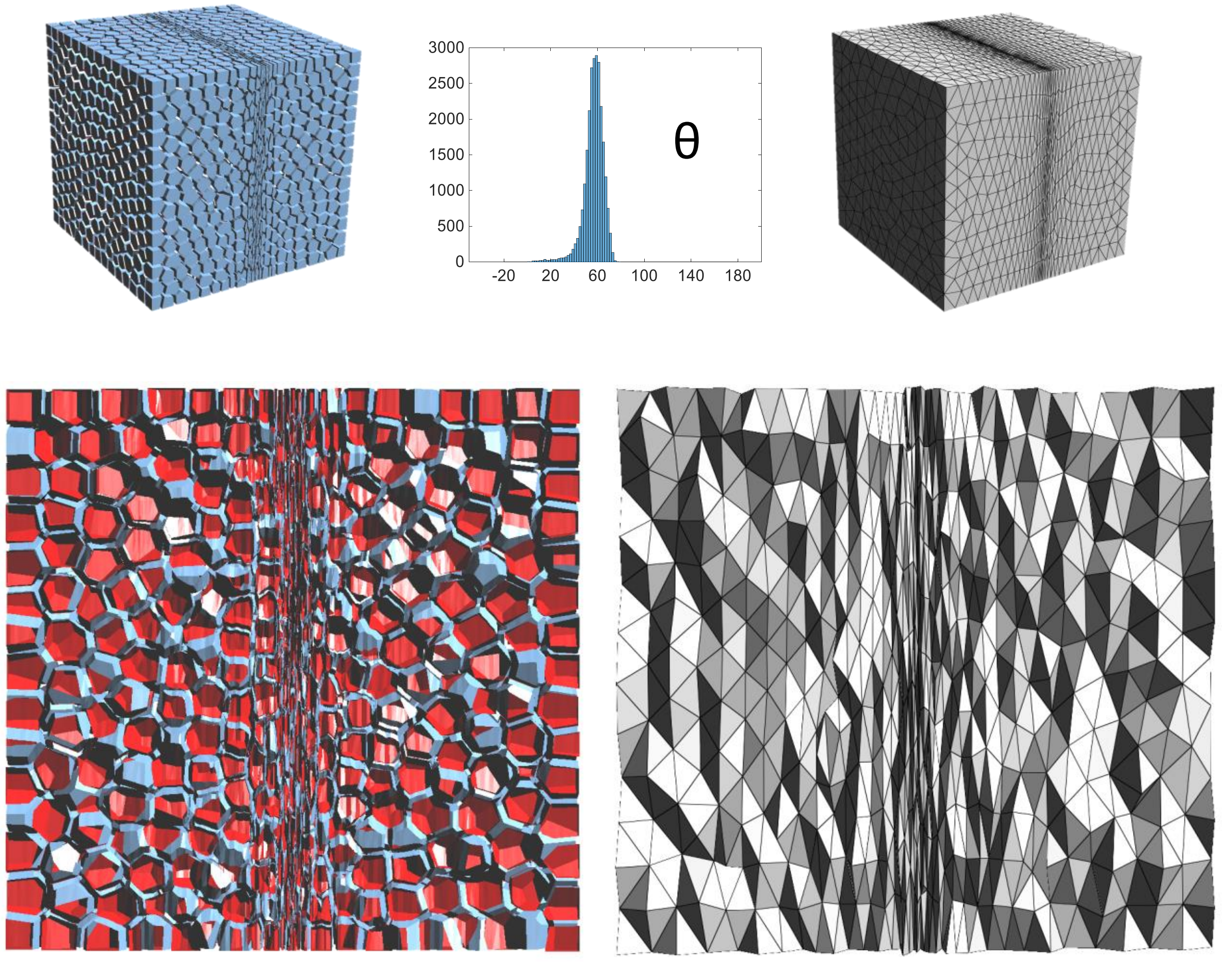
Hausdorff dist. = 0.004964%

With the curvature tensors

3D Volume RVD and Meshing: Analytic Metrics



Stretching ratio ∈ [1, 10] defined by a linear function



Stretching ratio ∈ [1, 25] defined by a highly nonlinear function

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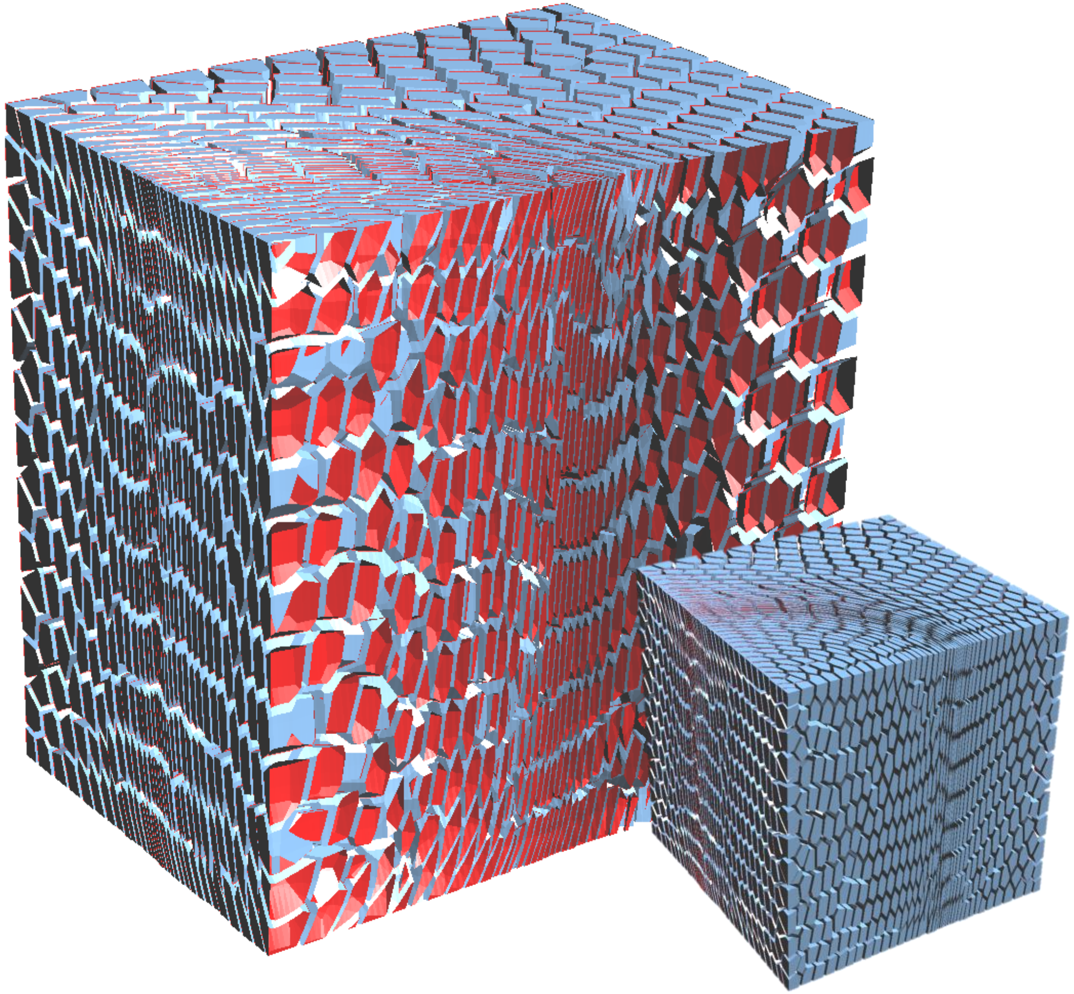
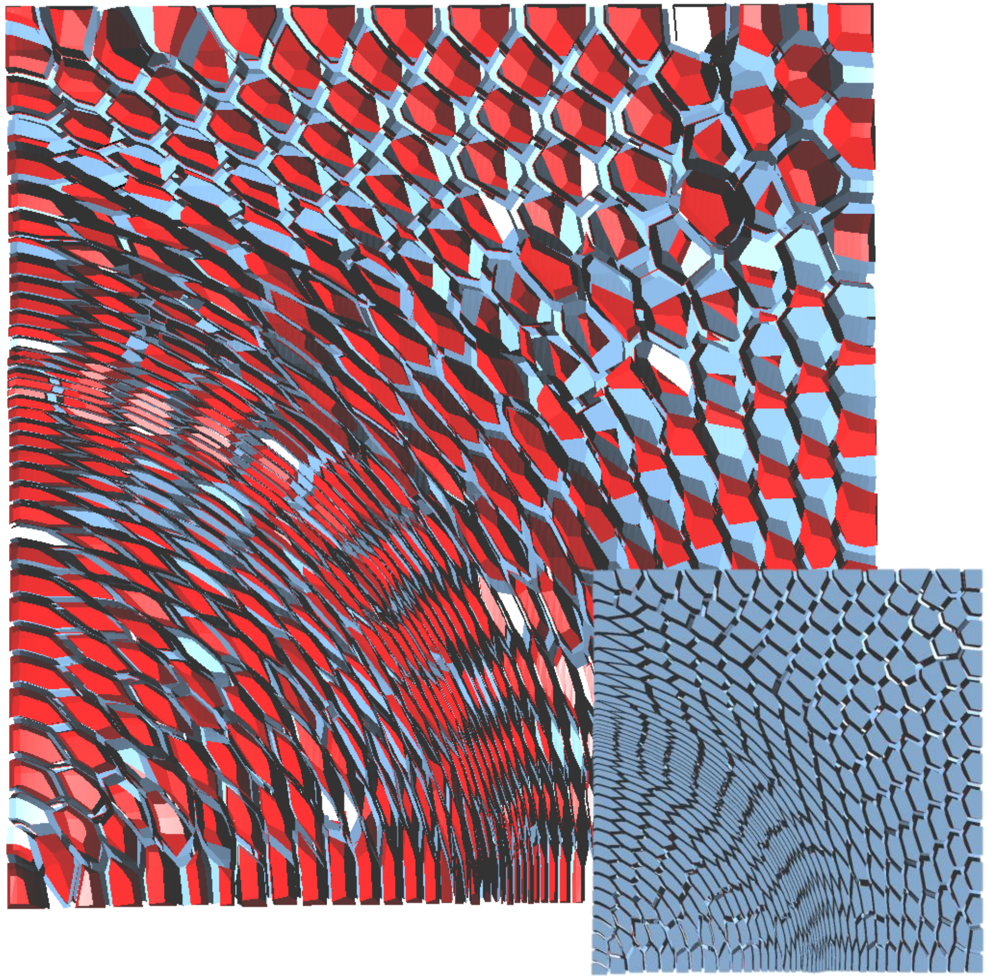
3D Volume RVD and Meshing: Analytic Metrics





Stretching ratio ∈ [1, 20] defined by a highly nonlinear function with a cylindrical rotation field

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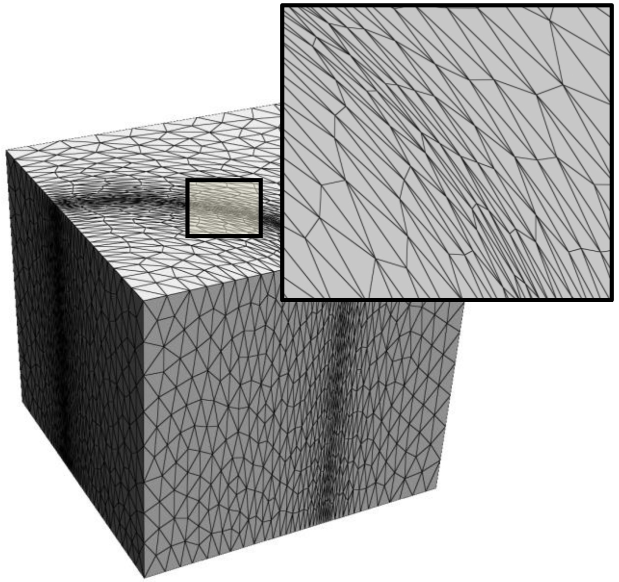
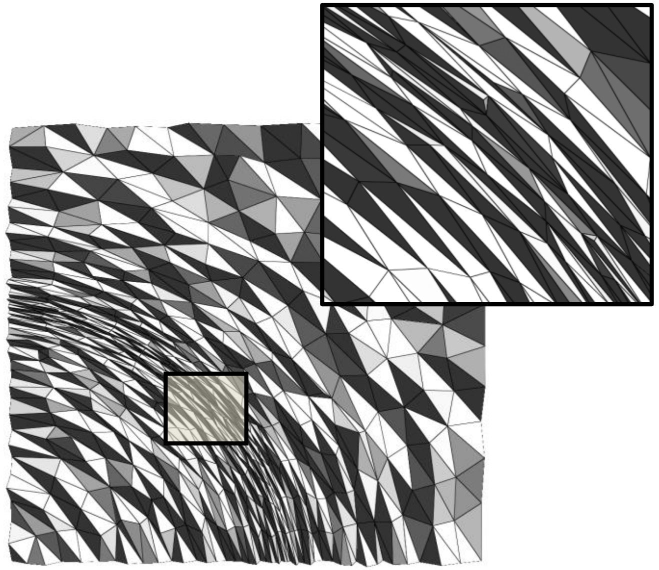
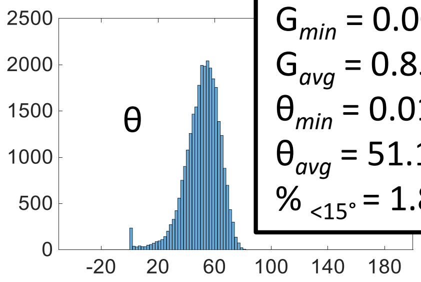
Comparison with Anisotropic Meshings



LCT

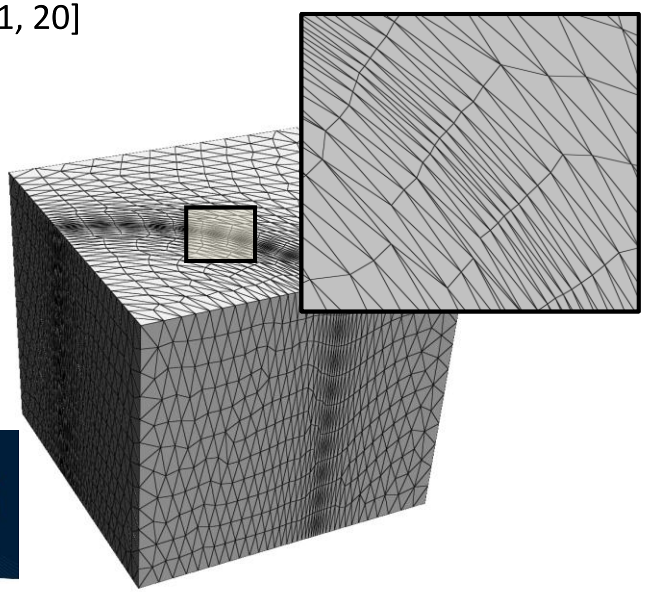
[Fu et al. 2014]



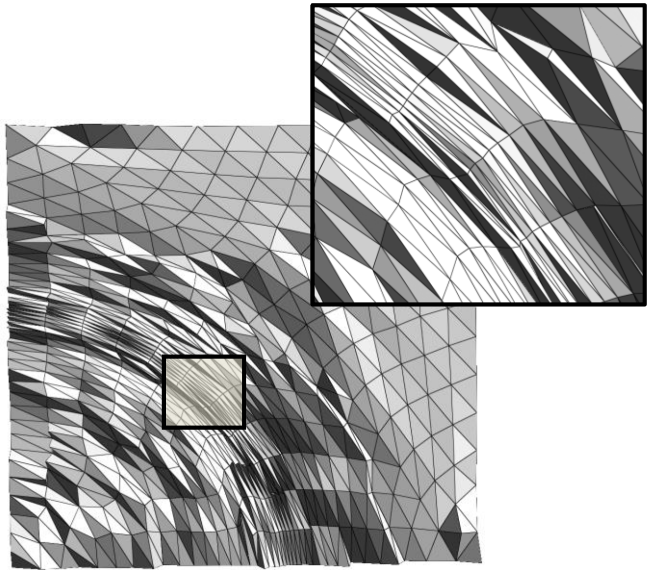


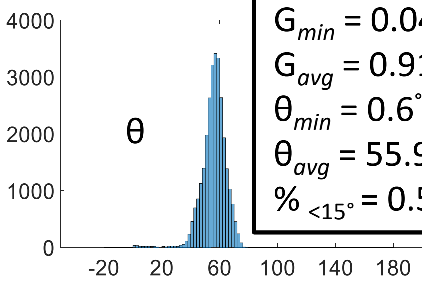
$G_{min} = 0.003$
$G_{avg} = 0.85$
$\theta_{min} = 0.01^{\circ}$
$\theta_{avg} = 51.16^{\circ}$
$\%_{<15^{\circ}} = 1.82\%$

Stretching ratio ∈ [1, 20]



Our method





$G_{min} = 0.04$
$G_{avg} = 0.91$
$\theta_{min} = 0.6^{\circ}$
$\theta_{avg} = 55.98^{\circ}$
$\%_{<15^{\circ}} = 0.57\%$

Note: both methods are computed without sliver elimination process

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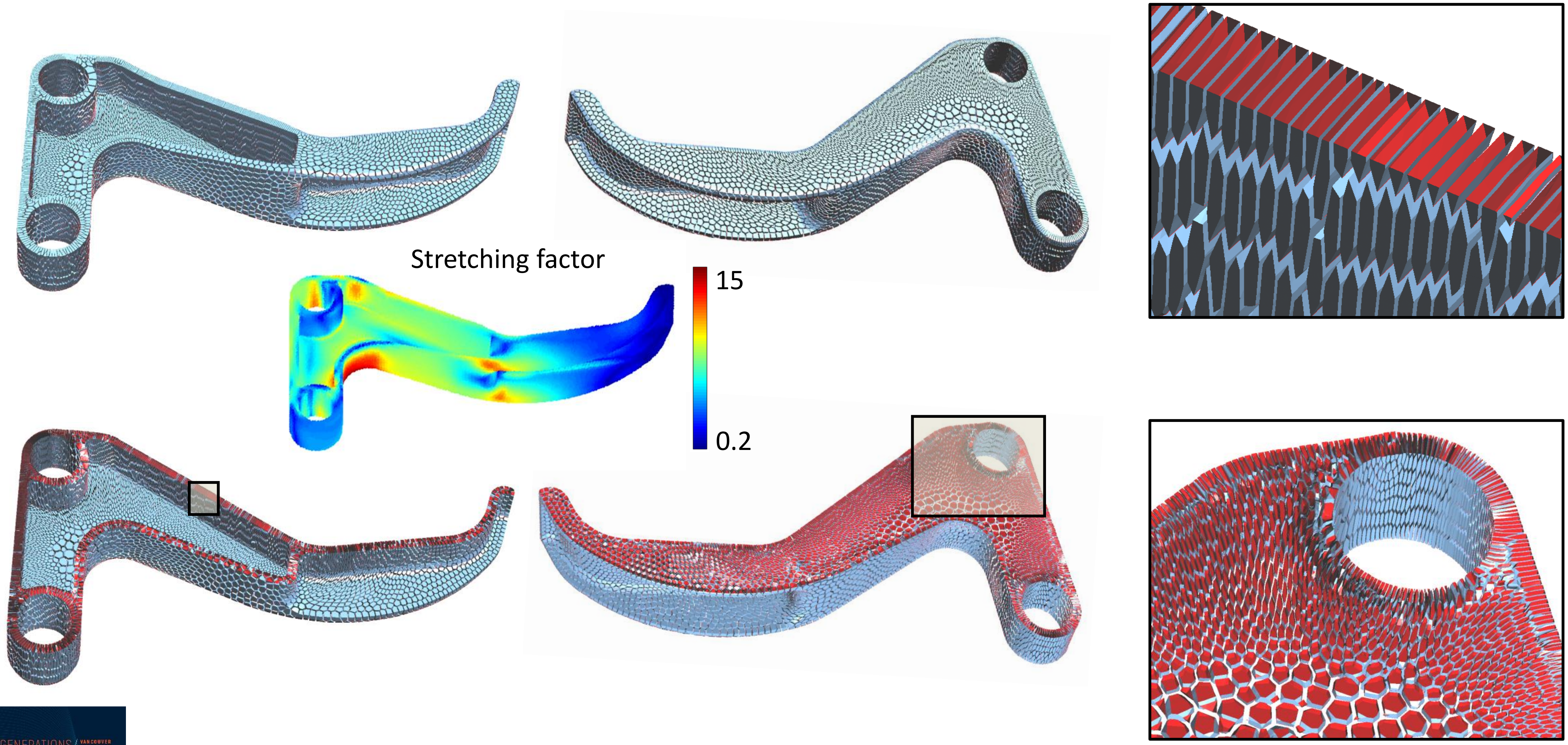
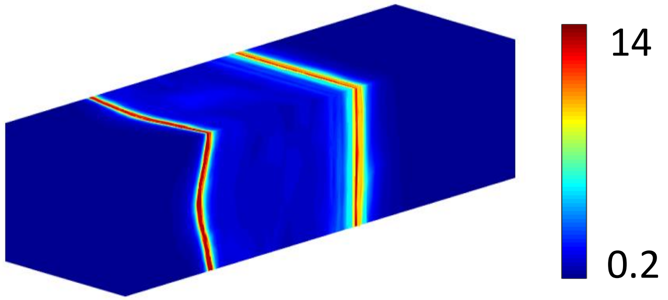
3D Volume RVD: Real Stress Tensor

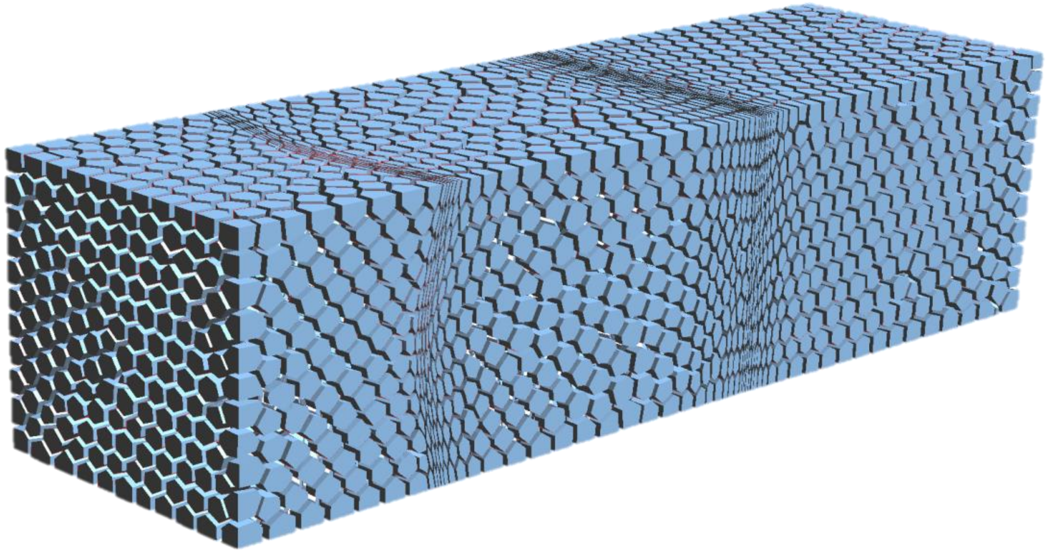


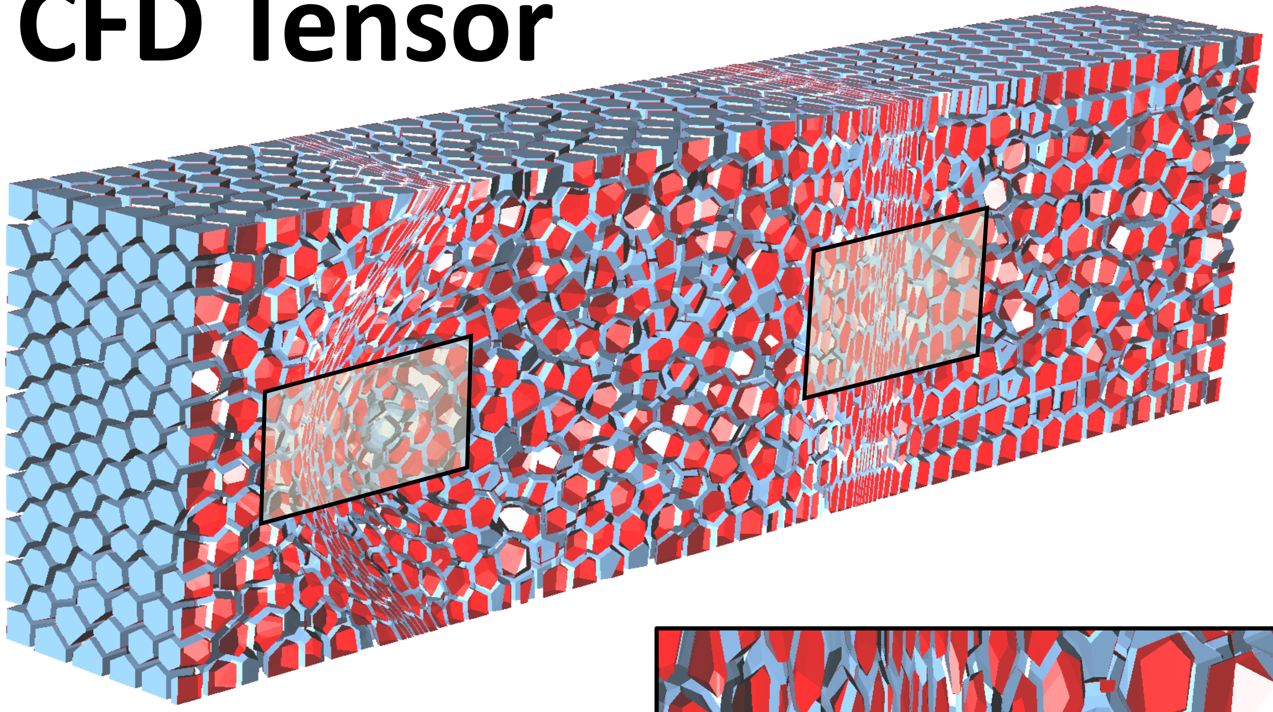
Tensor data downloaded from www.tensorvis.org

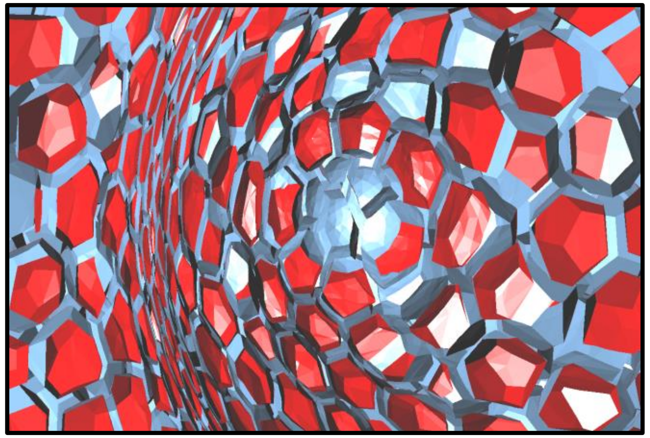
3D Volume RVD: Real CFD Tensor

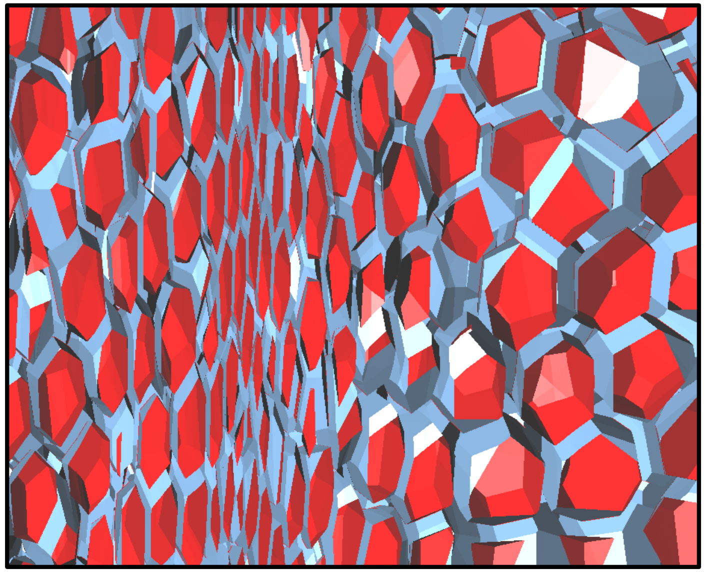
Stretching factor











Courtesy of Adrien Loseille for tensor data

Conclusion

- **A novel method for computing the self-intersection free Euclidean embedding in arbitrary dimensions and using it in Voronoi diagram, surface and volume meshing equipped with Riemannian metrics**

- **Limitations:**
  - ➢ The embedding computation is not a global approach
  - ➢ The convergence of the embedding computation is not theoretically guaranteed

- **Future work:**
  - ➢ Input metric with sudden discontinuities
  - ➢ GPU-based parallel algorithm and implementation
  - ➢ Simulations in medical imaging and computer animation

Acknowledgments

- Anonymous reviewers
- Joshua A. Levine, Adrien Loseille, Authors of [Fu et al. 2014], Authors of [Lévy and Bonneel 2012], Authors of [Boissonnat et al. 2015], etc.
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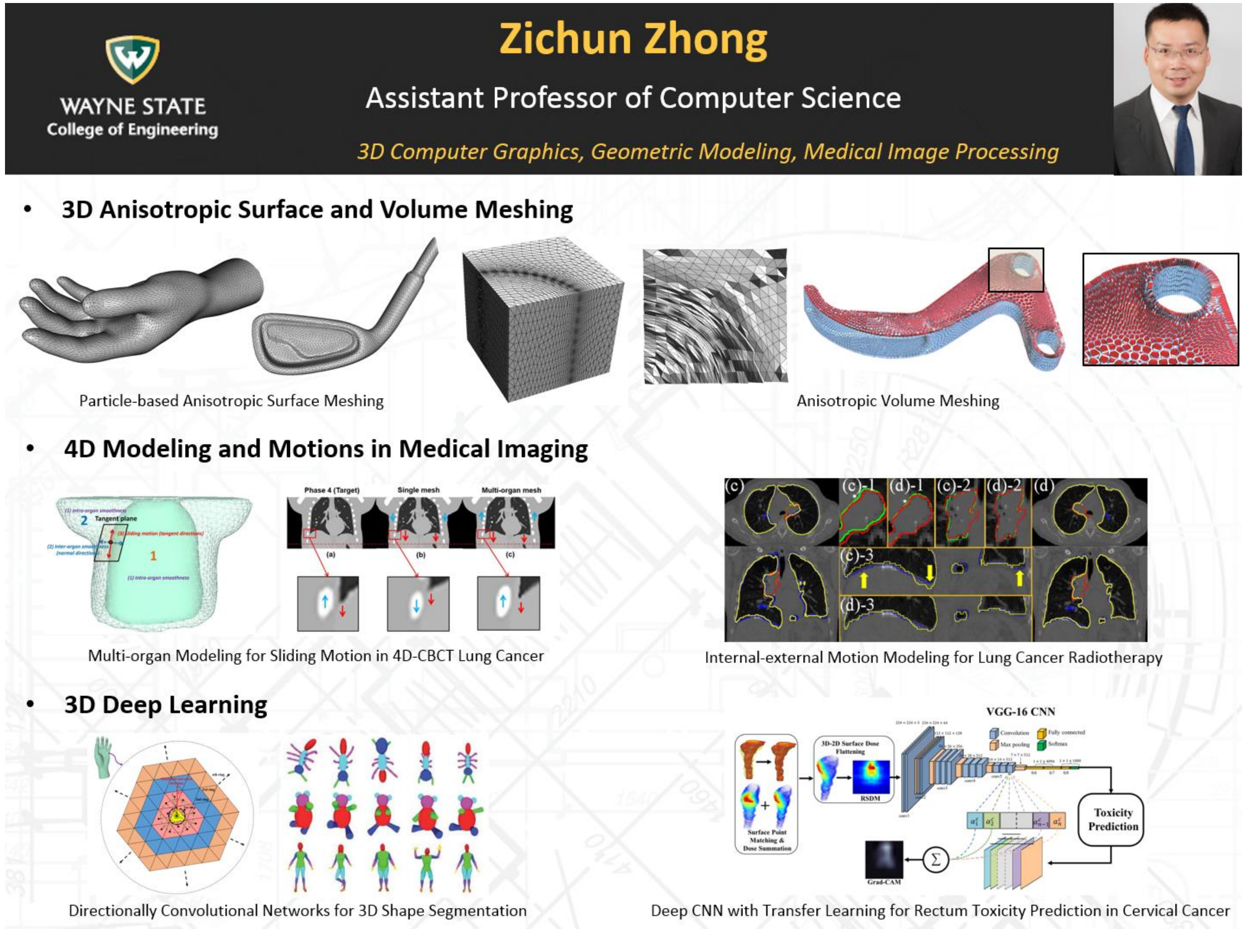
美国韦恩州立大学 (Wayne State University) 计算机建模与图像可视化实验室

- 招**全额奖学金博士生(Fellowship/RA/TA)**，优秀者可以破格免GRE成绩。

- 主要研究方向：**计算机图形学，几何建模**（尤其是曲面和体的网格重建），**医学图像处理**（尤其是可变形体的图像配准，三维和四维图像重建）,计算机视觉（尤其是三维形状重建和形状分析）,可视化，GPU并行算法。

- 感兴趣的同学请直接Email联系 **Zichun Zhong** 老师并把简历直接发至：zichunzhong@wayne.edu。具体信息可参阅钟老师的主页：http://www.cs.wayne.edu/zzhong/



- **韦恩州立大学(Wayne State University)**是美国的一所四年制公立大学，成立于1868年，位于美国汽车制造业中心的底特律市。该校现有本科生和研究生超过27000人，每年度的研究基金超过2亿美金，被美国国家科学基金会评为美国国内前50强的研究型公立大学，现与全世界40多所高等院校有合作关系。韦恩州立大学是目前在密歇根州的第三大公立大学，是美国30家规模最大的大学之一。

- **图片中的棕色大楼即为计算机系老师及博士生办公及科研的位置。**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