Weaving Geodesic Foliations

Etienne Vouga

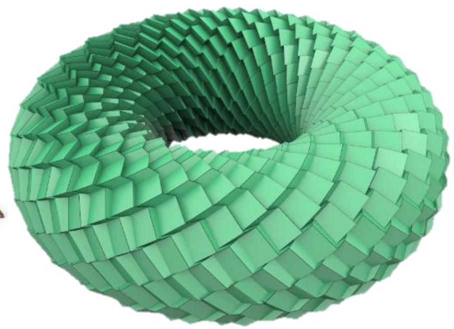
University of Texas at Austin

With: Josh Vekhter, Jiacheng Zhuo, Luisa Gil Fandino, Qixing Huang

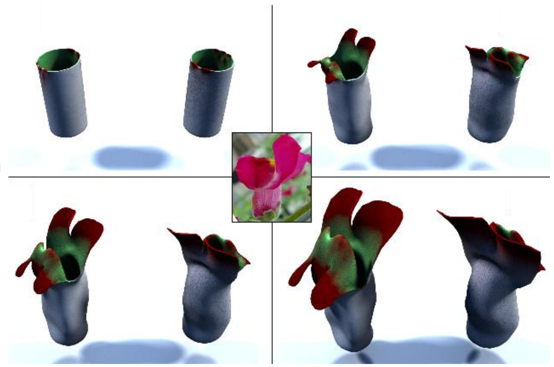
Research Overview

design

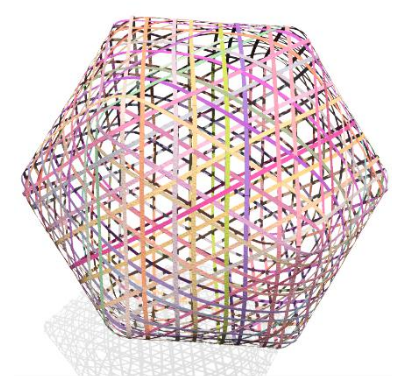
masonry



deployables

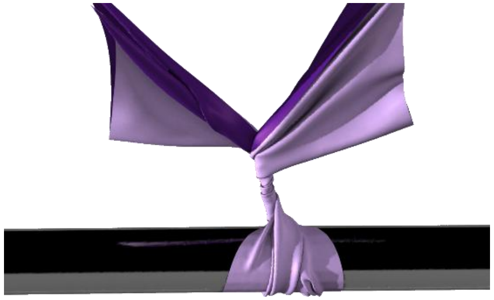


growth

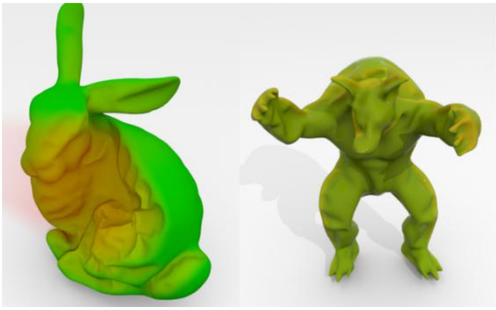


weaves

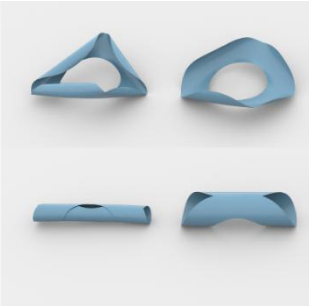
simulation



cloth, shells



crumpling,



swelling

contact, friction

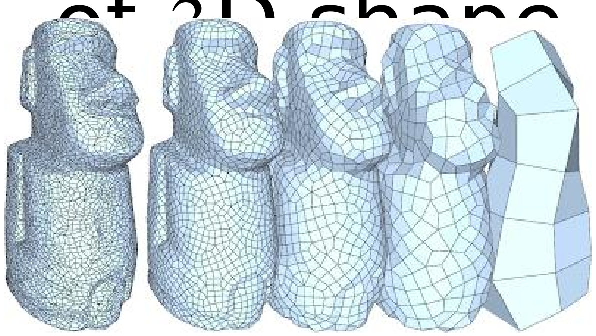
Why Study Discrete Geometry？

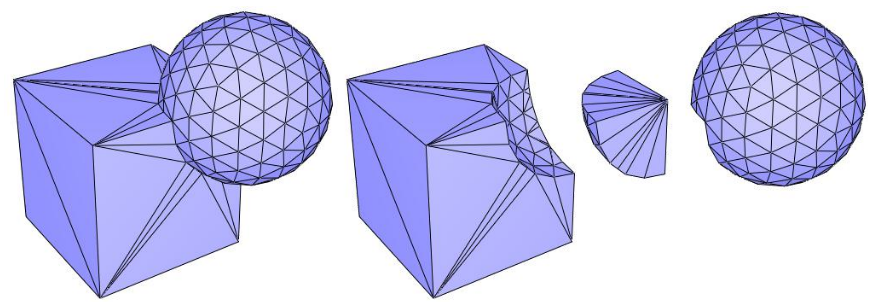
What is the relevance to machine learning？

Why Study Discrete Geometry？

What is the relevance to machine learning？

In computer graphics, we have developed a deep understanding of 3D shape





- how to reason about it, discretize

Why Study Discrete Geometry?

Differential geometry is the language of:

- shape
- deformation
- physics
- symmetries and mappings

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Differential geometry is the language of:

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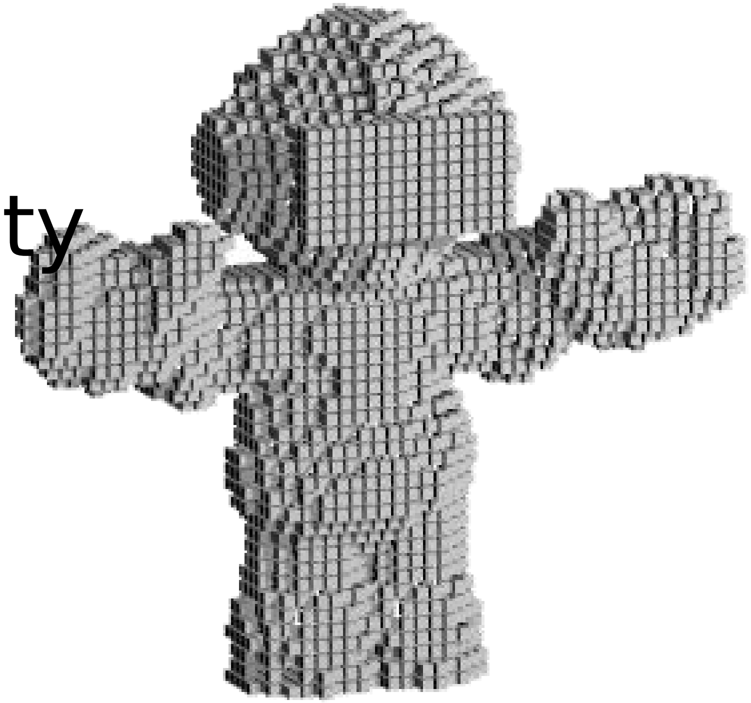
- symmetries and mappings

Yet the use of differential geometry in ML is still naive

Geometry in Machine Learning

Voxelization

- curse of dimensionality
- no surface geometry



Geometry in Machine Learning

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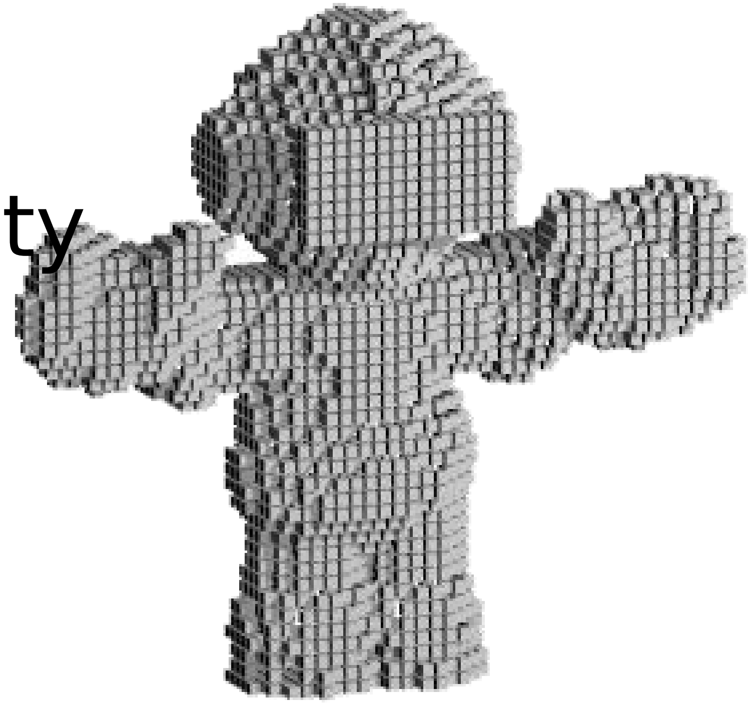


Projection onto planes

- inherently 2D…

ML Grand Challenge

How can we learn 3D shape, motion, deformation?

ML Grand Challenge

How can we learn 3D shape, motion, deformation?

Groundbreaking new techniques for learning 3D shape must use the **vocabulary** of shape: discrete differential geometry

This Talk: Key Topics

Discrete vector fields and integrability

Discrete foliations

Discrete geodesics and geodesic fields

Branched covering spaces

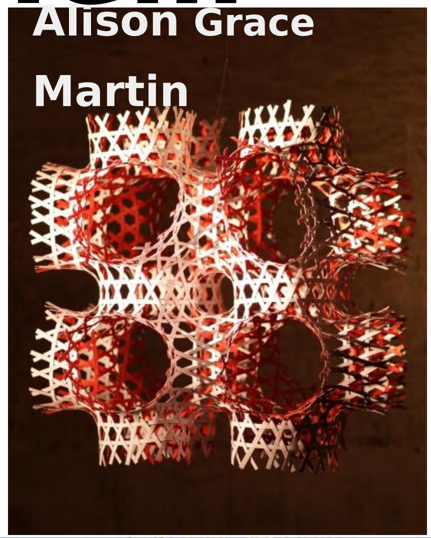
Woven Structures: From Small Scale…

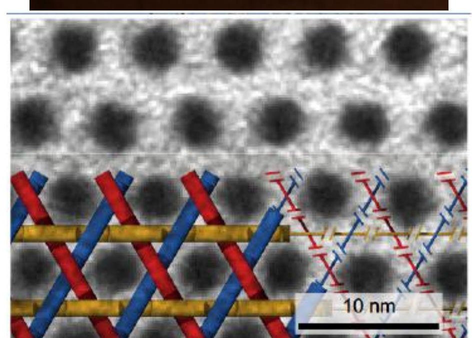
Martin

Puryear

Alison Grace

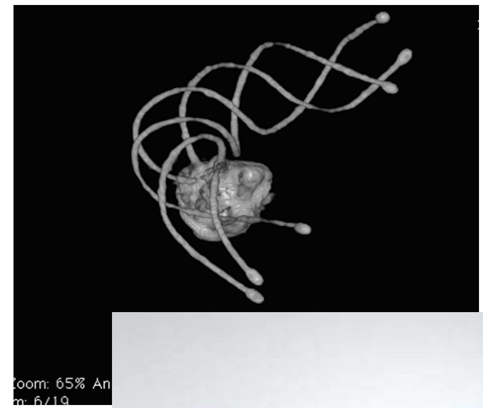
Martin

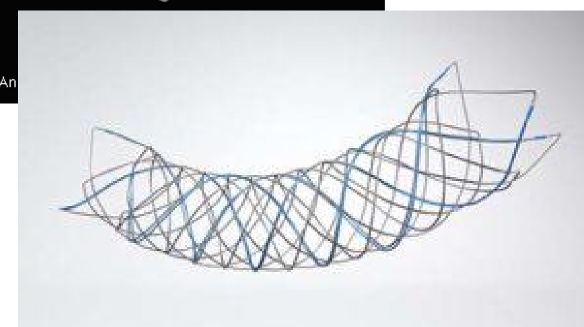




Nature Nanochemistry

“Quantum Spin Liquids” – Physics





LVIS/LVIS Jr. stents,

J. NeuroInterventional Surgery 2015

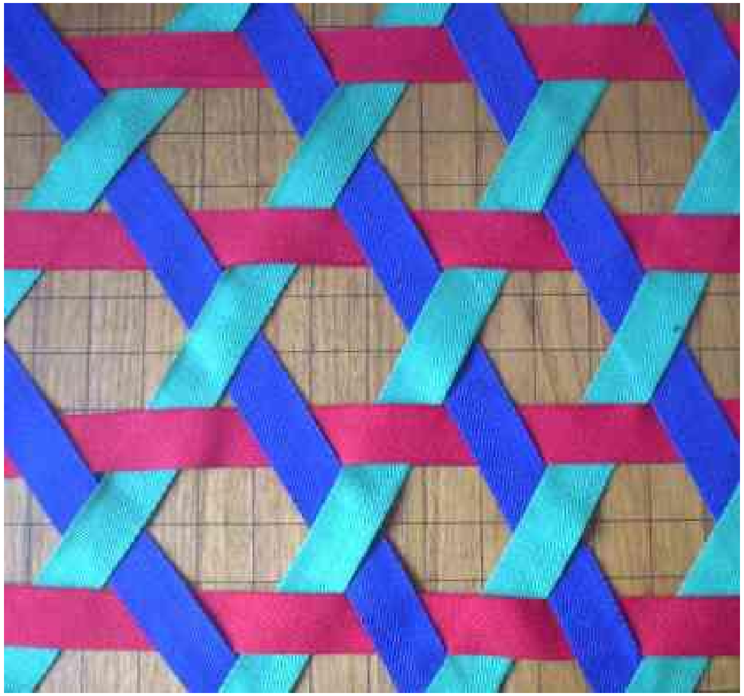
...to architectural

Centre Pompidou–Metz

MINIMA | MAXIMA
World Expo Pavillion

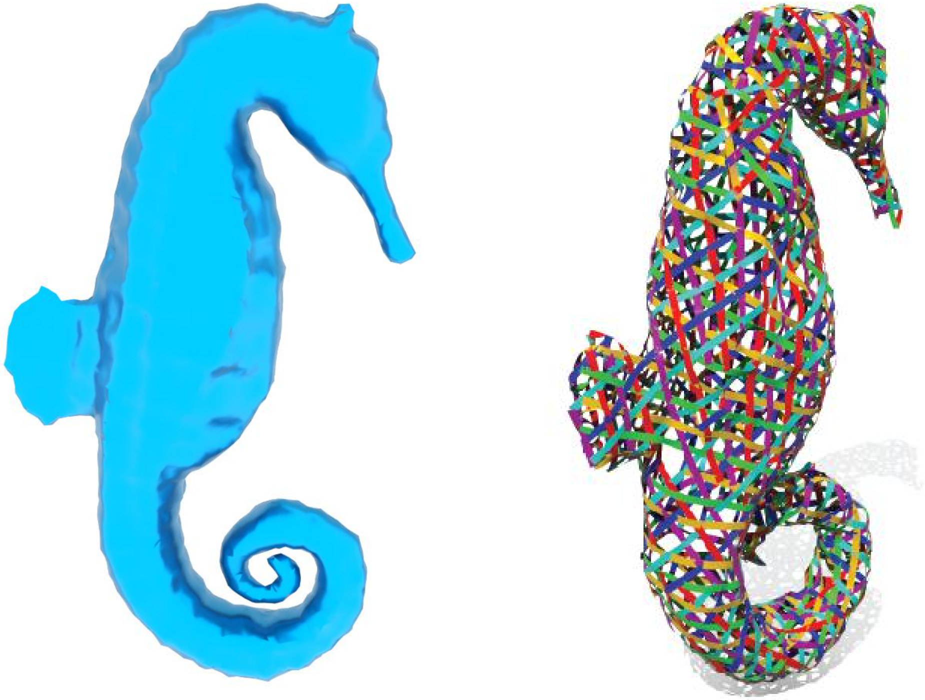
Elastic Ribbons Woven Triaxially

Can achieve wide array of shapes, using a wide array of materials.



Our Goal

Given a surface, figure out how to weave it



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Will attack this problem in two parts:

1. How do you lay out a **single** family of ribbons on a surface in a “nice” way?

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Will attack this problem in two parts:

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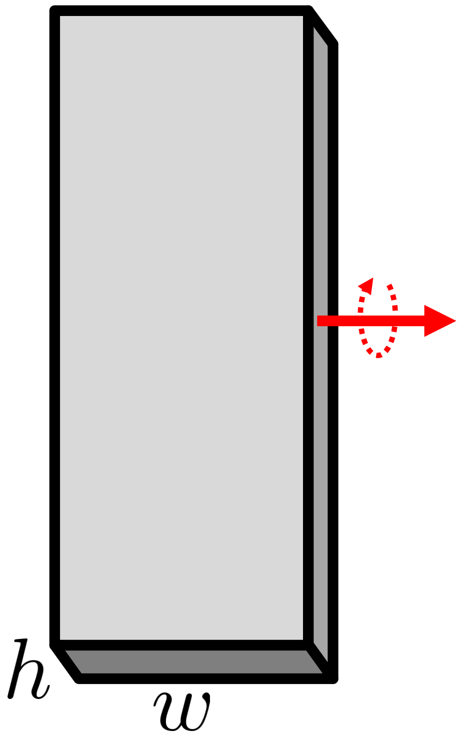
2. How do we extend to triaxial weaves?

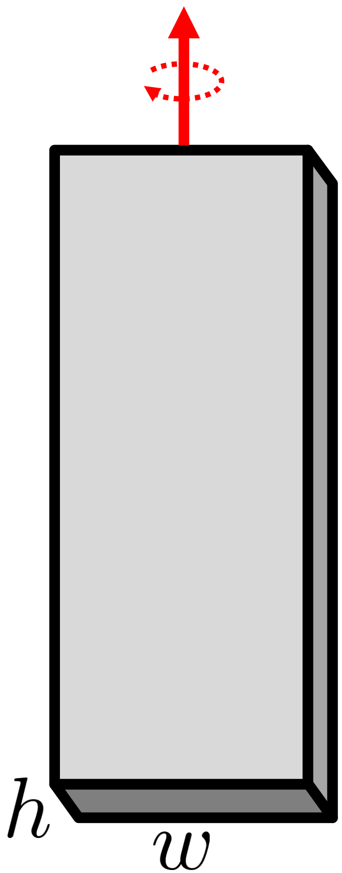
Physics of Ribbons



ribbon behaves as Eulerian beam

– small resistance to **out-of-plane** bending $\mathcal{O}(wh^3)$

Physics of Ribbons



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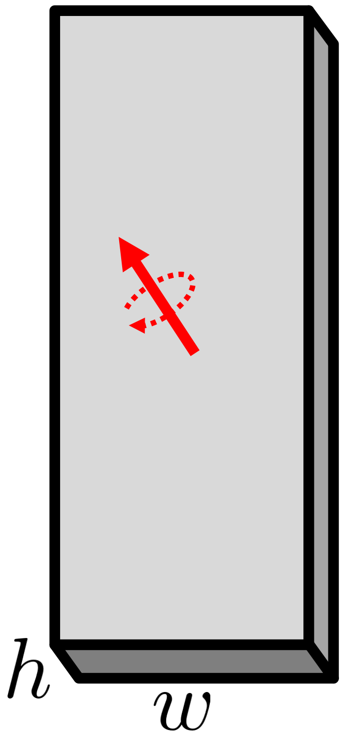
$O(wh^3)$

plane

bending

Physics of Ribbons



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- small resistance to **twist**
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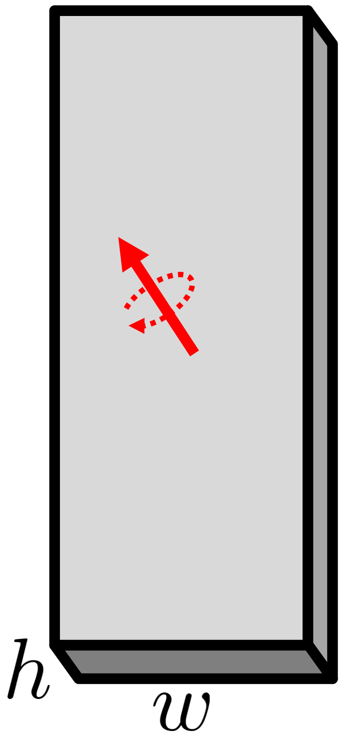

ribbon behaves as Eulerian beam

- small resistance to **out-of-plane** bending $O(wh^3)$
- small resistance to **twist** $O(wh^3)$
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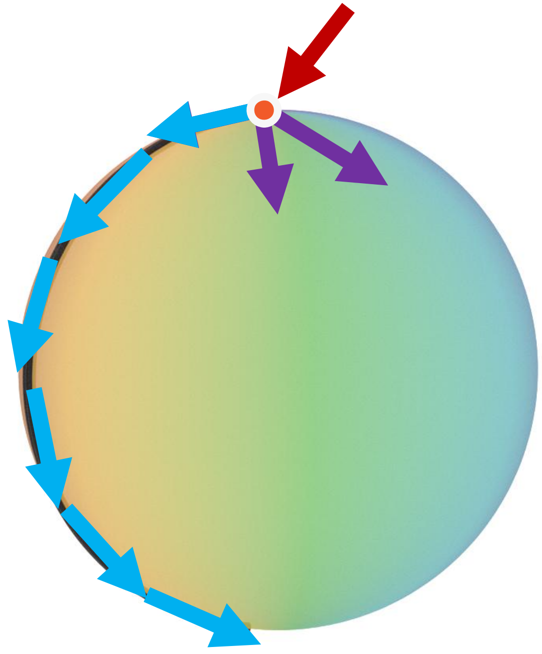
ribbons must follow

Geodesics Fundamentally Global

Geodesic segments determined by 3 degrees of freedom:

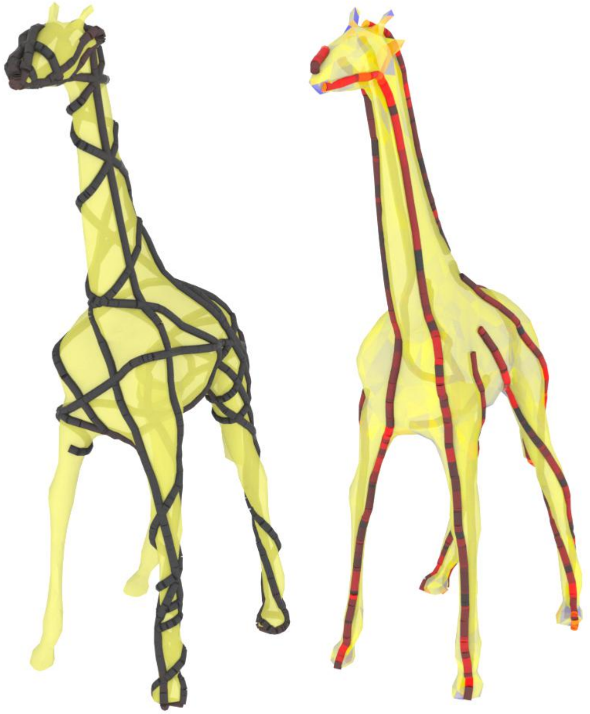
- Start point
- Direction
- Distance



Geodesic Layout Challenge

Tracing one geodesic for a long time "mummifies" the target surface

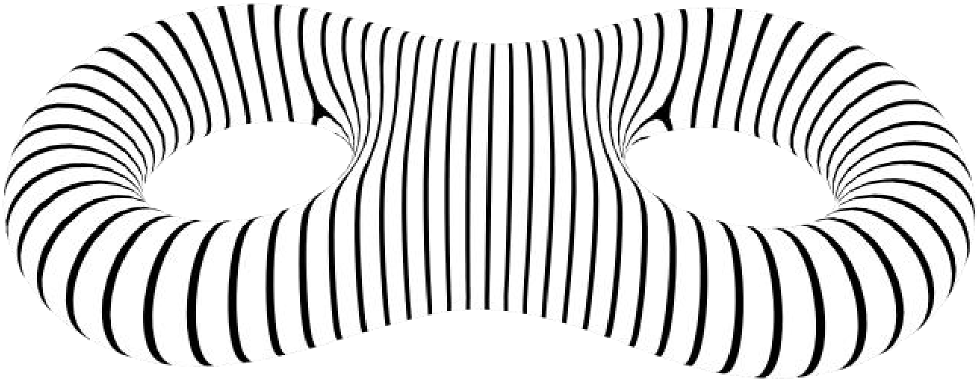
We want to "evenly" cover a surface with non–intersecting geodes

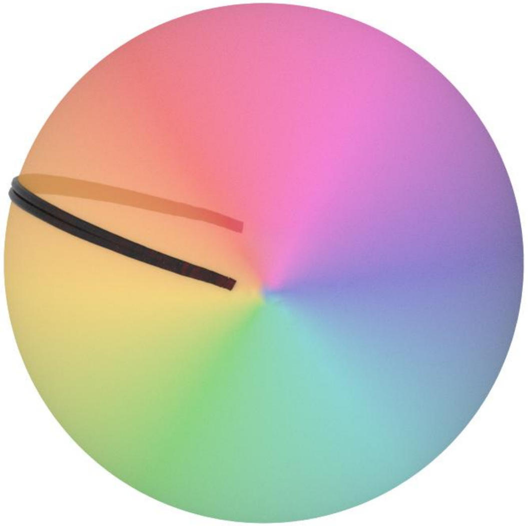


Foliations

A decomposition of a surface into a union of submanifolds, called leaves

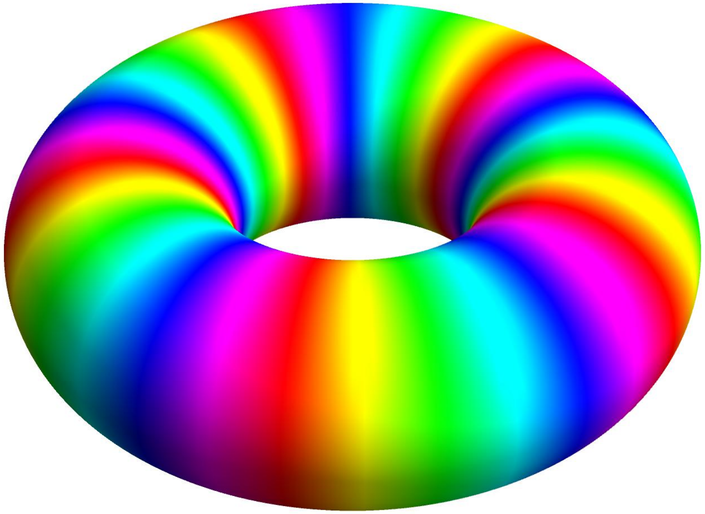
Or, a submersion $\theta : M \to S^1$





[Palmer. '15]

Geodesic Foliations: Two Views



submersion $\theta : M \to S^1$
with geodesic isolines

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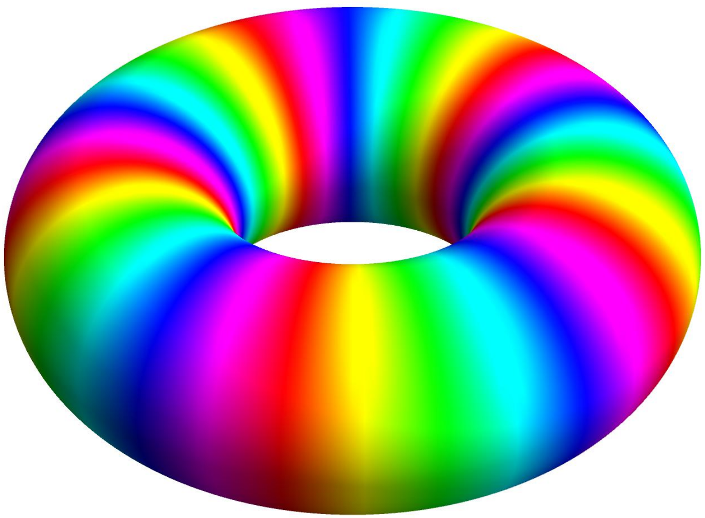

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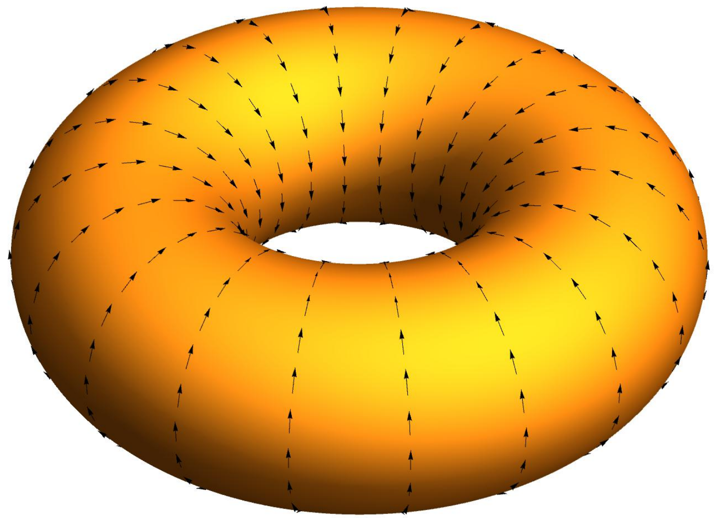

complete vector field $\mathbf{v} \in TM$ with closed geodesic integral curves

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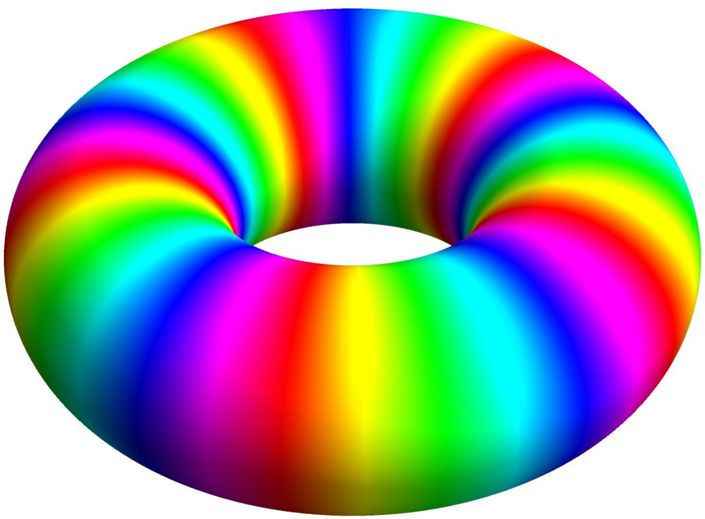

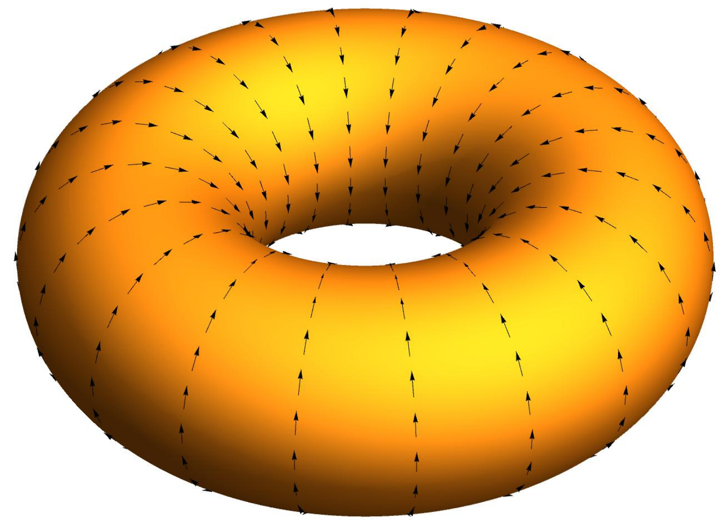

submersion $\theta : M \to S^1$ with geodesic isolines

(easier for applications)

complete vector field $\mathbf{v} \in TM$ with closed geodesic integral curves

Geodesic Foliation Relaxations

Issue: geodesic foliations usually don't exist (e.g. on the round sphere)

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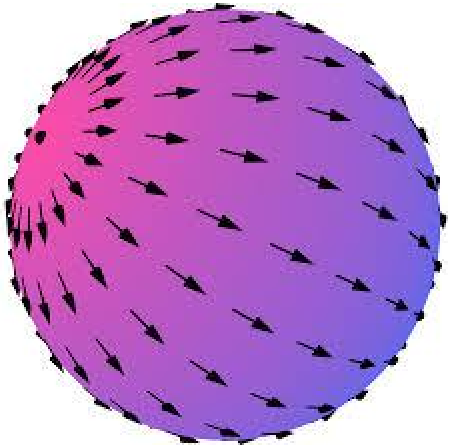

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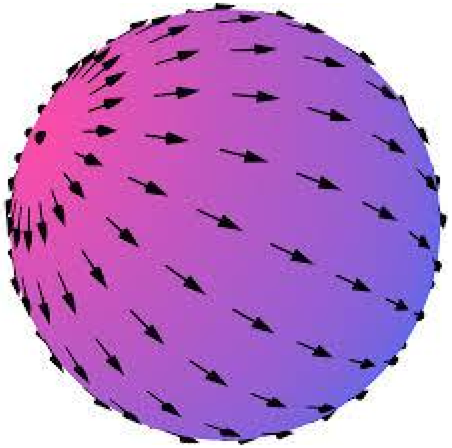
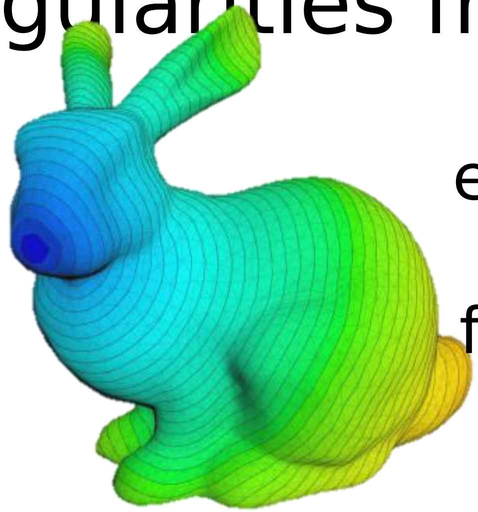
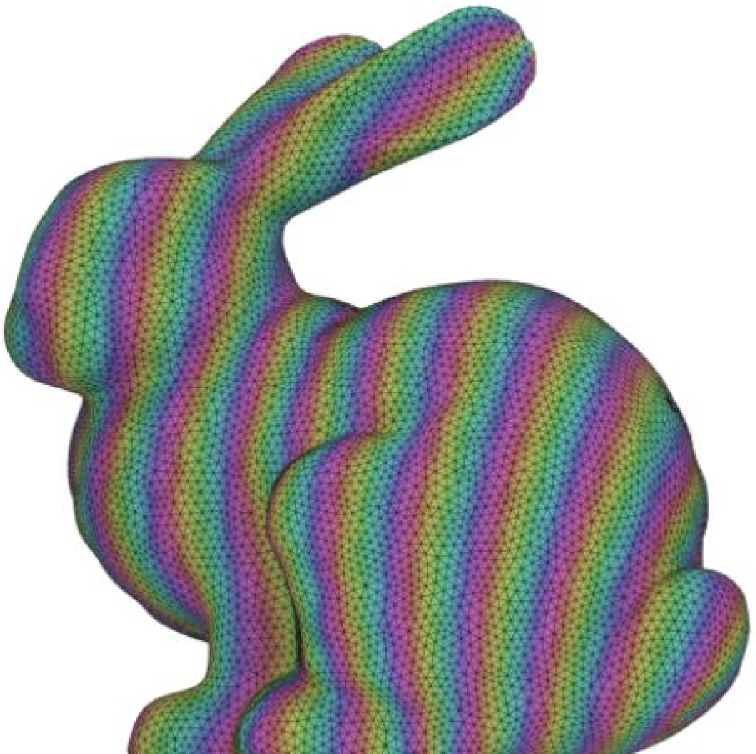




example: gradient of distance function from any point

Problem Overview

Ultimate goal: given (discrete) surface, find geodesic almost–θ foliation



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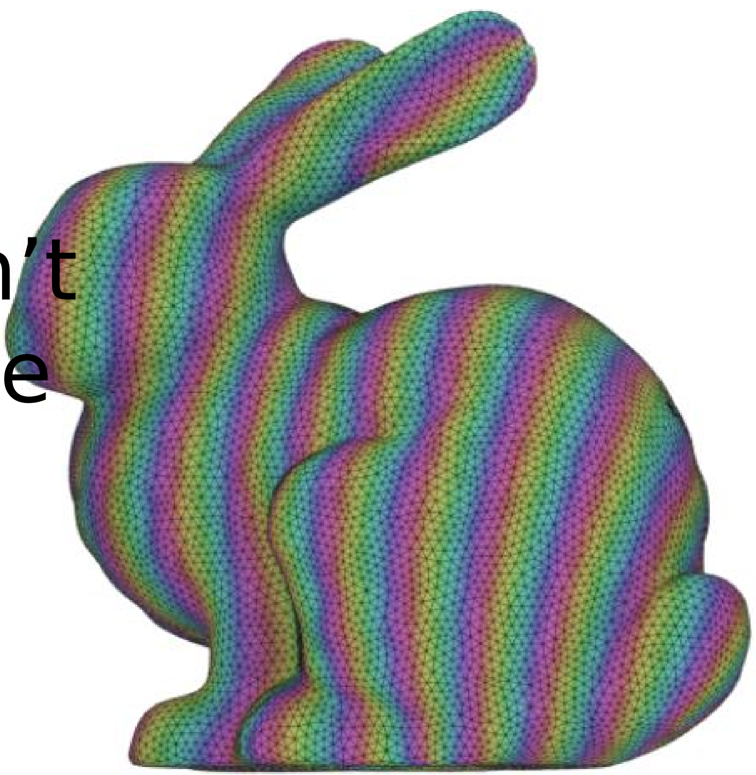
Ultimate goal: given (discrete) surface, find geodesic almost-θ foliation

This is too hard: we don't know how to discretize the isoline constraint



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Our steps:

1. Find vector field $\mathbf{v}$ that has geodesic integral curves
2. Recover θ by integrating the field:

$$\min_{\theta, s} \|\nabla\theta - s\mathbf{v}^{\perp}\| \quad \text{s.t.} \quad \|s\| = c.$$

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ban trivial solutio

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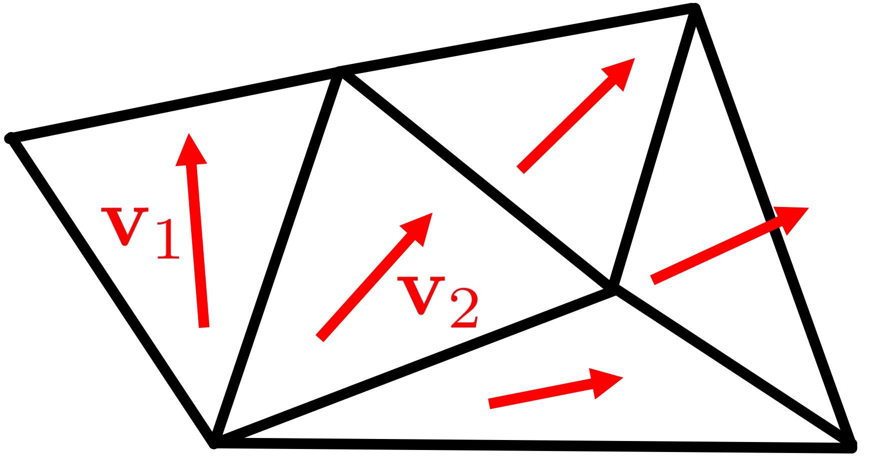
isolines and integral curves parallel

Geodesic Vector Fields

How can we tell if a discrete vector field "has geodesic integral curves"?



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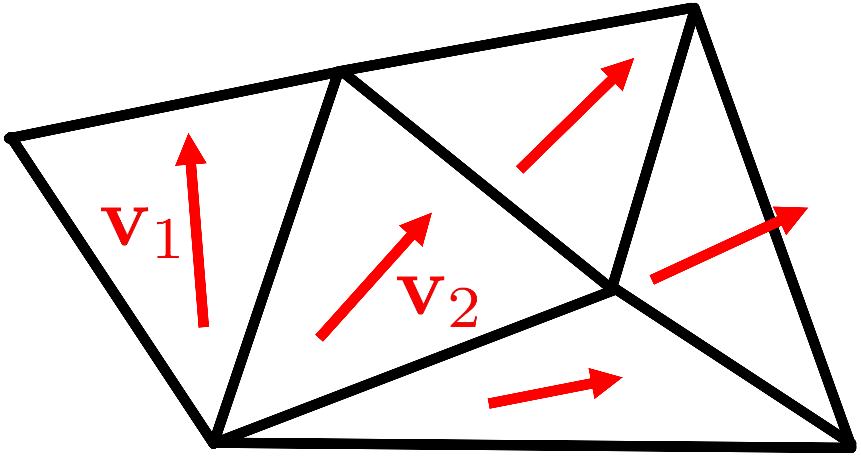

geodesic equation

$$\nabla_{\mathbf{v}}\mathbf{v} = 0?$$

Geodesic Singularities

Singularities are topologically necessary on surfaces of non–zero genus

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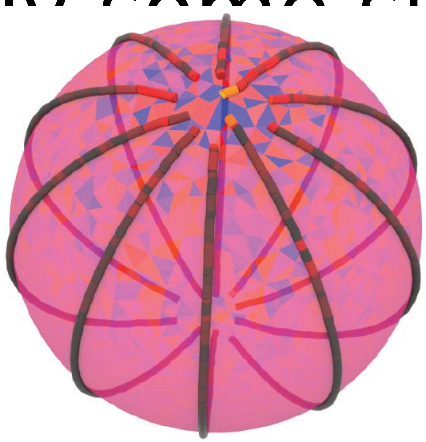
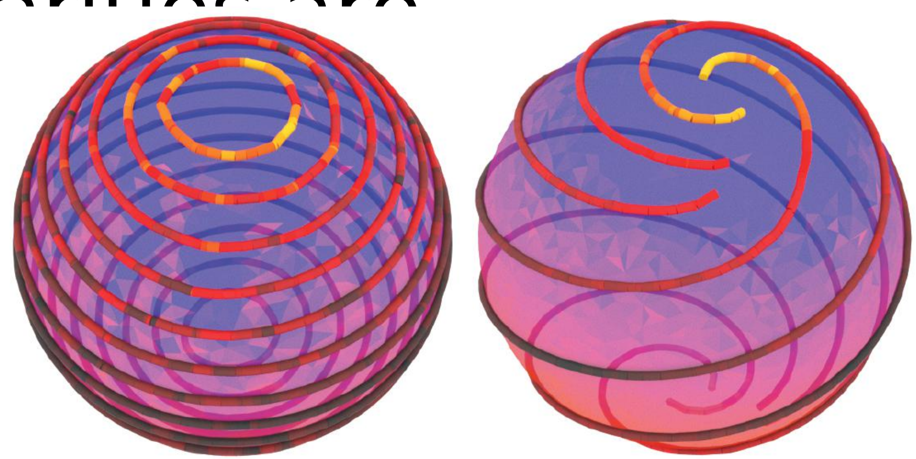
Only some singularities are
a



ok
geodesic almost everywhere



not geodesic

Geodesic Singularities

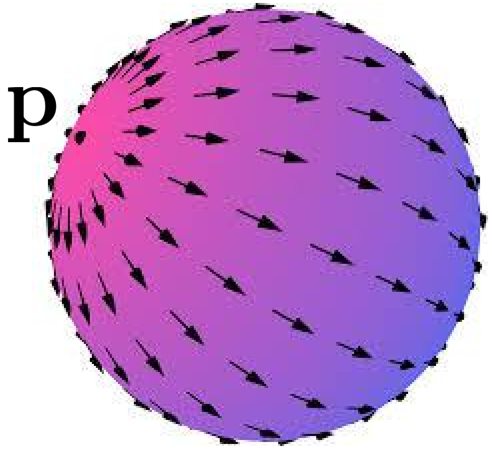
Singularities are topologically necessary on surfaces of non–zero genus

Only some singularities are acceptable

Need a definition of discrete geodesic field that is well–defined at “good” singularities

Vector Field Integrability

A vector field is **integrable** if it is the gradient of a potential function $\mathbf{v} = \nabla f$



$$f(\mathbf{q}) = d(\mathbf{p}, \mathbf{q})$$

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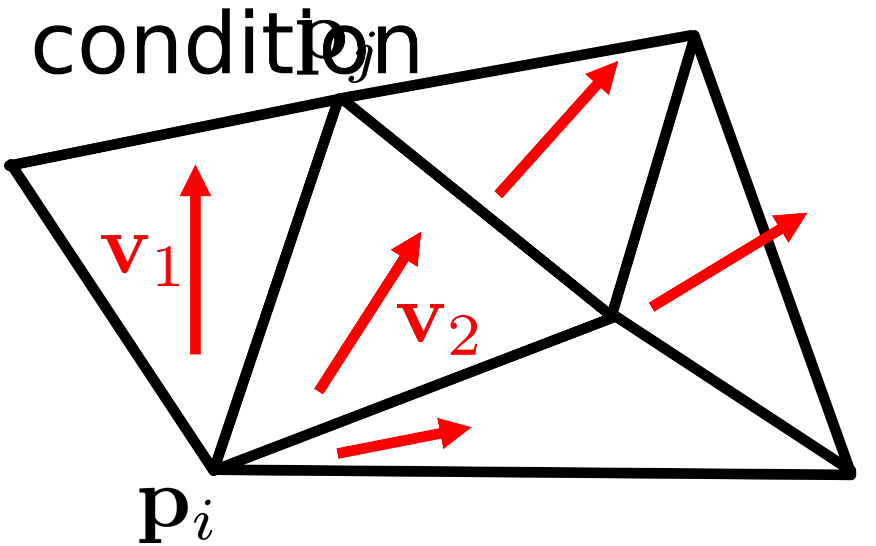
Discrete integrability: per–edge condition



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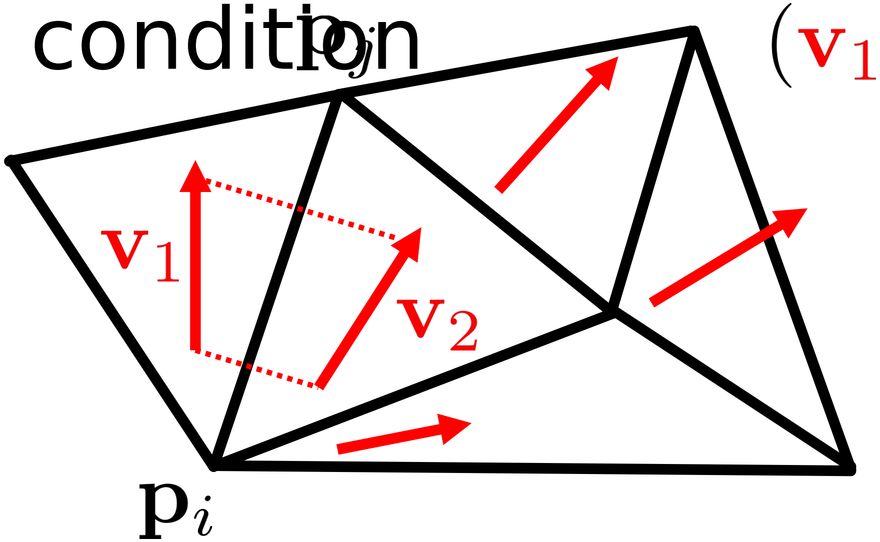

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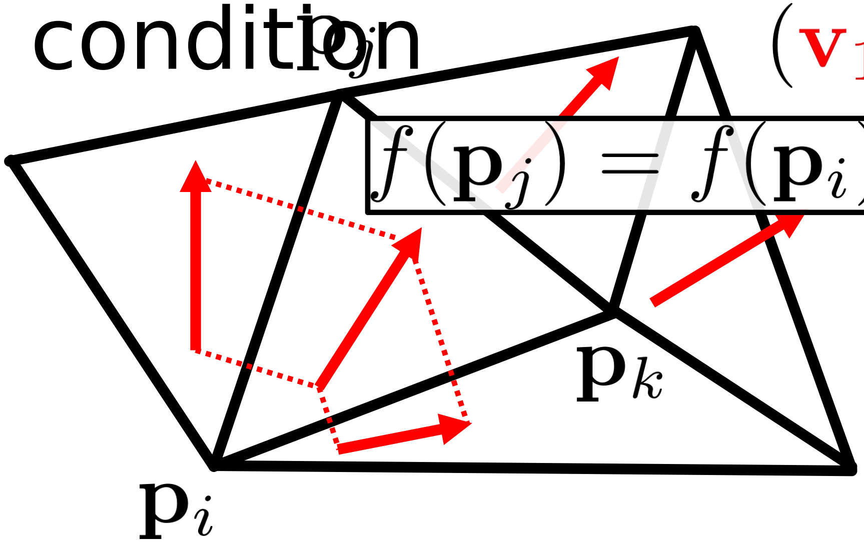
Discrete integrability: per–edge condition

$$(\mathbf{v}_1 - \mathbf{v}_2) \cdot (\mathbf{p}_j - \mathbf{p}_i) = 0$$

$$\boxed{f(\mathbf{p}_j) = f(\mathbf{p}_i) + \mathbf{v}_1 \cdot (\mathbf{p}_j - \mathbf{p}_i)}$$



Discrete Curl

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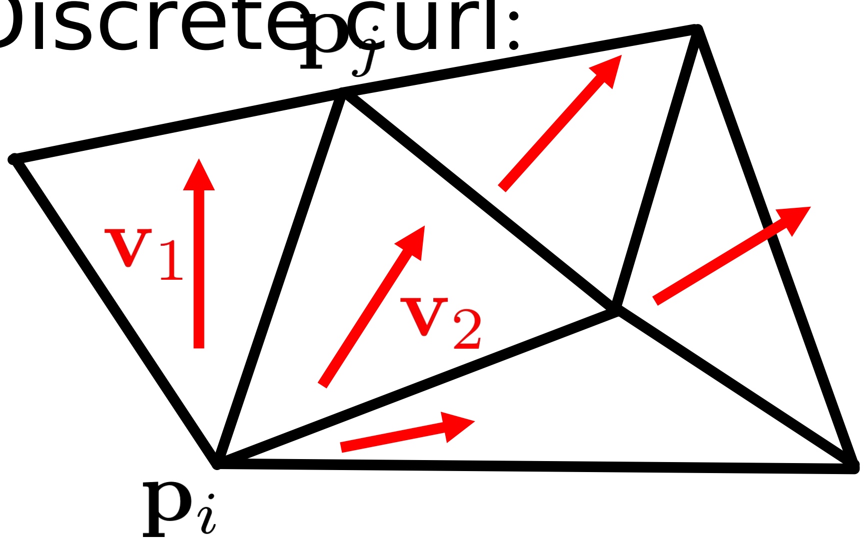
($\nabla \times \mathbf{v} = 0$ locally equivalent condition:)

Discrete Curl

A vector field is **integrable** if it is the gradient of a potential function $\mathbf{v} = \nabla f$

(locally equivalent condition: $\nabla \times \mathbf{v} = 0$)

Discrete curl: $$(\nabla \times \mathbf{v})_{ij} = (\mathbf{v}_1 - \mathbf{v}_2) \cdot (\mathbf{p}_j - \mathbf{p}_i)$$



Discrete Curl: Who Cares?

Geodesic condition can be written in terms of vector curl:

$$\nabla \times \mathbf{v} = 0$$

$$\|\mathbf{v}\| = 1$$

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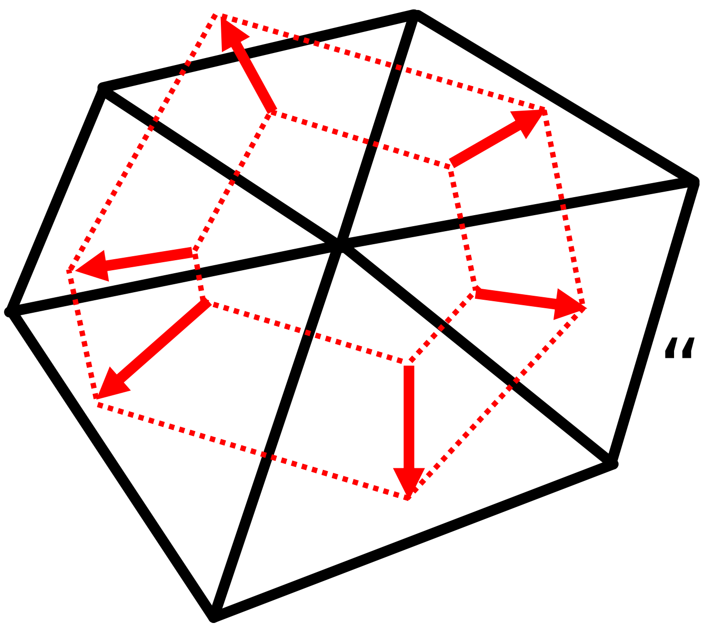


Discrete curl does not "see" good singularities!

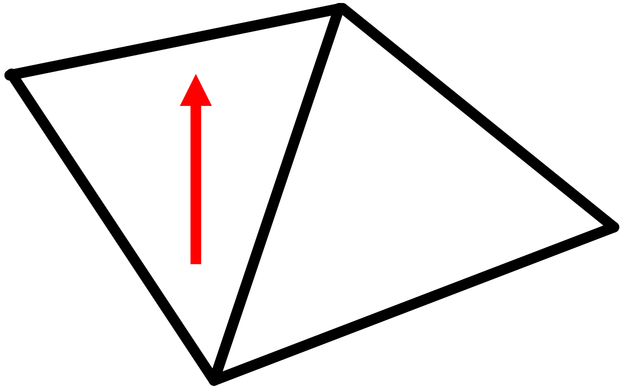
$$\nabla \times \mathbf{v} = 0$$

$$\|\mathbf{v}\| = 1$$

Discrete Geodesic Fields

In the smooth setting: there are many curl-free unit fields

Problem: discretization **overconstrained**



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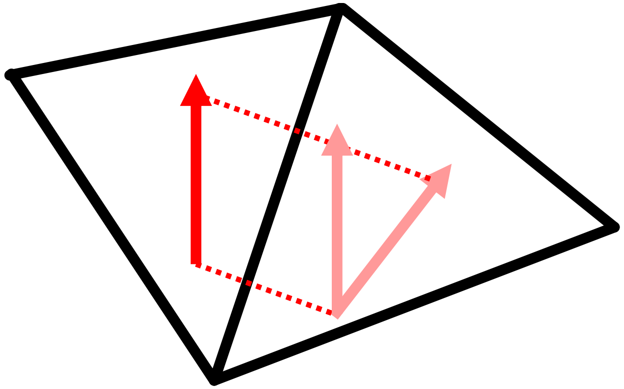

$$\nabla \times \mathbf{v} = 0$$

$$\|\mathbf{v}\| = 1$$

Discrete Geodesic Fields

We define a (discrete, approximately) geodesic field as $\hat{\mathbf{v}}$ any solution to:

$$\min_{\hat{\mathbf{v}}, \delta\mathbf{v}} \|\delta\mathbf{v}\|^2 \quad \text{s.t.} \quad \begin{aligned} \nabla \times (\hat{\mathbf{v}} + \delta\mathbf{v}) = 0 \\ \|\hat{\mathbf{v}}_i\| = 1 \end{aligned}$$

We show: in the smooth setting $\nabla \times \mathbf{v} = 0, \delta\mathbf{v} = 0$ solutions are exactly those with

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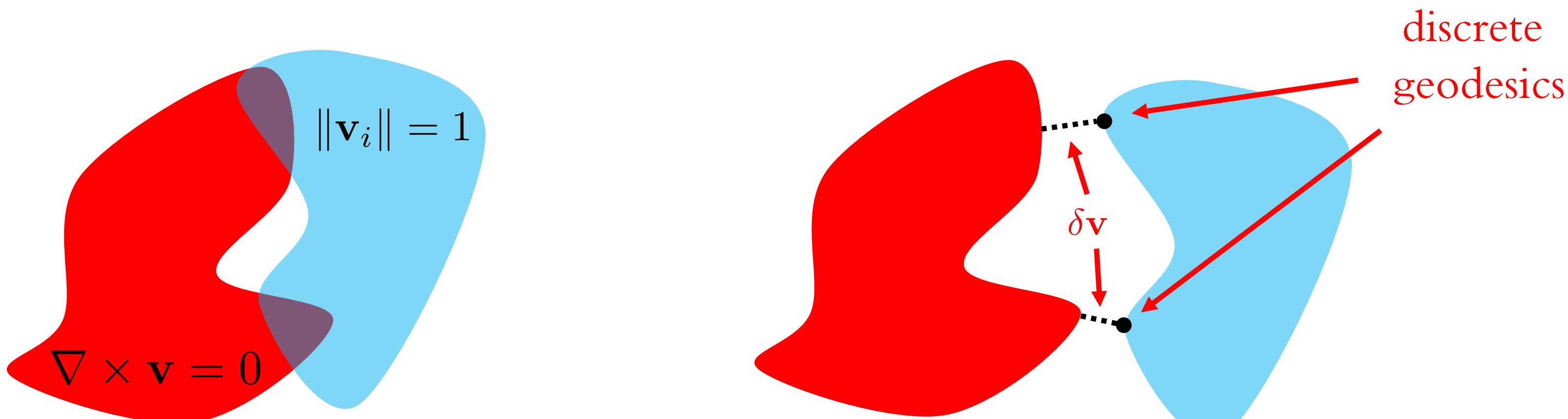

smooth setting

discrete setting

Geodesic Field Design

1. Start with initial unit field $\hat{\mathbf{v}}^0$
2. Descend using energy

$$\min_{\hat{\mathbf{v}},\delta\mathbf{v}} \|\delta\mathbf{v}\|^2 + \lambda\|\nabla(\hat{\mathbf{v}}+\delta\mathbf{v})\|^2 \text{ s.t. } \begin{array}{c} \nabla\times(\hat{\mathbf{v}}+\delta\mathbf{v}) = 0 \\ \|\hat{\mathbf{v}}_i\| = 1 \end{array}$$

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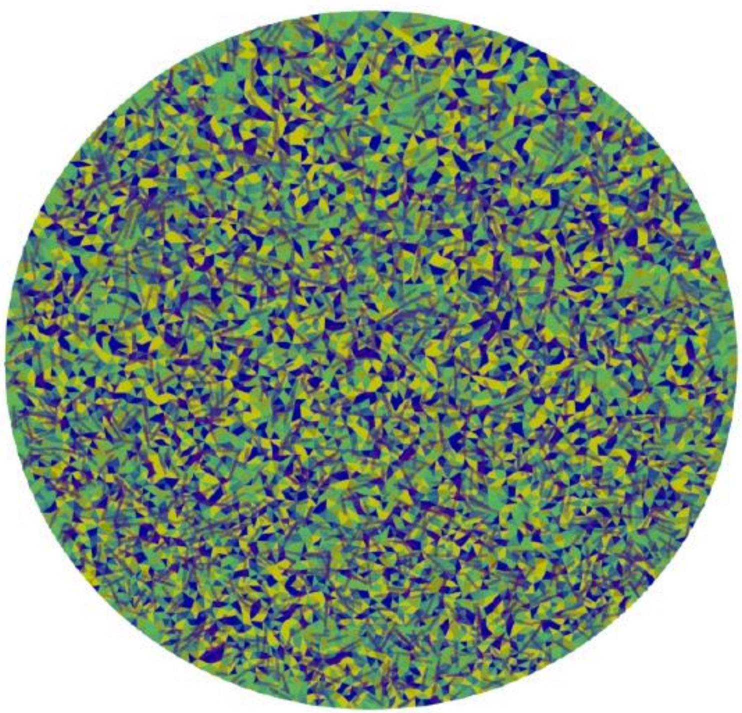
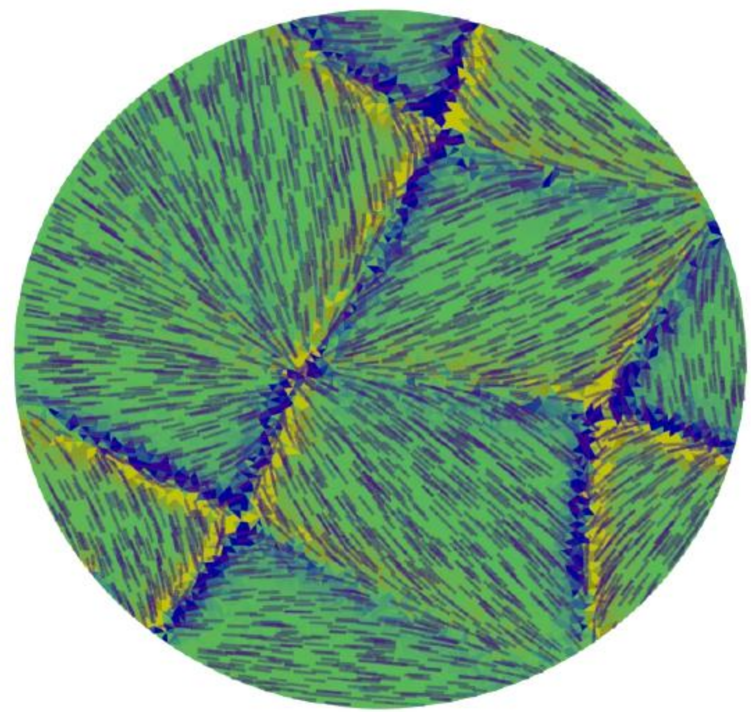
trades off smoothness and geodesic−ness

Results on Disk

For random initial field:



$\lambda = 0$



$\lambda = 10^{-7}$

$$\min_{\hat{\mathbf{v}},\delta\mathbf{v}} \|\delta\mathbf{v}\|^2 + \lambda\|\nabla(\hat{\mathbf{v}} + \delta\mathbf{v})\|^2 \text{ s.t. } \begin{array}{c} \nabla \times (\hat{\mathbf{v}} + \delta\mathbf{v}) = 0 \\ \|\hat{\mathbf{v}}_i\| = 1 \end{array}$$

Geodesic Field Design

1. Start with initial unit field $\hat{\mathbf{v}}^0$
2. For $j = 0, \ldots$
   - fix $\hat{\mathbf{v}}^j$, compute $\delta\mathbf{v}^{j+1/2}$

$$\min_{\hat{\mathbf{v}},\delta\mathbf{v}} \|\delta\mathbf{v}\|^2 + \lambda\|\nabla(\hat{\mathbf{v}}+\delta\mathbf{v})\|^2 \text{ s.t. } \begin{array}{c} \nabla\times(\hat{\mathbf{v}}+\delta\mathbf{v}) = 0 \\ \|\hat{\mathbf{v}}_i\| = 1 \end{array}$$

Geodesic Field Design

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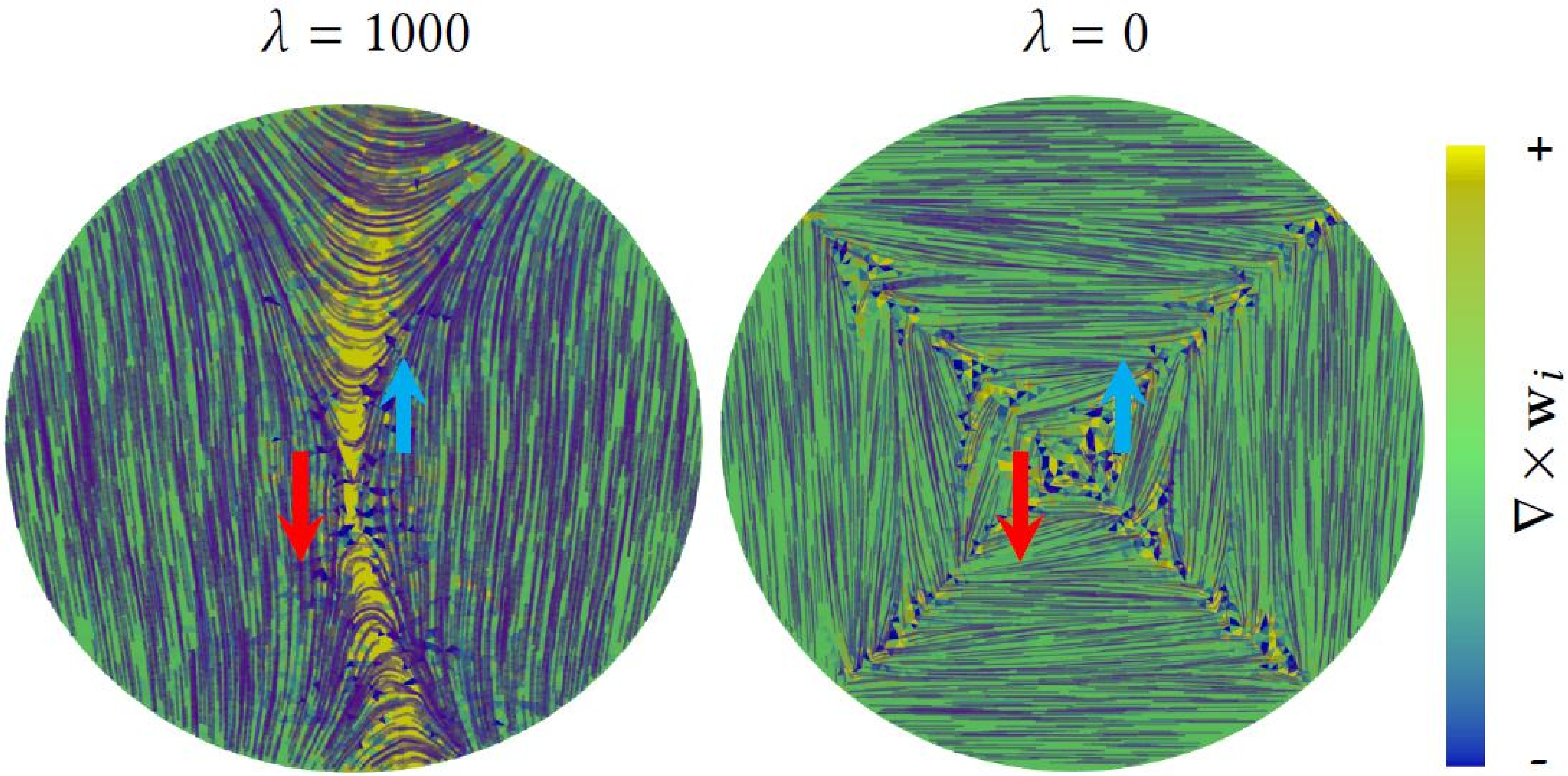
   - fix $\hat{\mathbf{v}}^j$, compute $\delta\mathbf{v}^{j+1/2}$
   - set

$$\hat{\mathbf{v}}^{j+1} = \frac{\hat{\mathbf{v}}^j + \delta\mathbf{v}^{j+1/2}}{\|\hat{\mathbf{v}}^j + \delta\mathbf{v}^{j+1/2}\|}$$

$$\delta\mathbf{v}^{j+1} = \mathbf{v}^j + \delta\mathbf{v}^{j+1/2} - \mathbf{v}^{j+1}$$

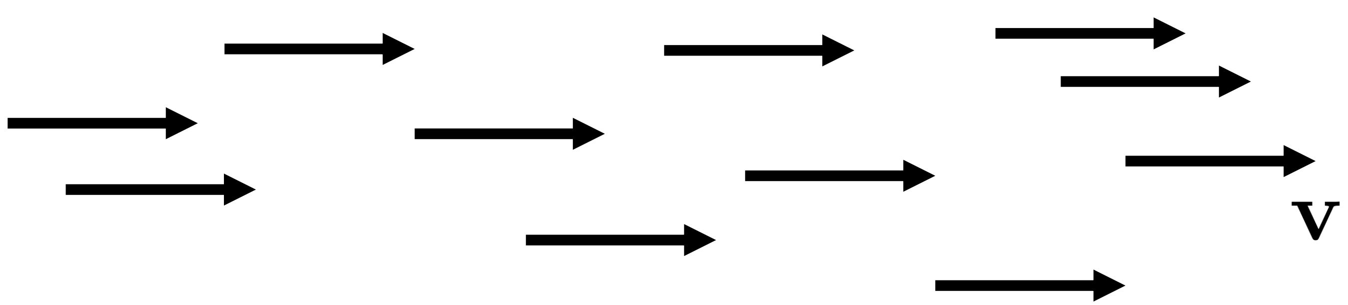
Effect of Smoothness Term



Results in 3D

Extracting Integral Curves

Once we have the vector field, how to trace out the integral curves?



Usual approach: find scalar function $\nabla f = \mathbf{v}^{\perp}$

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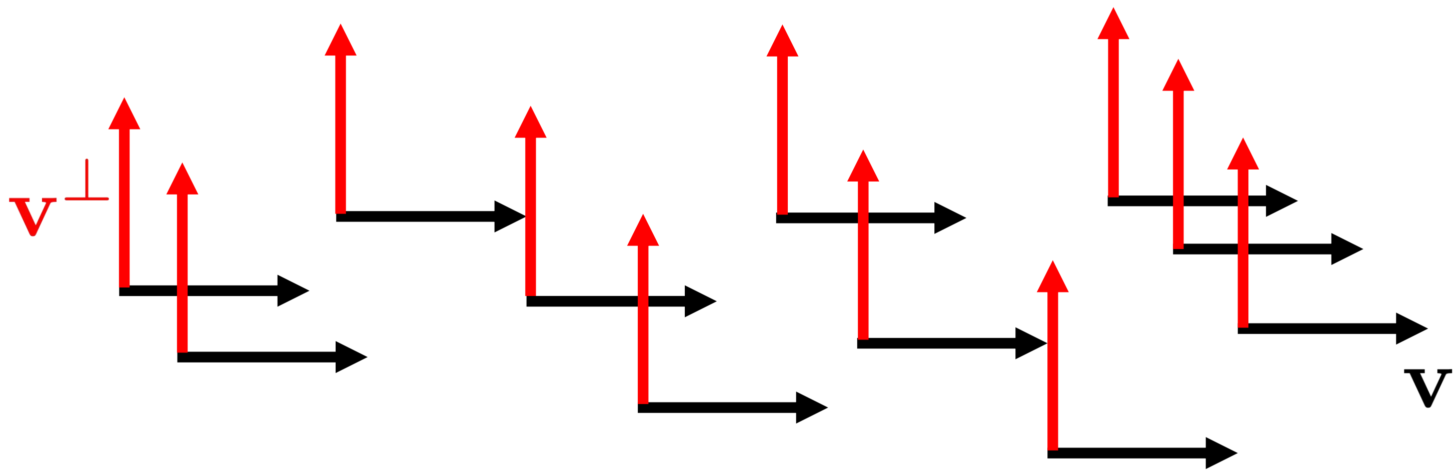

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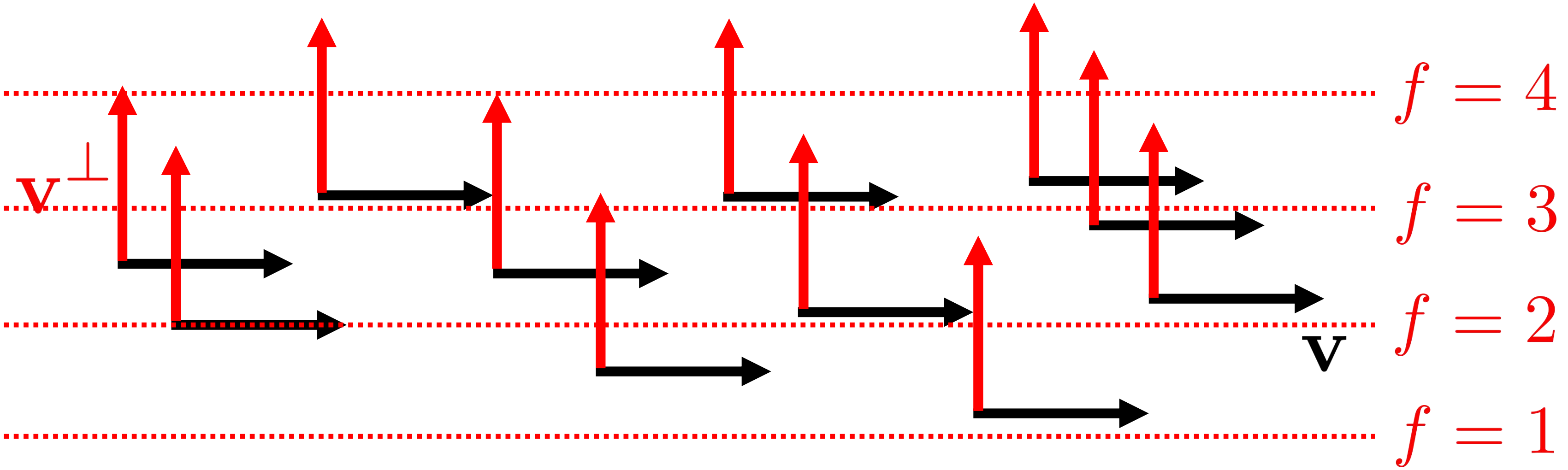

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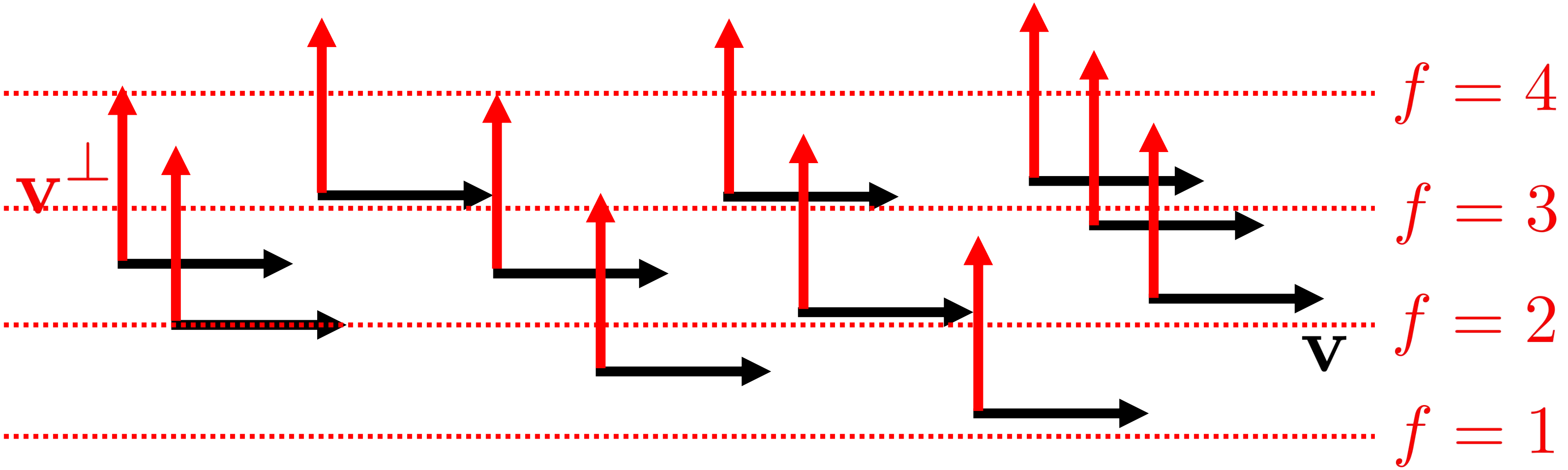


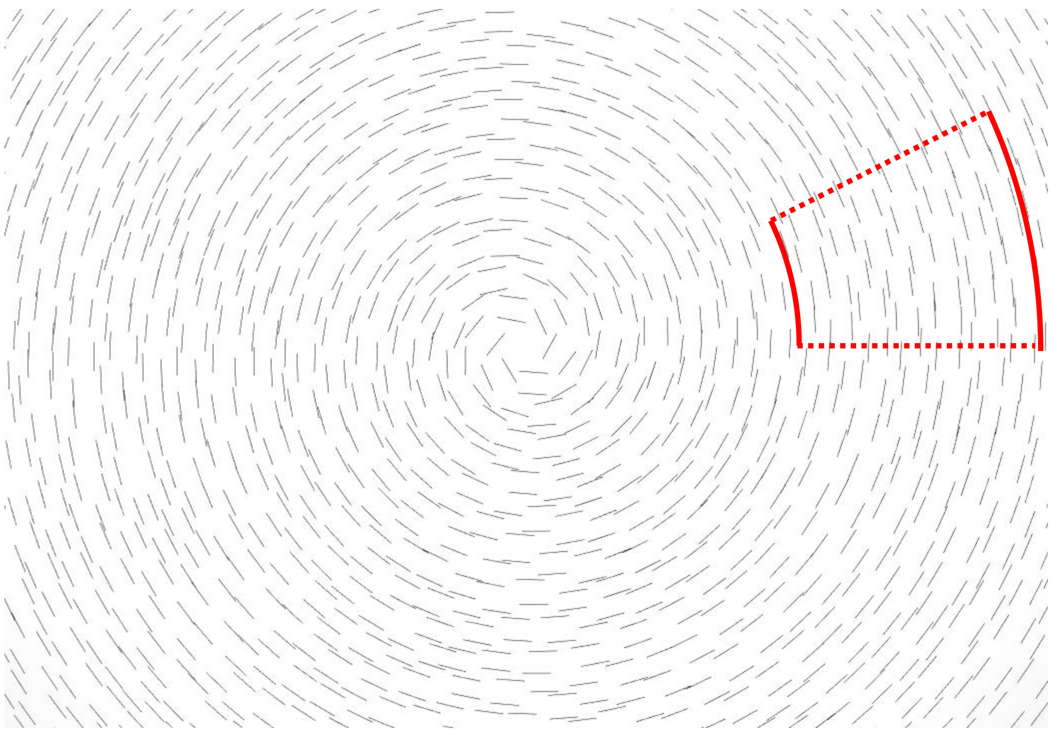
Usual approach: find scalar function $\nabla f = \mathbf{v}^\perp$

Extract level sets of f

Extracting Integral Curves

Two possible obstructions:

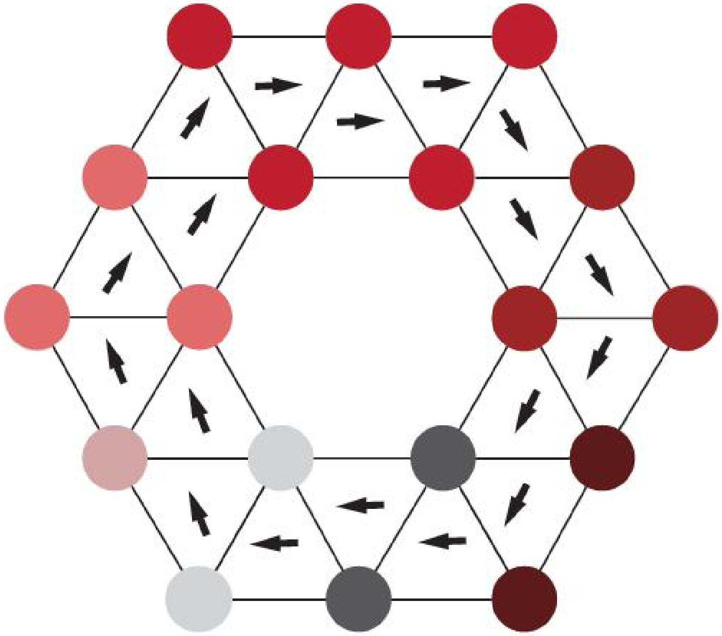
– local: failure of $\mathbf{v}^{\perp}$ to be curl-free



Extracting Integral Curves

Two possible obstructions:

- local: failure of $v^{\perp}$ to be curl-free
- global:

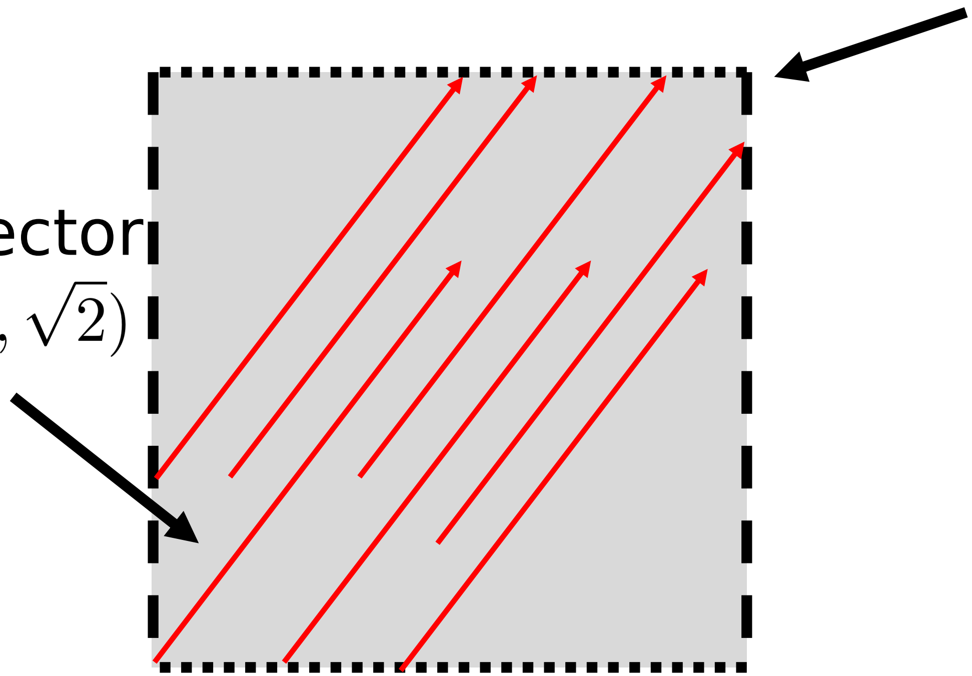


Extracting Integral Curves

Sometimes no solution:

flat 2x2 torus

constant vector field $\mathbf{v}^\perp = (1, \sqrt{2})$



Fixing Local Integrability Failure

Main idea: we care only about the direction of the geodesic field, not the magnitude

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Find a scalar function s with

$$\nabla \times (s\mathbf{v}^{\perp}) = 0$$

Fixing Local Integrability Failure

Reuse idea from geodesic field design:

$$\min_{s,\delta\mathbf{v}^{\perp}} \int \|\delta\mathbf{v}^{\perp}\|^2 \quad \text{s.t.} \quad \nabla \times (s\mathbf{v}^{\perp} + \delta\mathbf{v}^{\perp}) = 0$$

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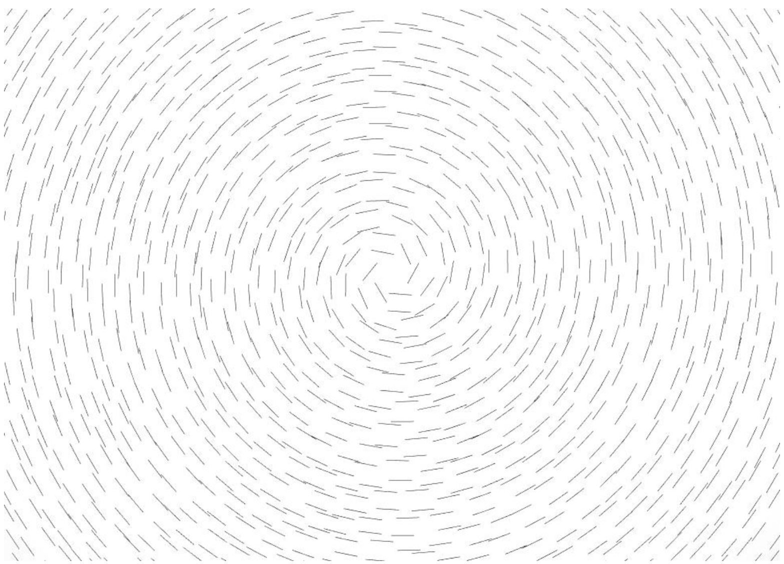
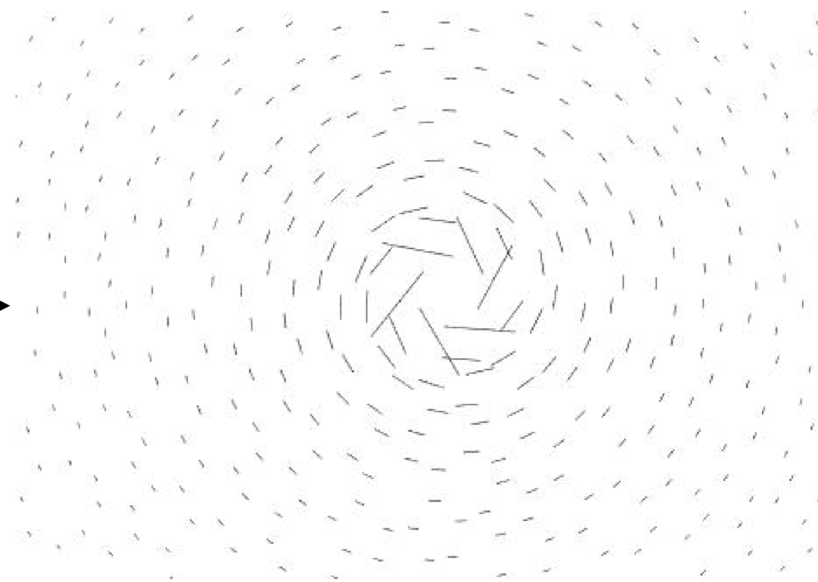
Turns into **generalized eigenvector** problem

Fixing Local Integrability Failure

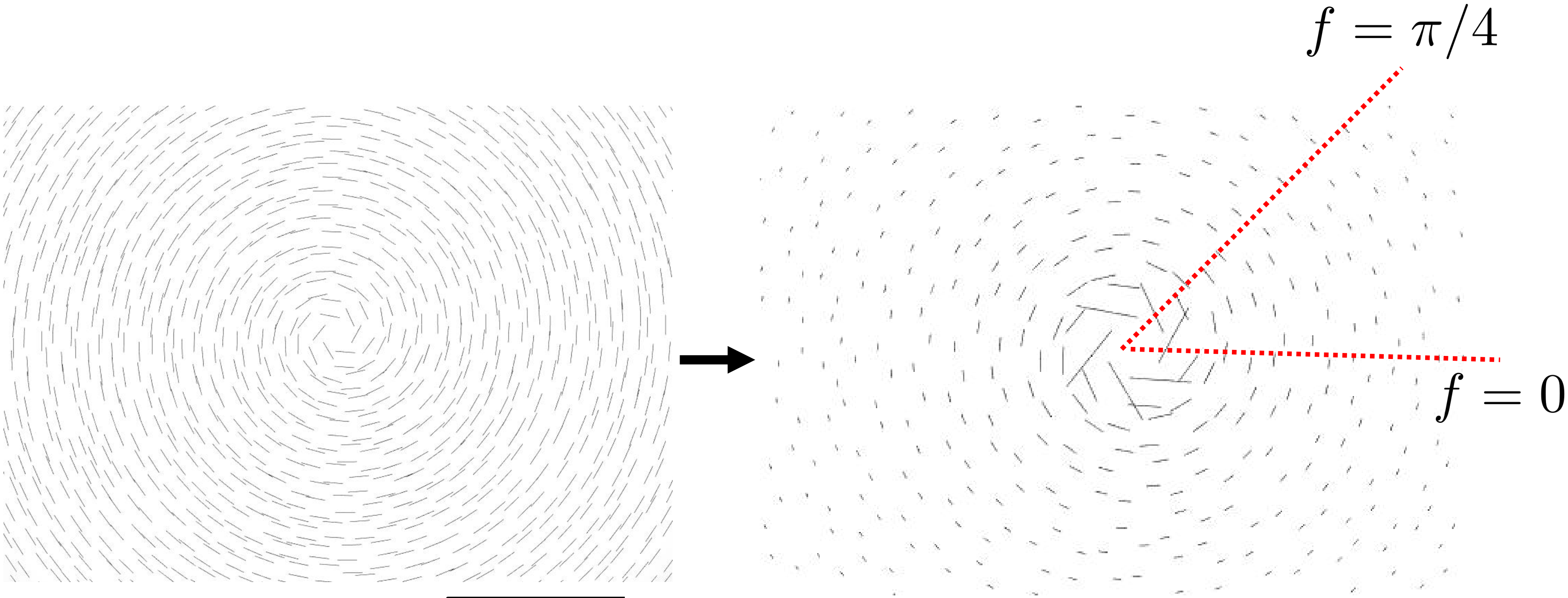


$$\mathbf{v}^{\perp} = (-y, x)/\sqrt{x^2 + y^2}$$



$$s\mathbf{v}^{\perp} = (-y, x)/(x^2 + y^2)$$

Fixing Local Integrability Failure

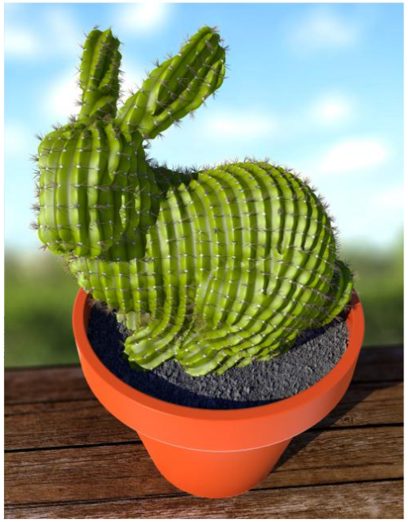


$\mathbf{v}^{\perp} = (-y, x)/\sqrt{x^2 + y^2}$

$s\mathbf{v}^{\perp} = (-y, x)/(x^2 + y^2)$

Fixing Global Integrability Failure

- Very challenging; no fully satisfying solution exists
- We use global nonlinear optimization technique initialized with $s\mathbf{v}^\perp$



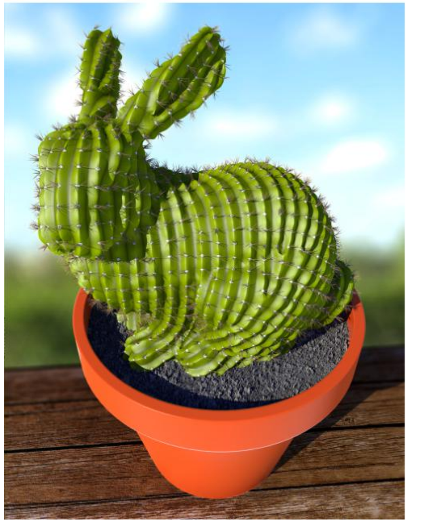
Knöppel et al

Integrated Vector Fields

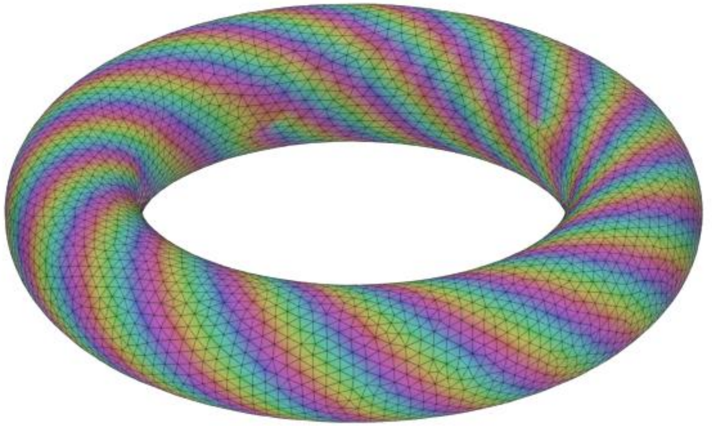
Fixing Global Integrability Failure

Very challenging; no fully satisfying solution exists

We use global nonlinear optimization technique initialized with $s\mathbf{v}^{\perp}$



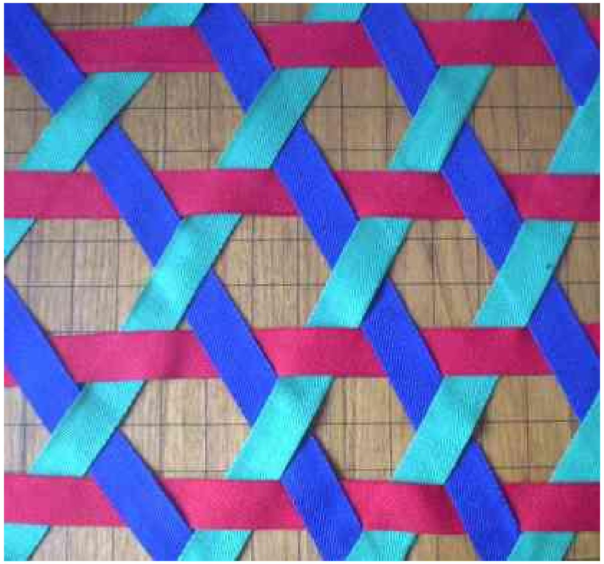
Knöppel et al



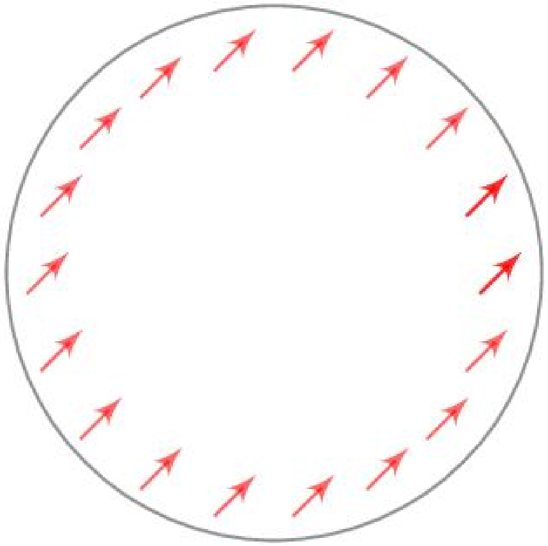
Result: function $\nabla\theta \approx s\mathbf{v}^{\perp}$ whose isolines are the designed geodesics

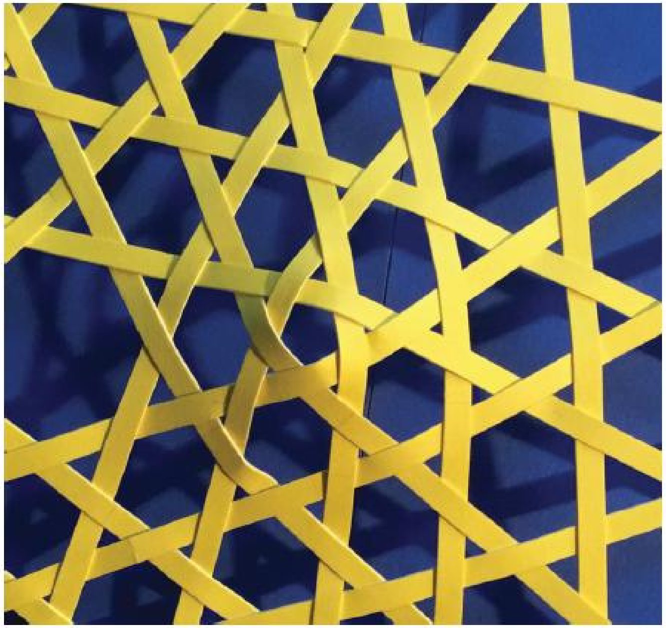
Back to Basketweaving

In a **triaxial weave** ribbons are laid out in three near-parallel families



Topological Weave Singularities

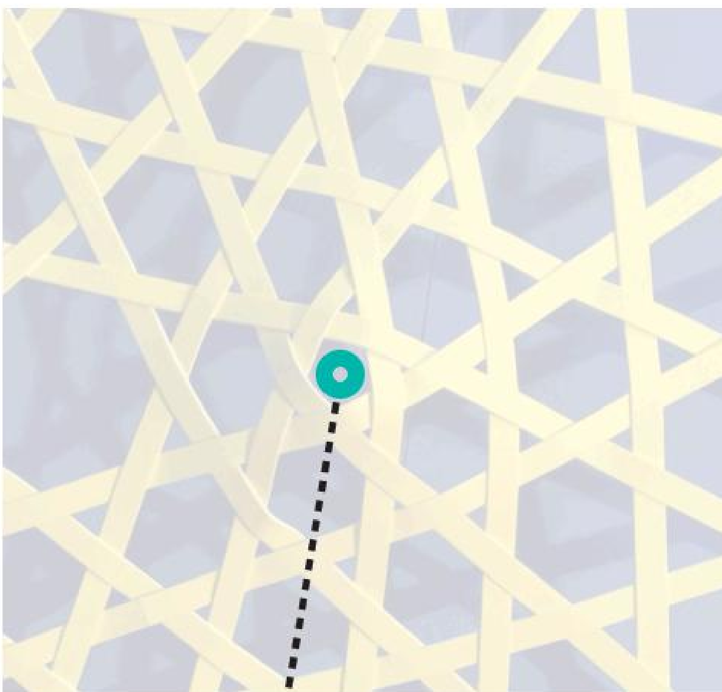


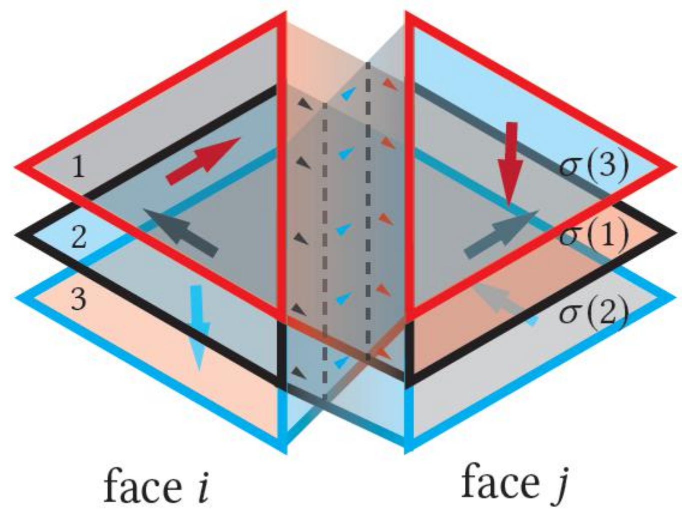


http://images.math.cnrs.fr/Visualiser-la-courbure.html

Topological Weave Singularities

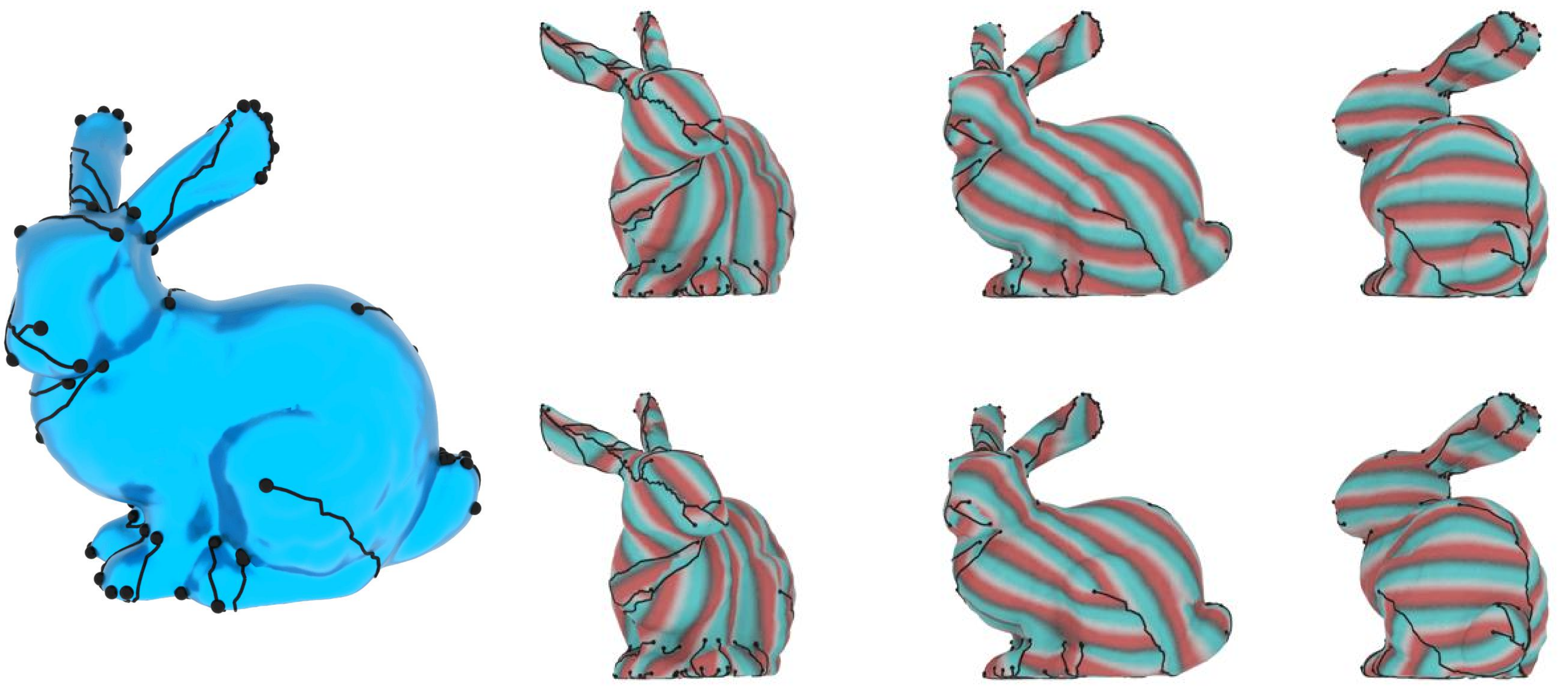
Circulating around singularity permutes the six weave families





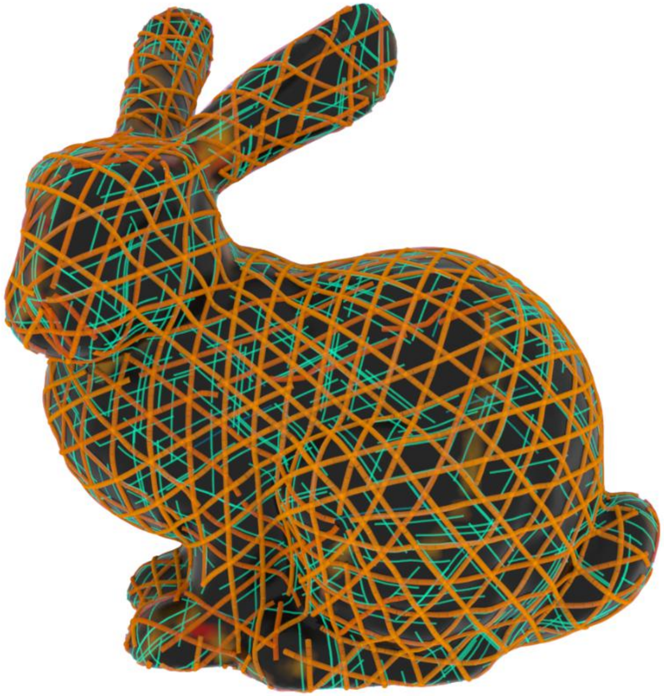
Triaxial Weave Design

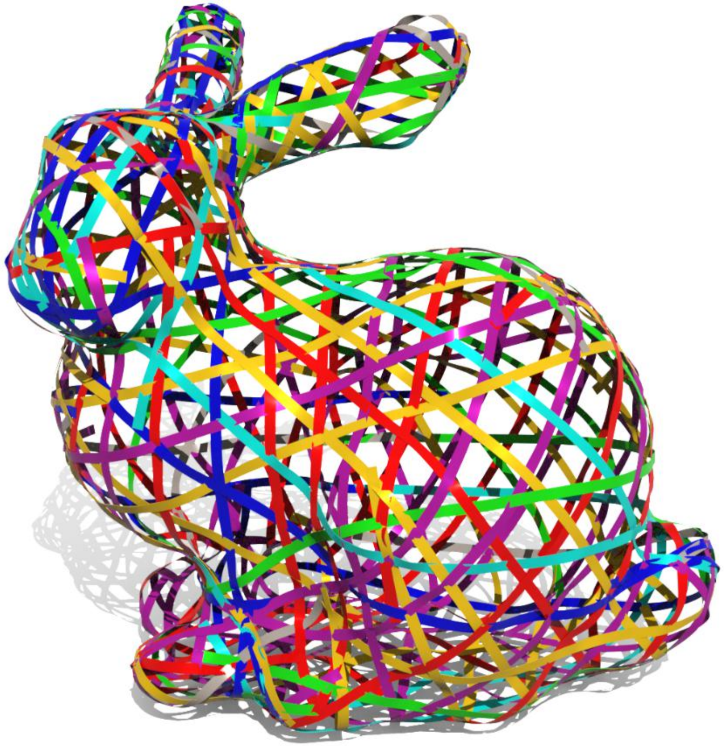
Design **single** geodesic foliation on **sixfold cover** of original surface



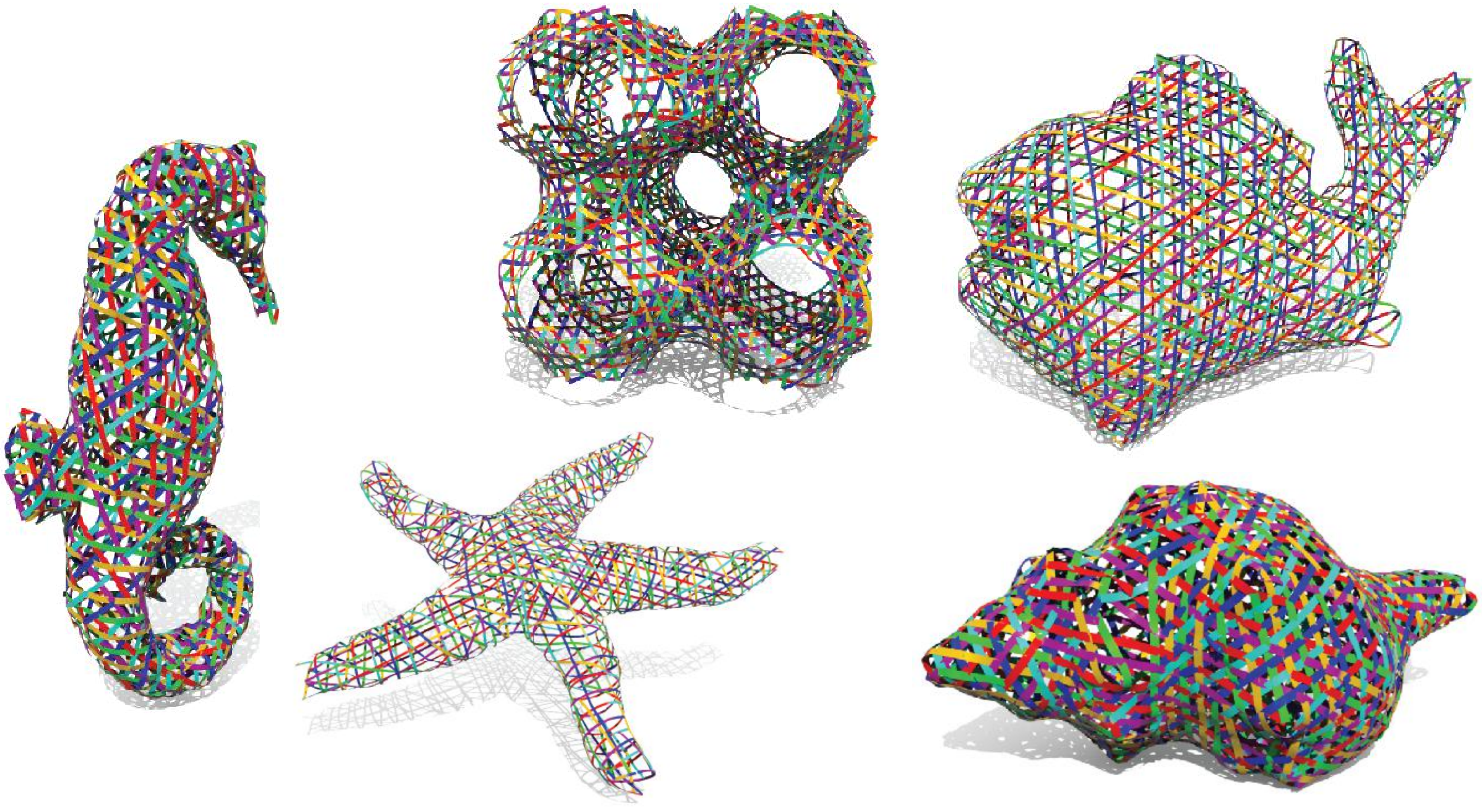
Triaxial Weave Design

Then extract isolines, polish with sim

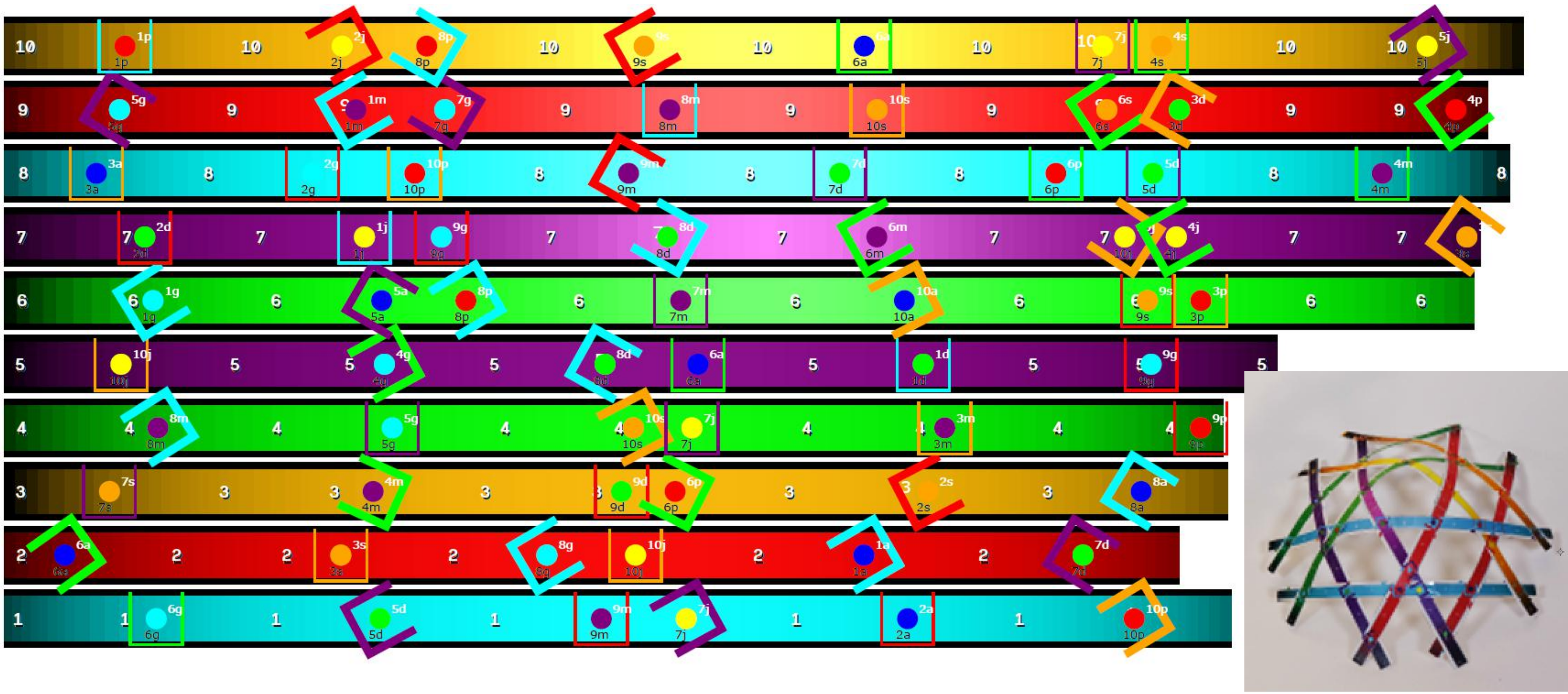




More Results



A Real Design File



Fabricated Examples