Utrecht University

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אוניברסיטת בר-אילן

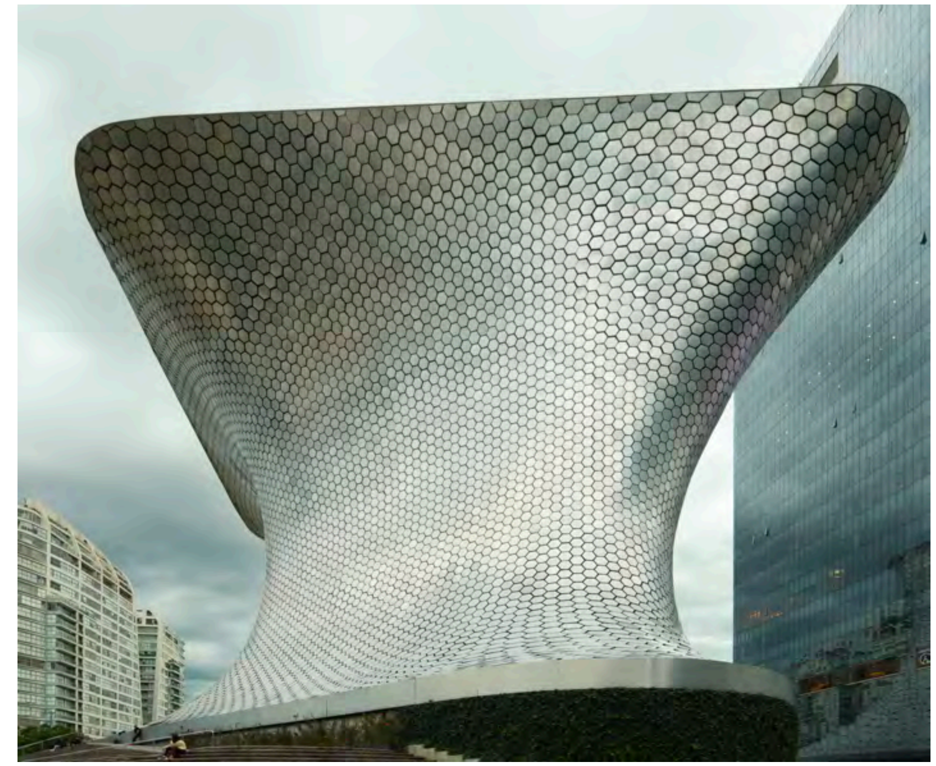
Möbius Geometry Processing

Amir Vaxman

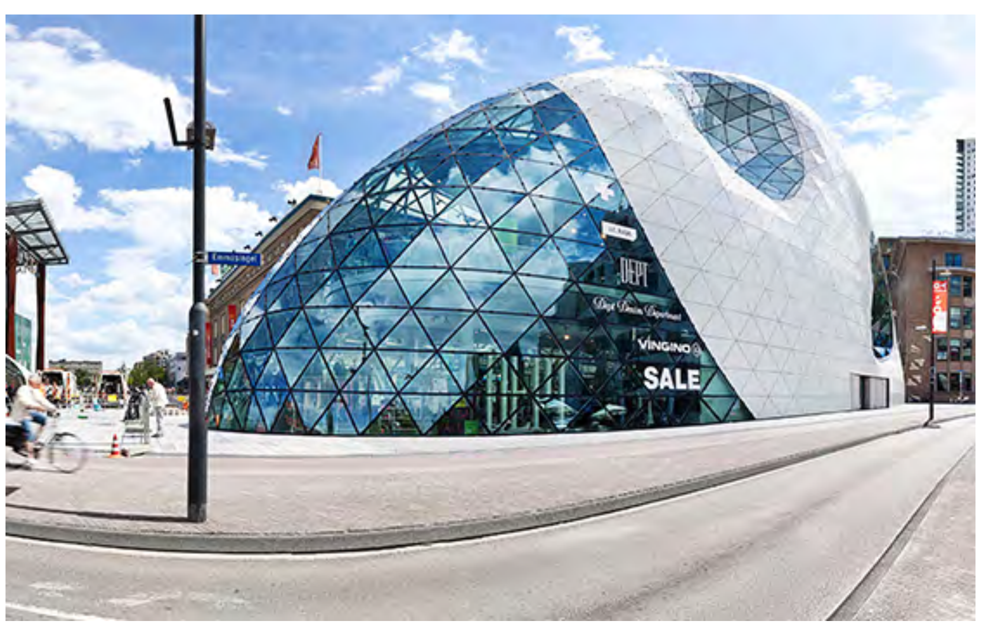
Collaboration with Christian Müller and Ofir Weber

Motivation: Architectural Geometry

Eye Museum, Amsterdam



Museo Soumaya, Mexico City

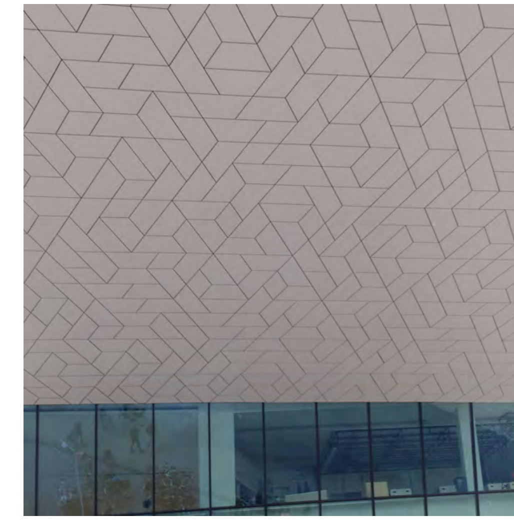
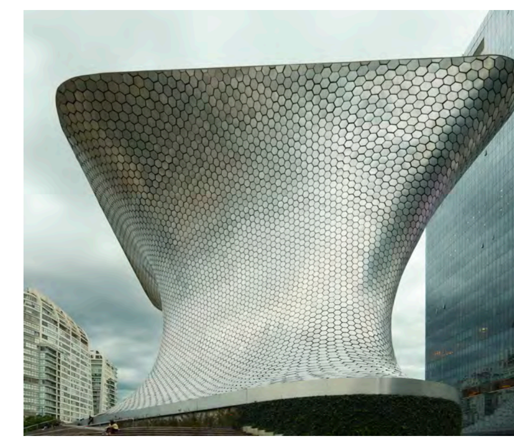
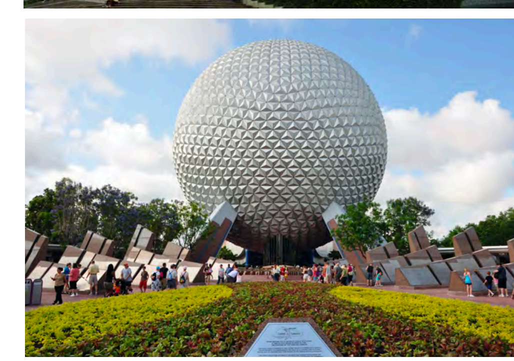
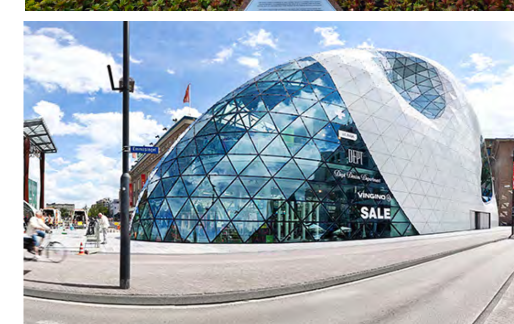


De Blob, Eindhoven

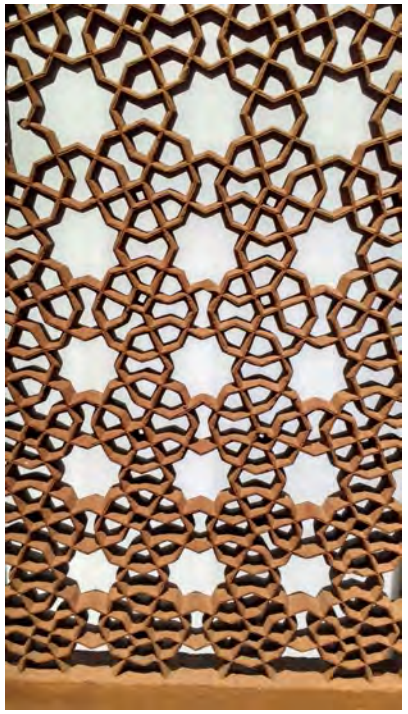
Epcot Theme Park, Bay Lake, Florida

Architectural Geometry

- Striking features:
  - Unconventional patterns
  - Regularity
  - Spherical and circular aesthetics

Unconventional Patterns



Islamic Museum, Louvre

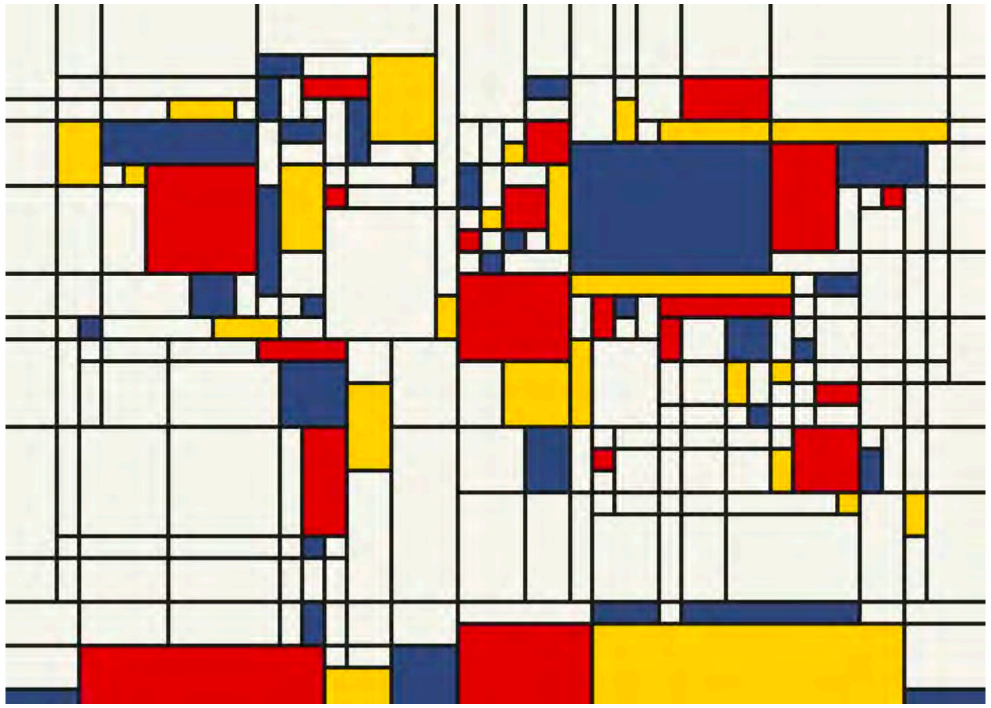
N.E.R.V.O.U.S. Systems

sidewalk patterns

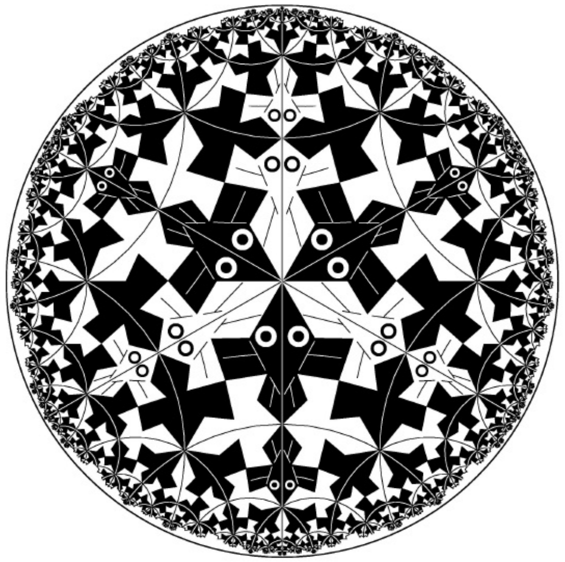
Penrose Tilings

Dutch Unconventional Patterns

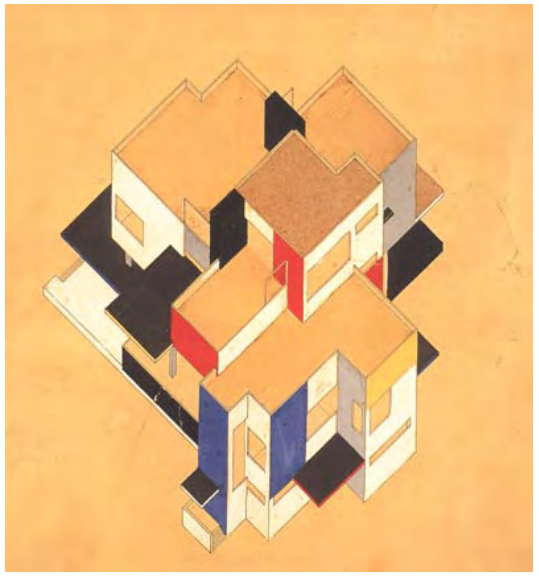
Westergasfabriek, Amsterdam



Mondriaan

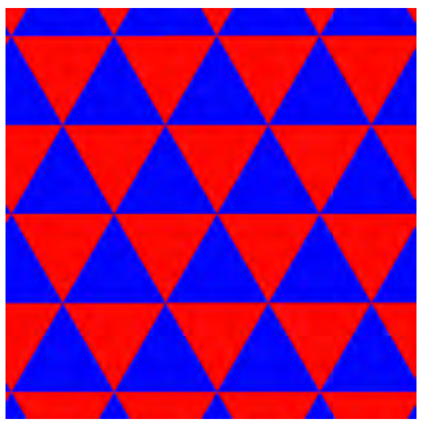


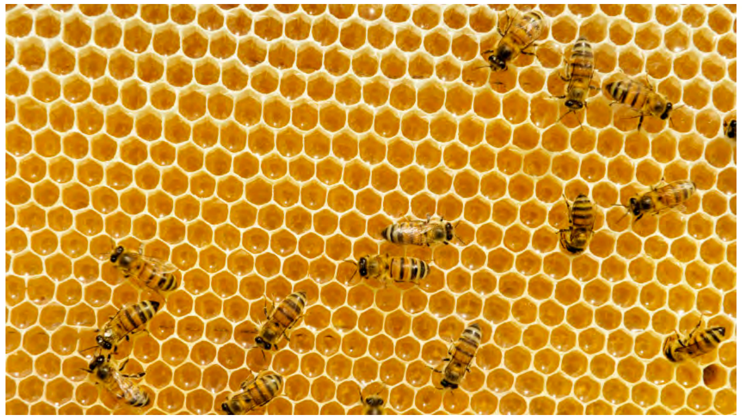
Escher



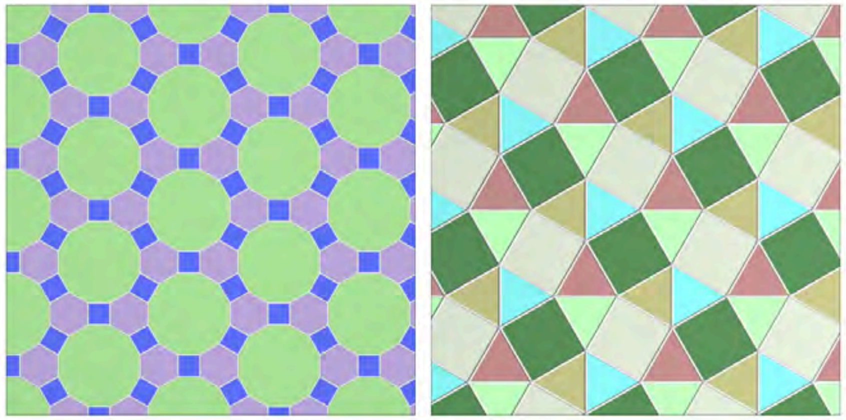
van Doesburg

(Semi-) Regular Patterns

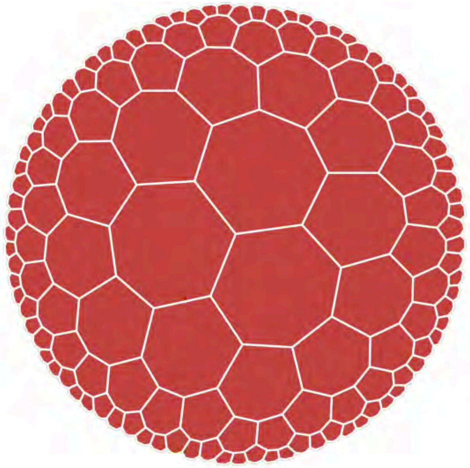




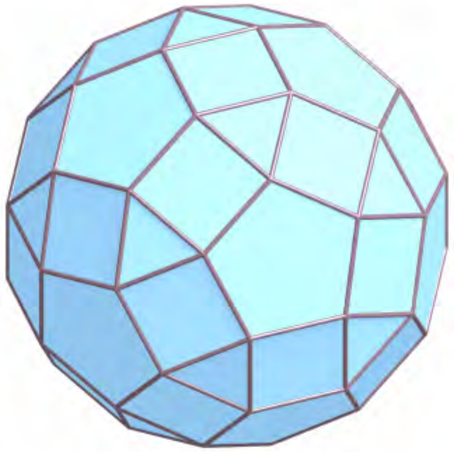
Pure



Mixed, Archimedean

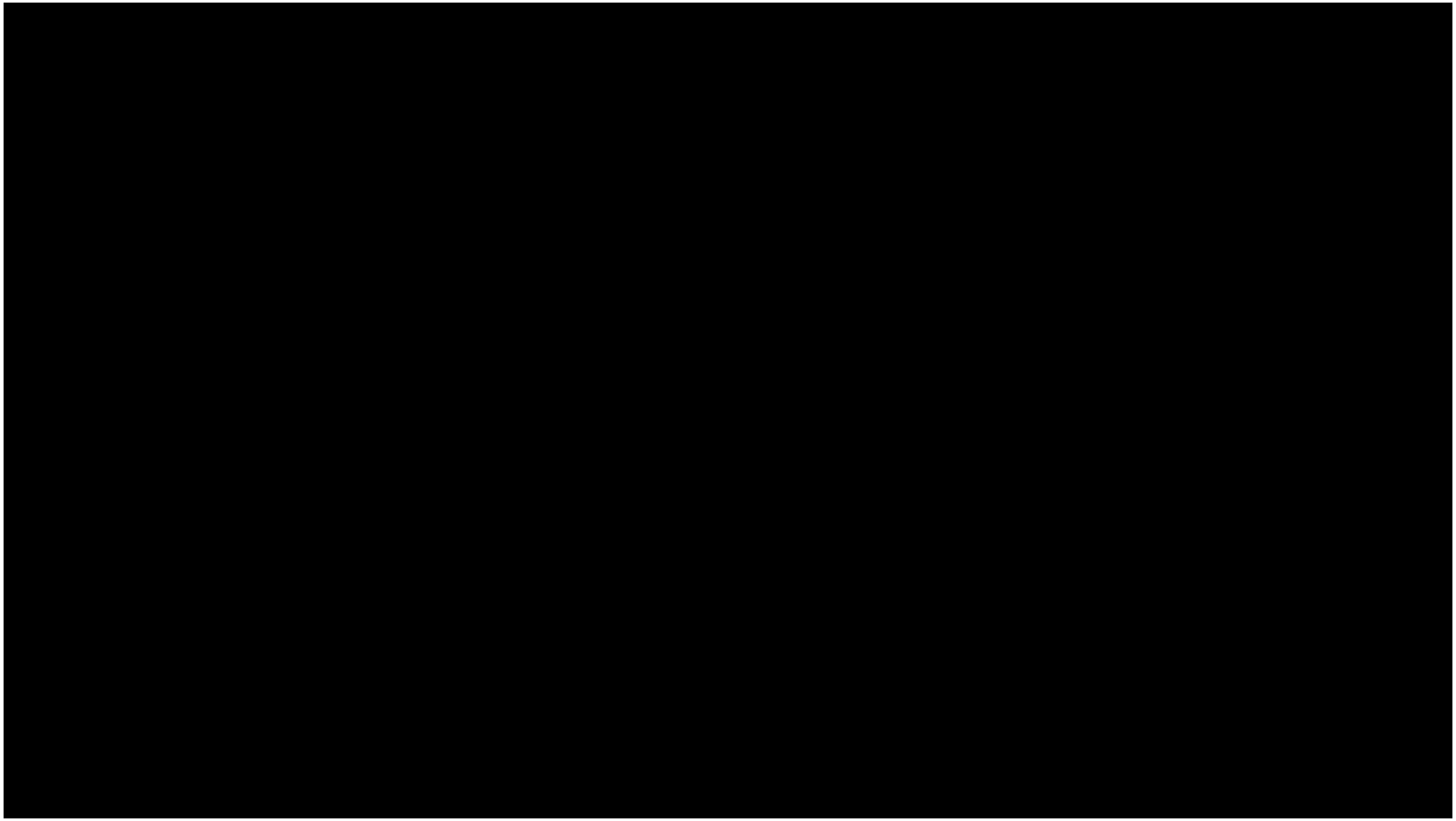


Hyperbolic



Spherical

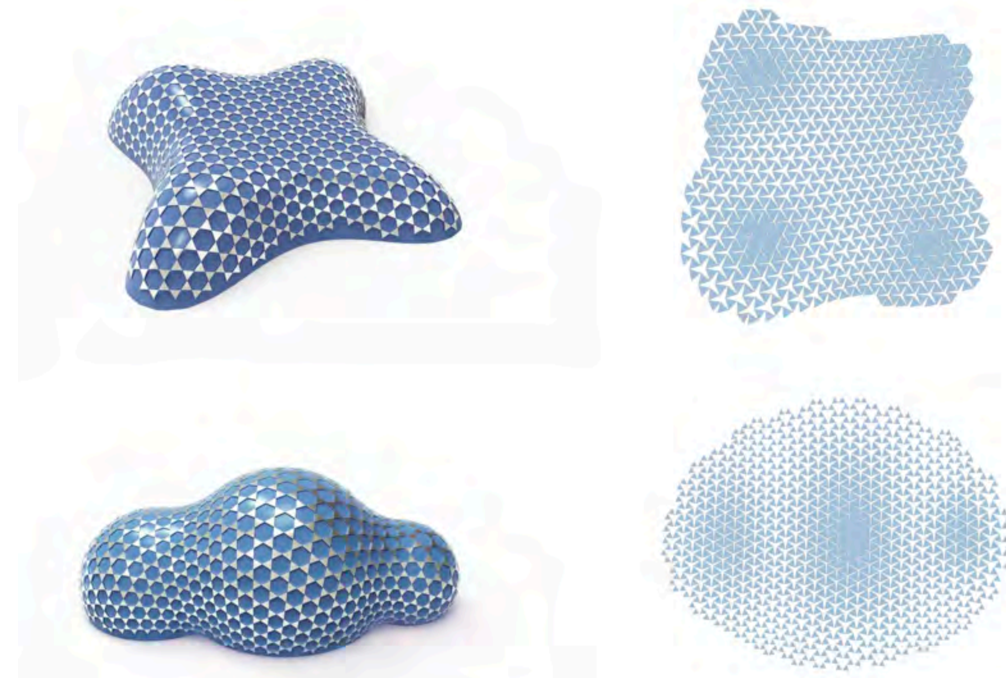
Design Paradigm: Geometry from Combinatorics



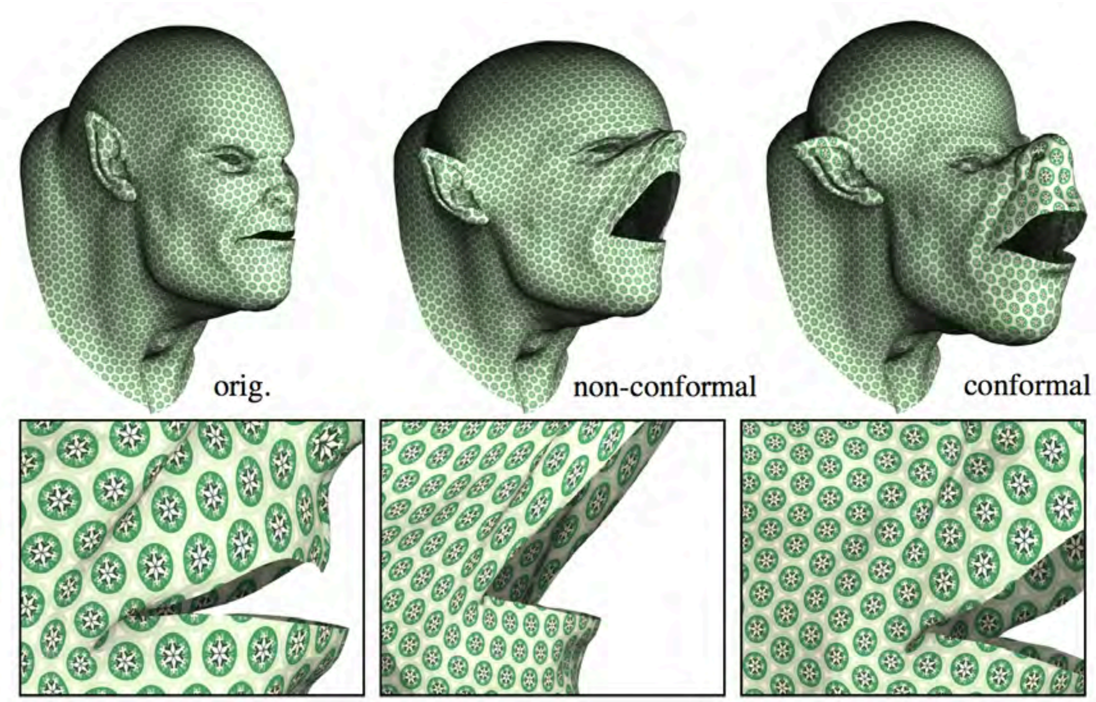
Handle-based deformation + optimization

Conformal Equivalence

- Local scales + rotation
  - no shear
- Preserves features



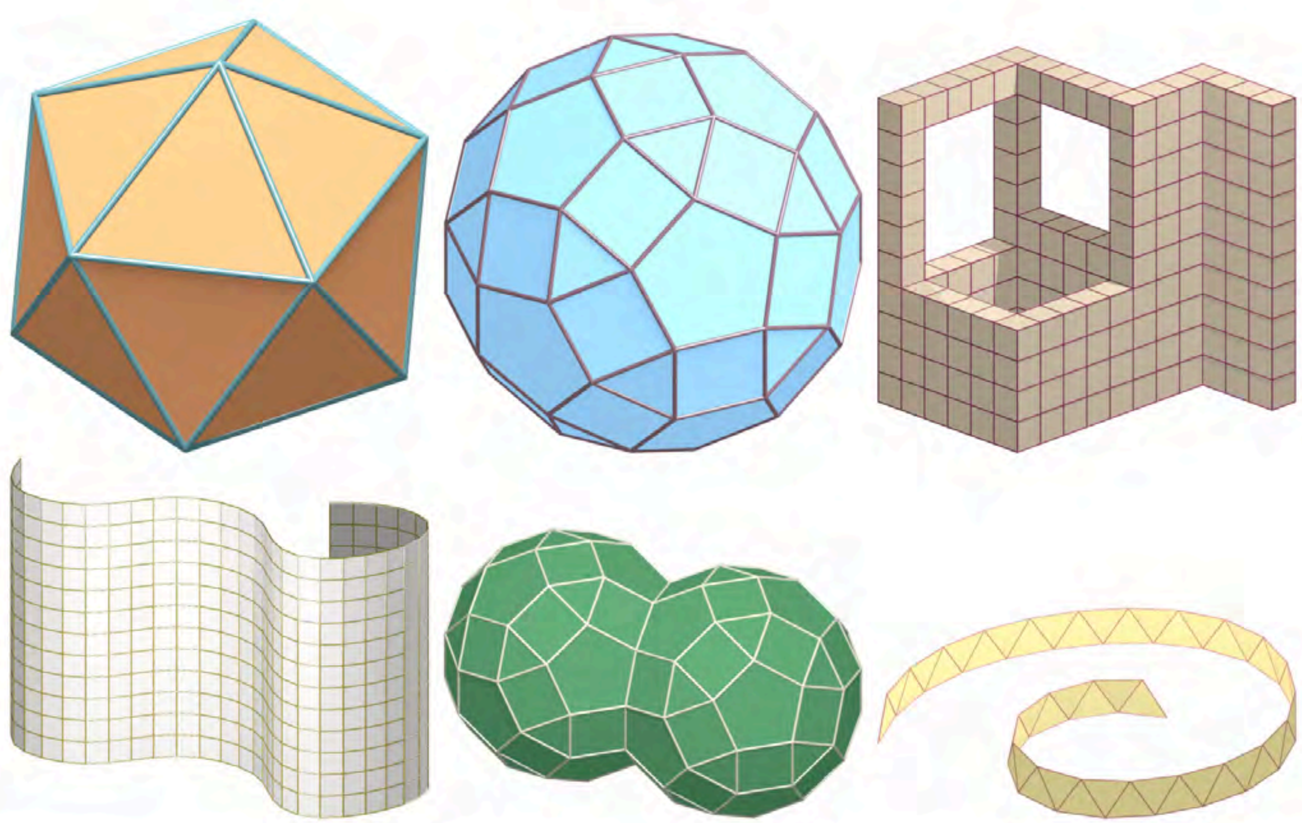
[Konakovic-Lukovic et al. 2018]



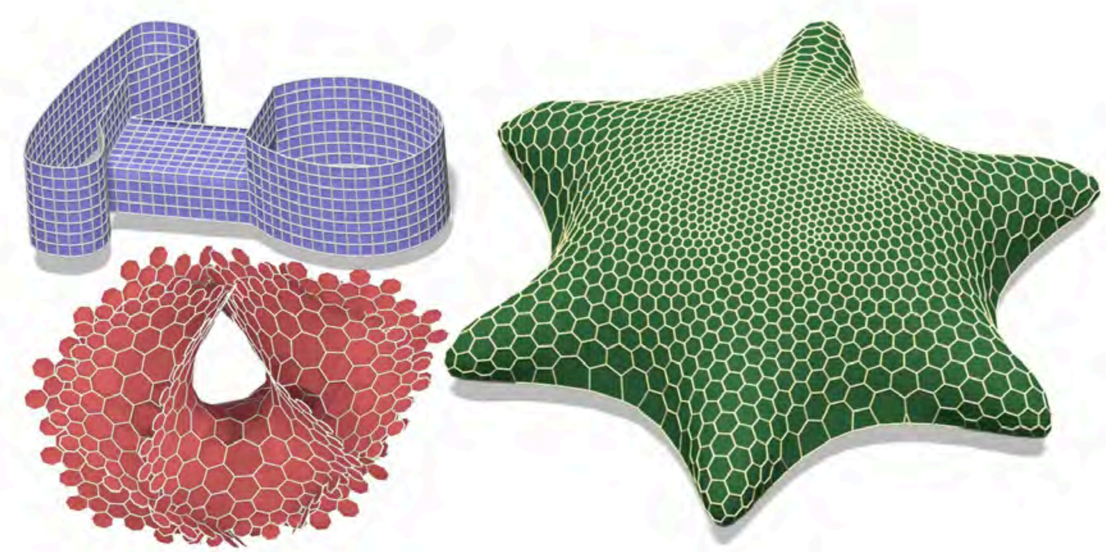
“Kreod” Pavillons, London

Regular Meshes from Polygonal Patterns

- What is the “best” mesh for a given regular pattern?
- As-regular-as-possible
- Regular = conformal + original regular pattern.



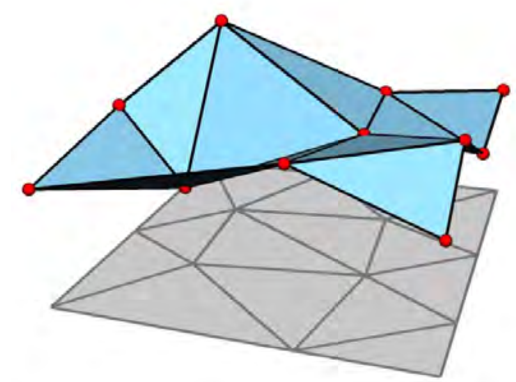
Perfect



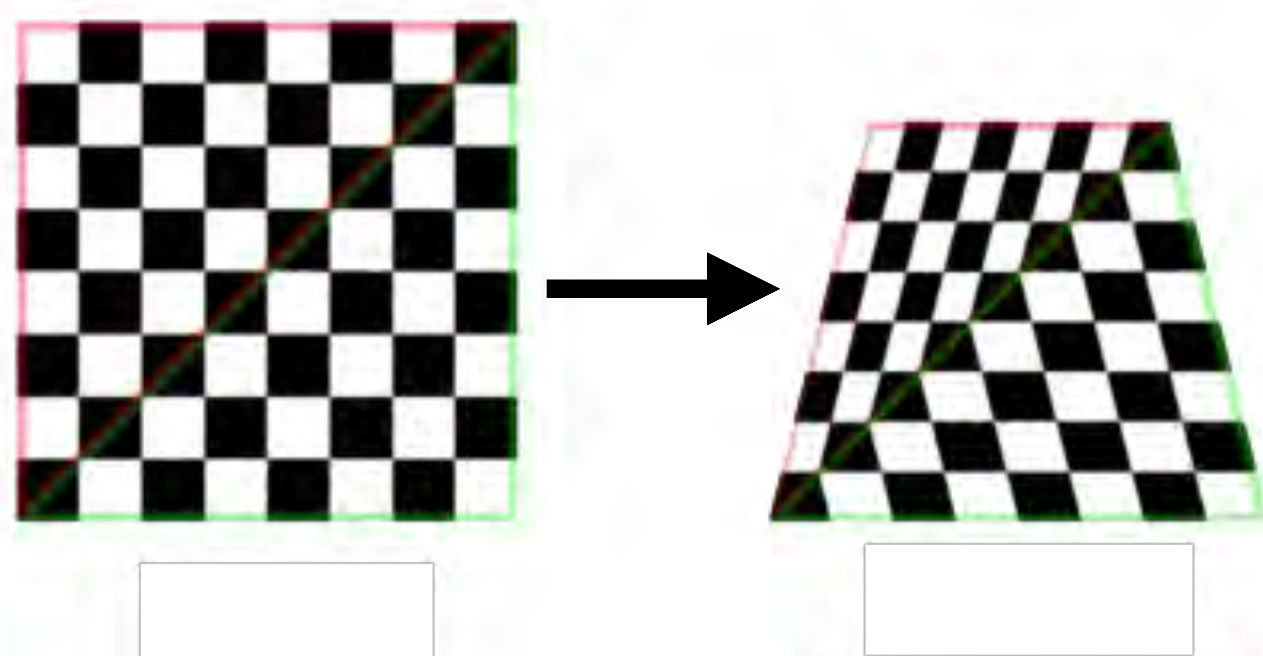
As-possible

Piecewise Linearity: the FEM Paradigm

- Staple of geometry processing
- (Mostly) triangle-based
- Scalar function space: vertex-based
- Transformations: piecewise affine

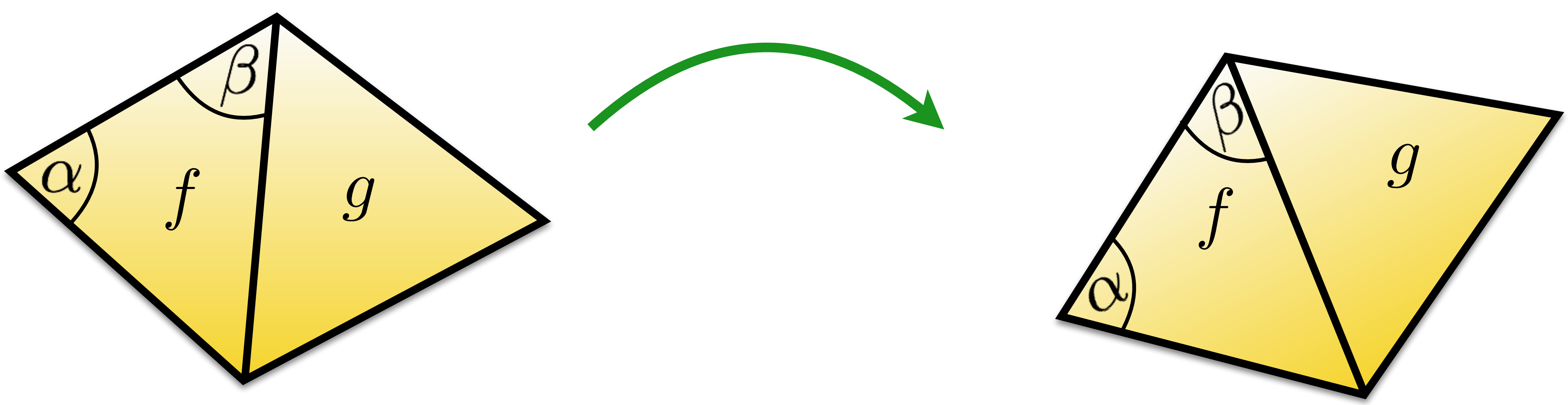


[Nieser 2012]



FEM Conformality

- Conformal = preservation of angles.
- Piecewise affine transformations ==> no “true conformal” but global similarities.

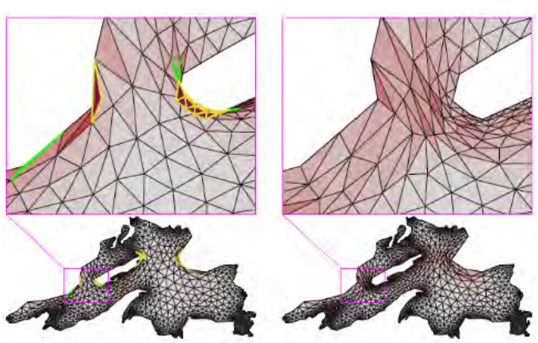


FEM Conformality

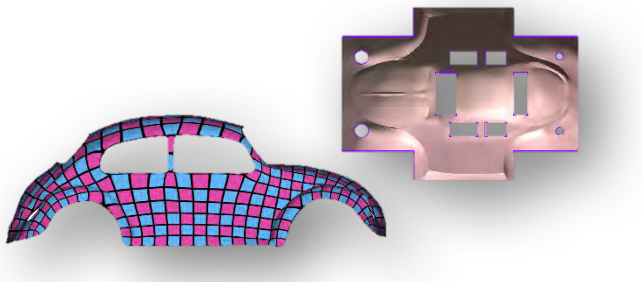
- Problem: no “true conformal” but global similarities.
  - Only “as-possible”, bounded or approximate.
  - Limited support for polygonal meshes.

[Levy et al. 2002]

[Crane et al. 2011]



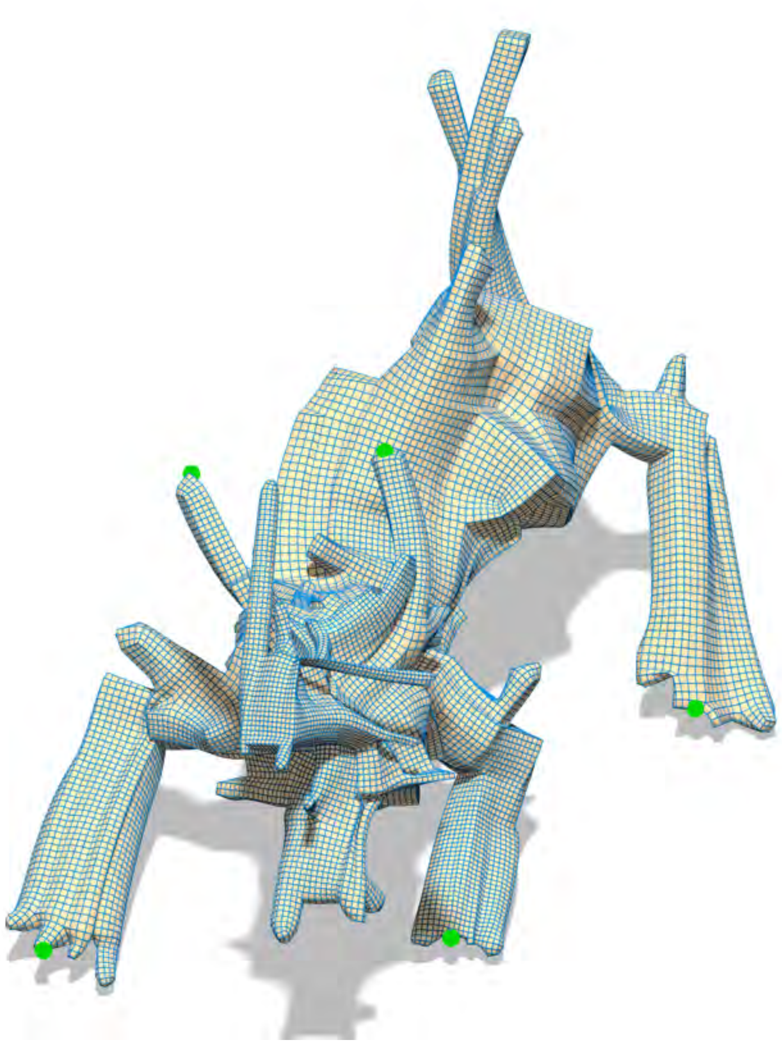
[Lipman 2012]

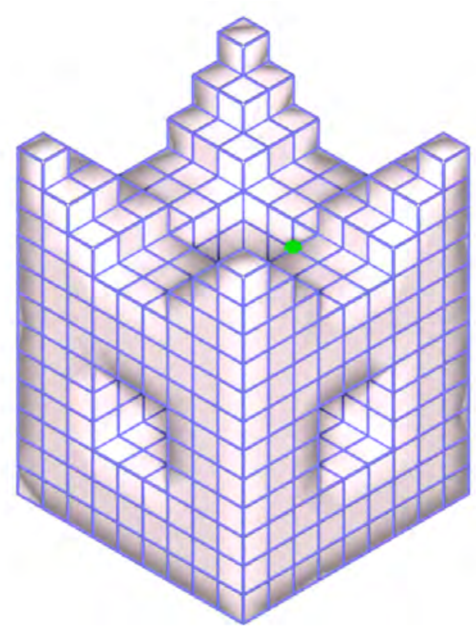


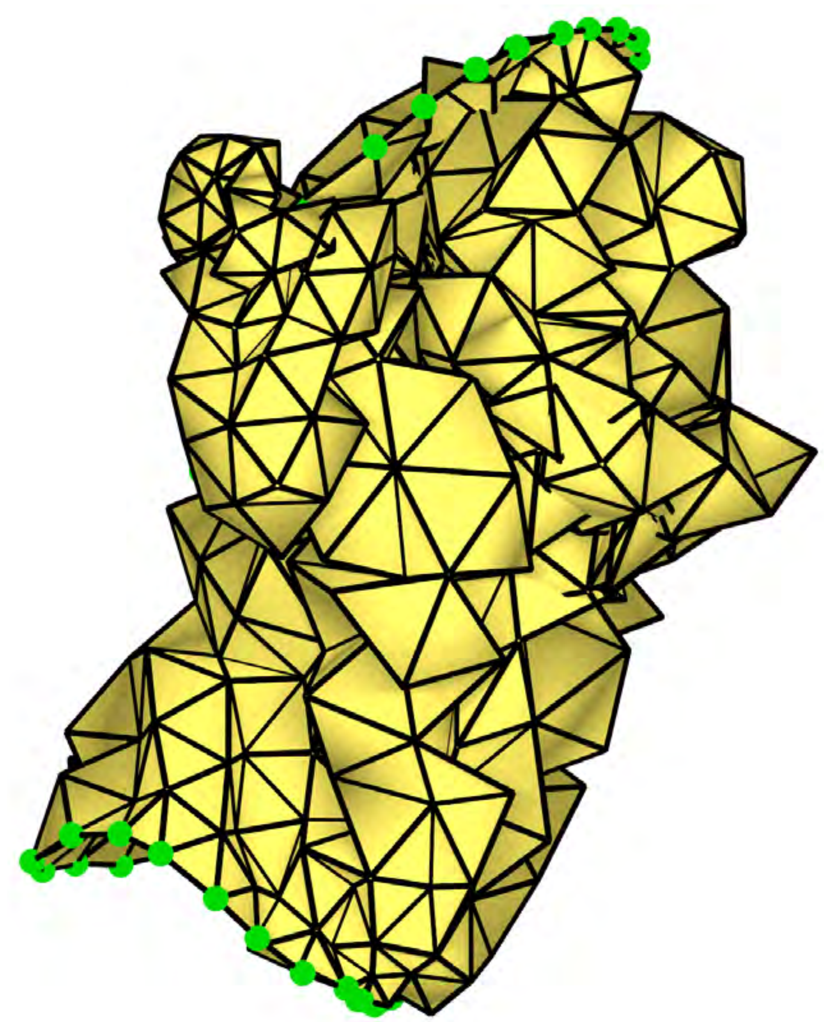
[Weber et al. 2002]

FEM Regularity

- Every face as regular as possible?
- For quad meshes: developable surfaces.
- Problematic for other types.

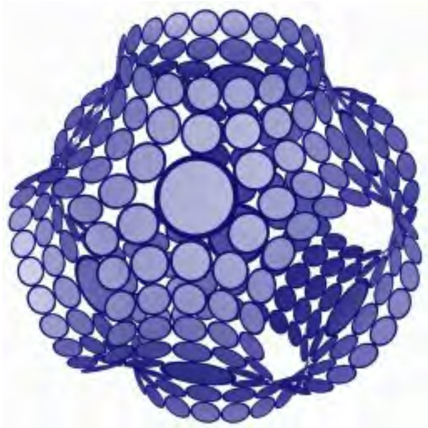


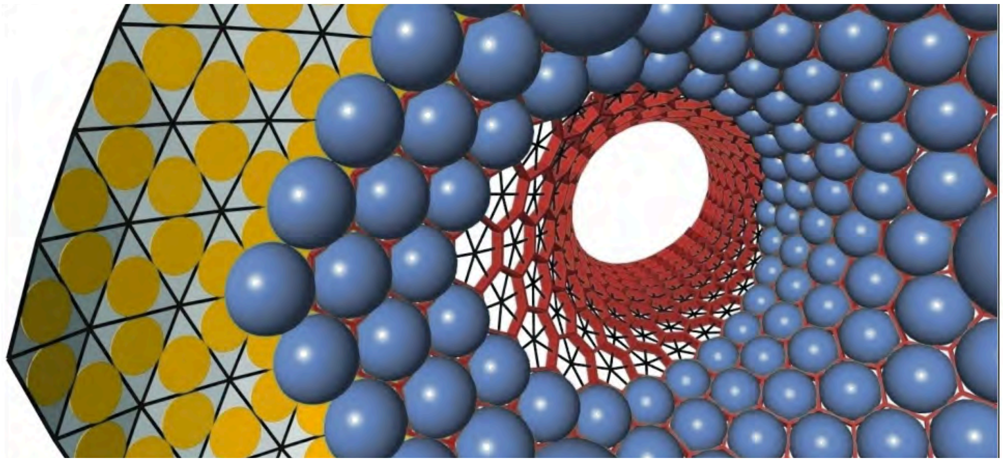




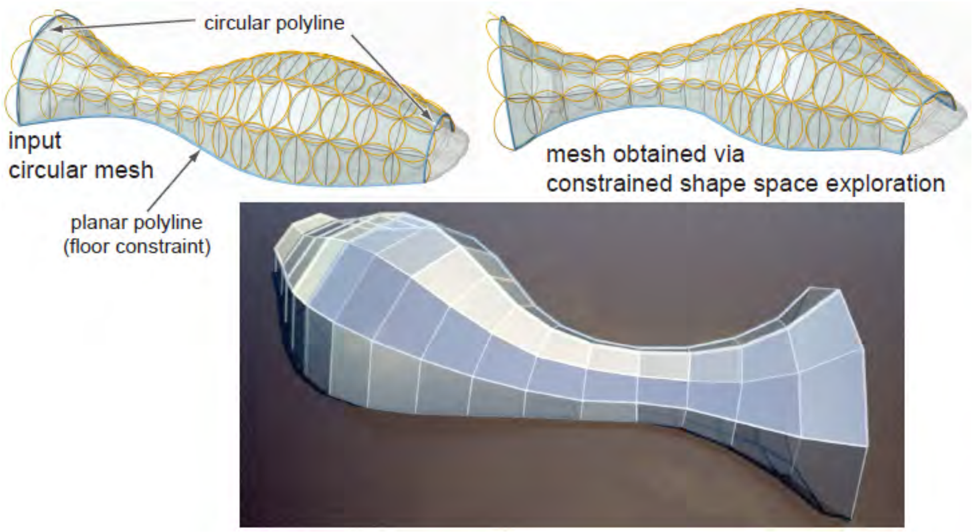
Alternative paradigm: Surfaces from Circles

- Circumcircle per face
- Discrete differential geometry

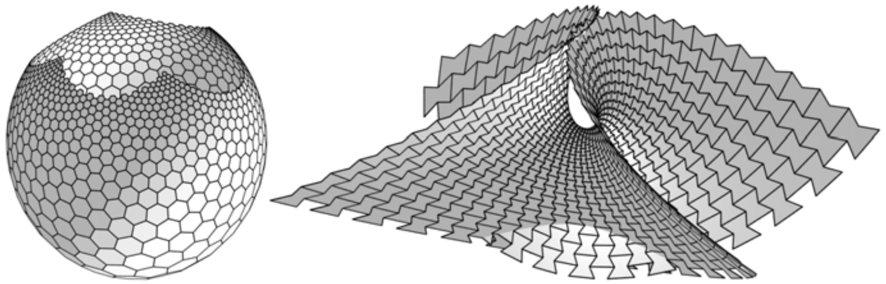




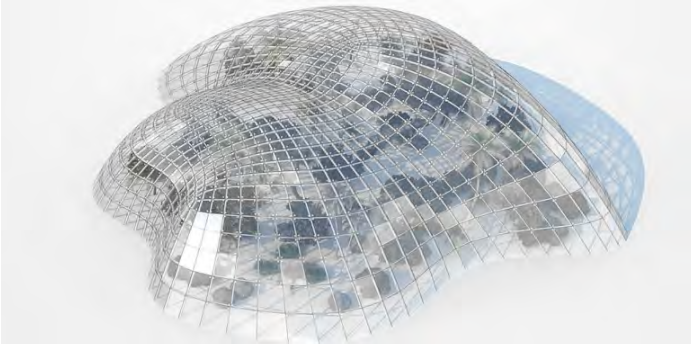
[Schiftner et al. 2009]



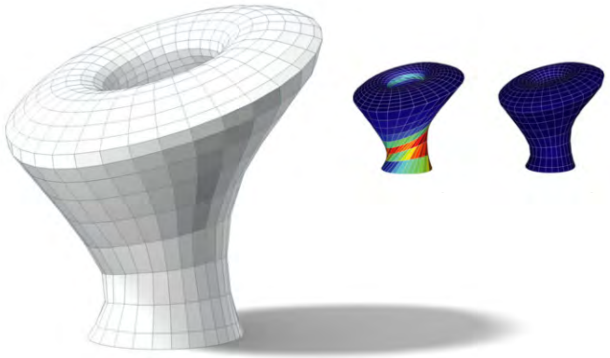
[Yang et al. 2011]



[Müller 2011]

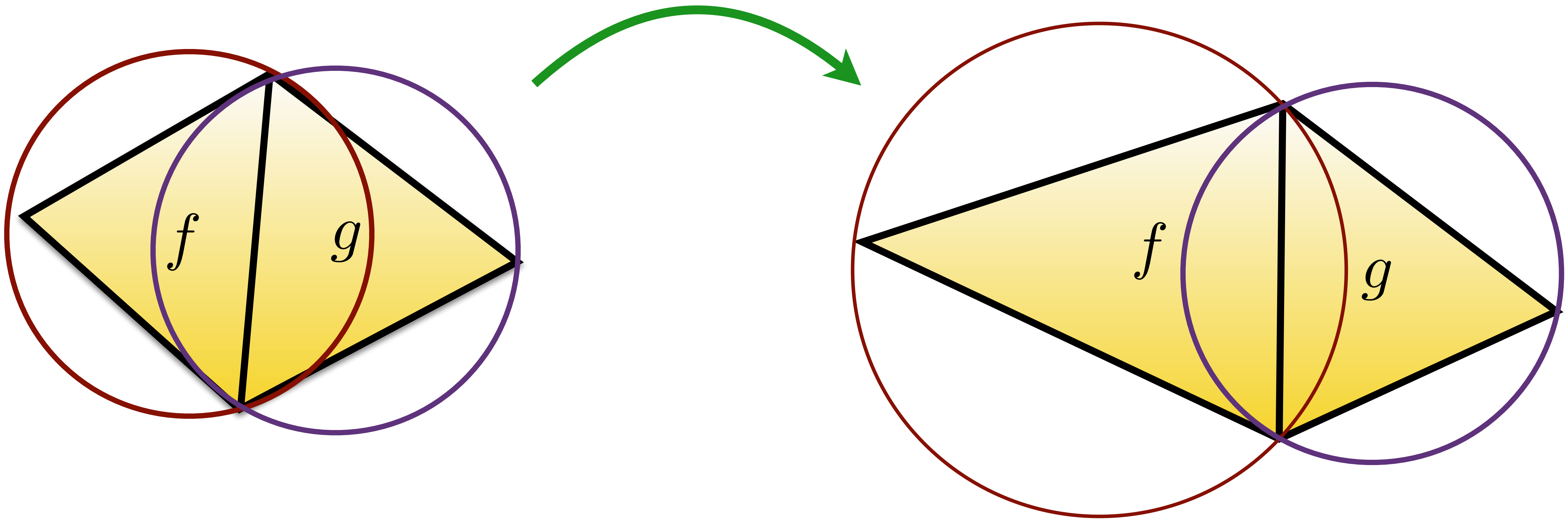


[Tang et al. 2014]



[Bouaziz et al. 2012]

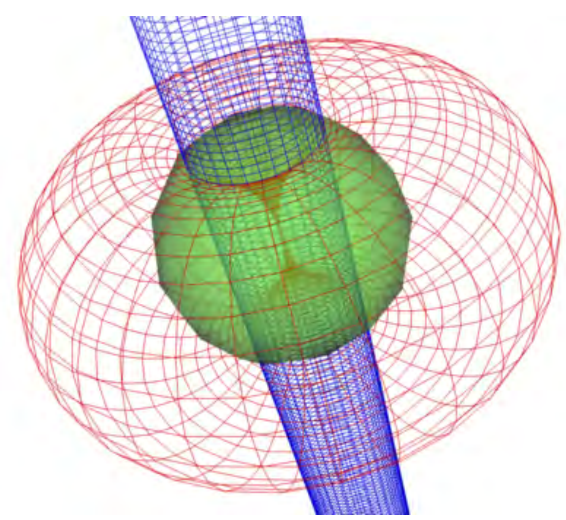
Circle-Pattern Transformations

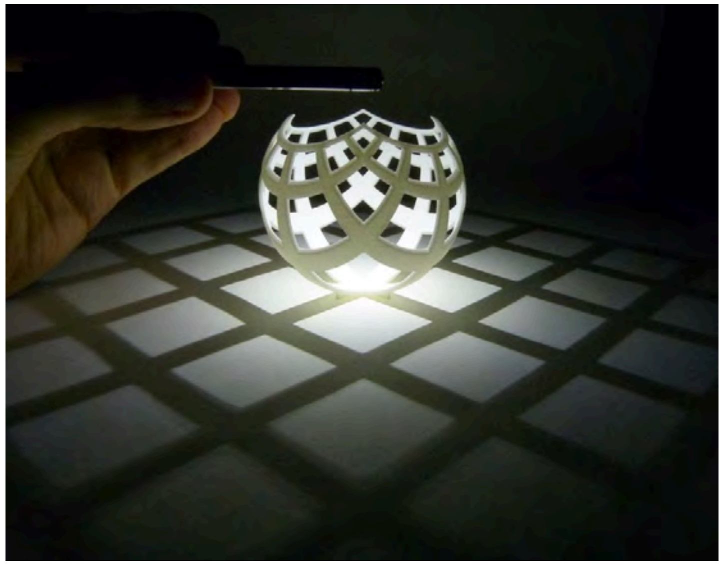


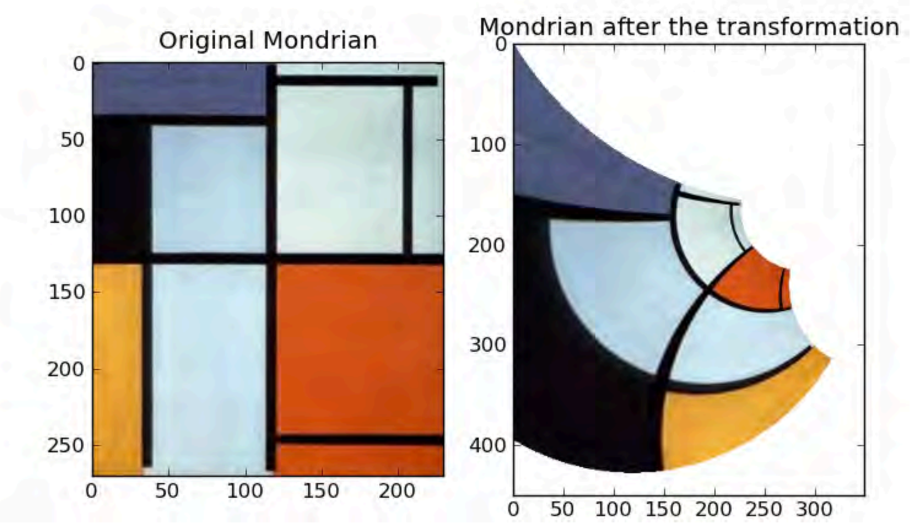
What is conformal?

Möbius Transformations

- *n*-spheres to *n*-spheres
    - Generalized spheres (+planes)
- Comprising:
    - Similarities
    - Inversion in spheres
- Conformal
    - Except at poles
    - *Only* conformal transformations in $n \geq 3$







http://glowingpython.blogspot.co.il/2011/08/applying-moebius-transformation-to.html

Quaternionic Transformations

$$\mathbb{R}^3 \longleftrightarrow q = (0, x, y, z) \in Im\mathbb{H}$$

Imaginary Preserving: $m(q) : Im\mathbb{H} \to Im\mathbb{H}$

$$q = s\left(cos\left(\phi\right), \hat{v} sin\left(\phi\right)\right)$$

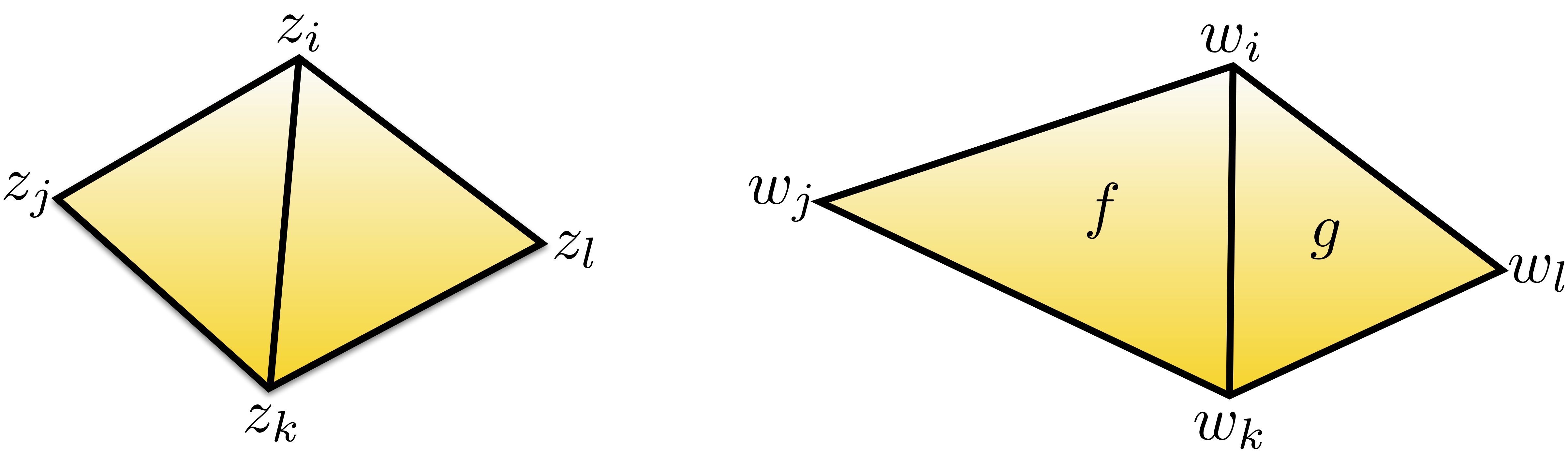
Modulus

Phase

Unit Direction

Cross Ratio

Principle: define conformal by preserved invariants



$$cr_z[i,j,k,l] = \frac{z_{ij}z_{kl}}{z_{jk}z_{li}}$$

Same Möbius Transformation ⟷ cross-ratio preserved

3D Cross ratio

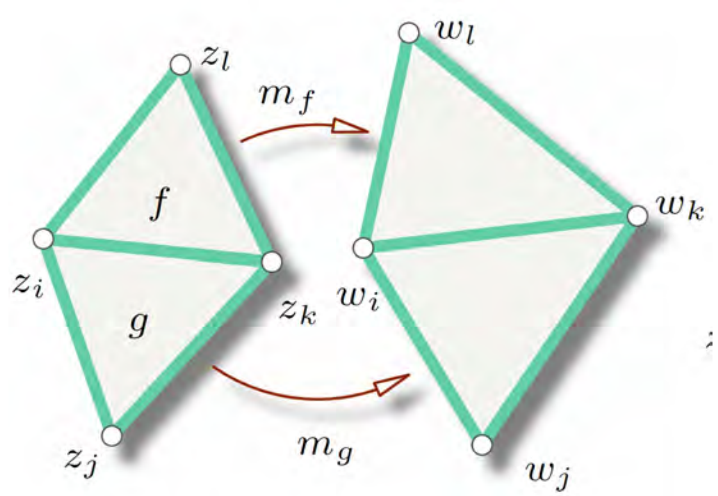
$$cr_q[i,j,k,l] = q_{ij}\,(q_{jk})^{-1}\,q_{kl}\,(q_{li})^{-1}$$

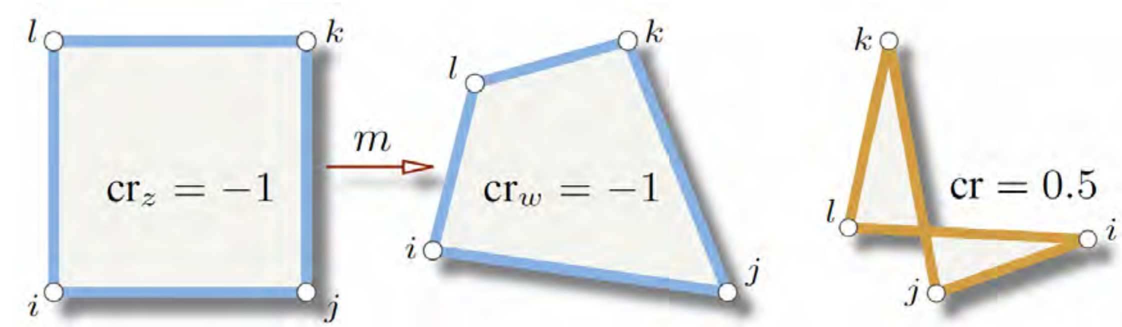
$$cr_w[i,j,k,l] = \boxed{(cq_i + d)}\,cr_q[i,j,k,l]\,\boxed{(cq_i + d)^{-1}}$$

Same Möbius Transformation ⟷ cross-ratio **conjugated**

Piecewise Möbius Paradigm

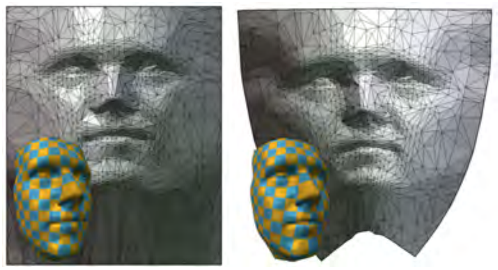
- Single Möbius transformation per face
- Conformality measured by change in cross-ratio.
    - at edges and on faces





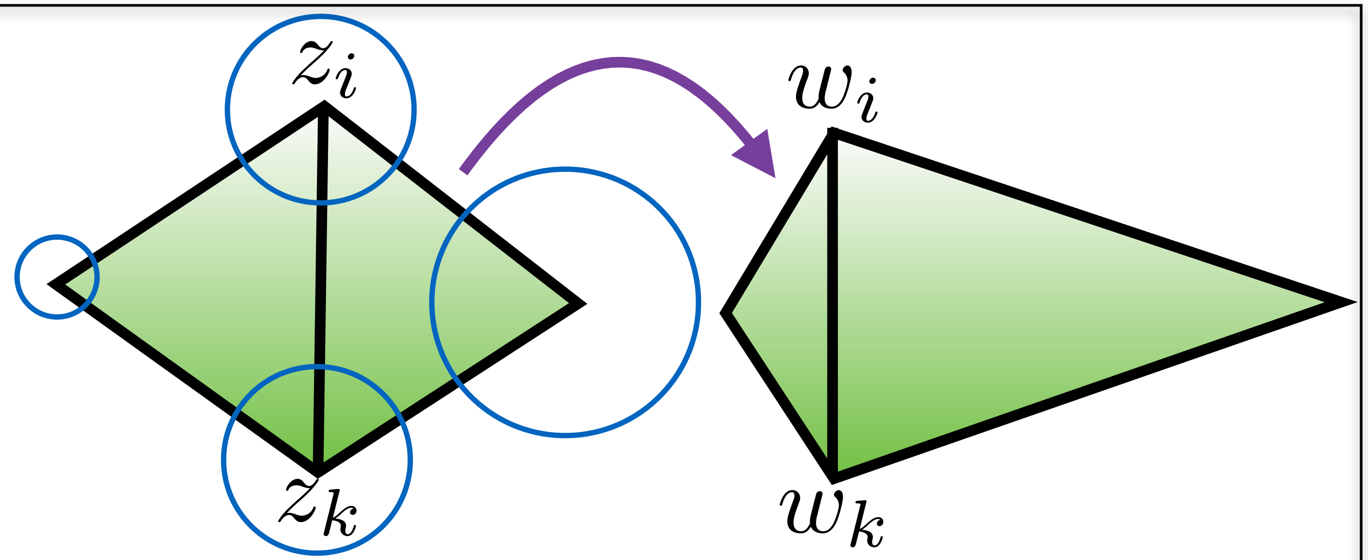
Discrete Conformality

$$cr = |cr|\, e^{i(\pi - \phi_{ik})}$$

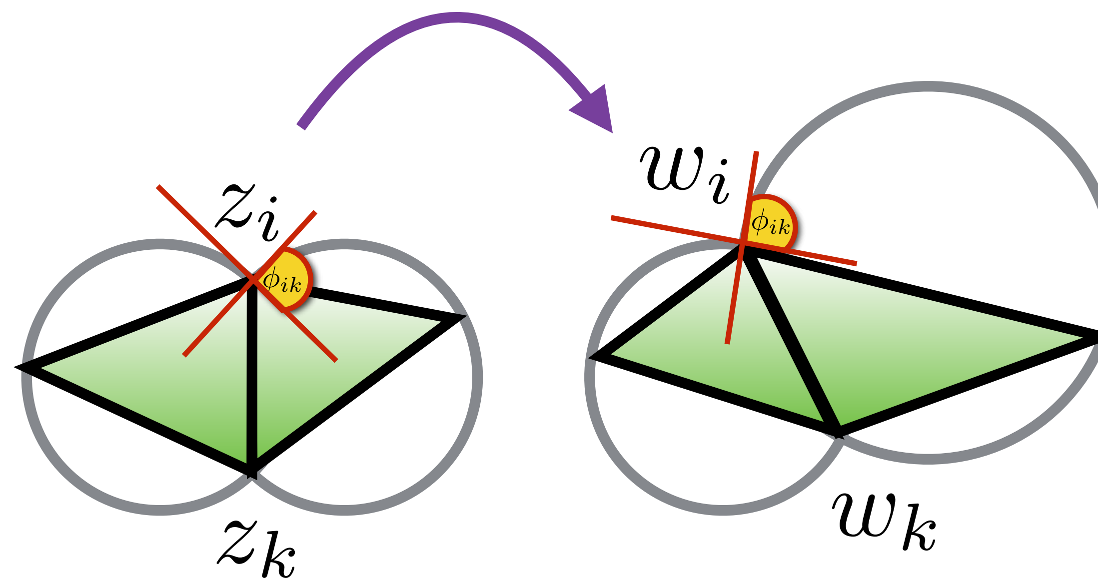


[Springborn *et al*. 2008]

[Kharevych *et al*. 2006]



$$|w_{ik}| = |z_{ik}|\, e^{((u_i+u_k)/2)}$$



(Discrete) metric conformal (**MC**)

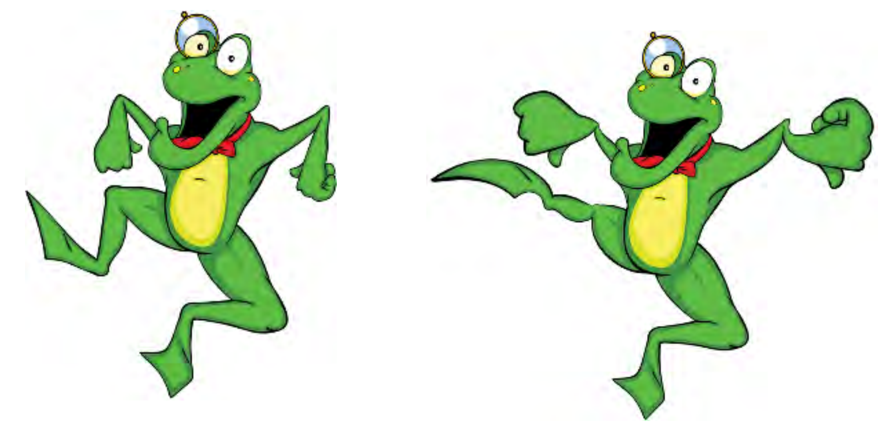
Intersection-angle preserving (**IAP**)

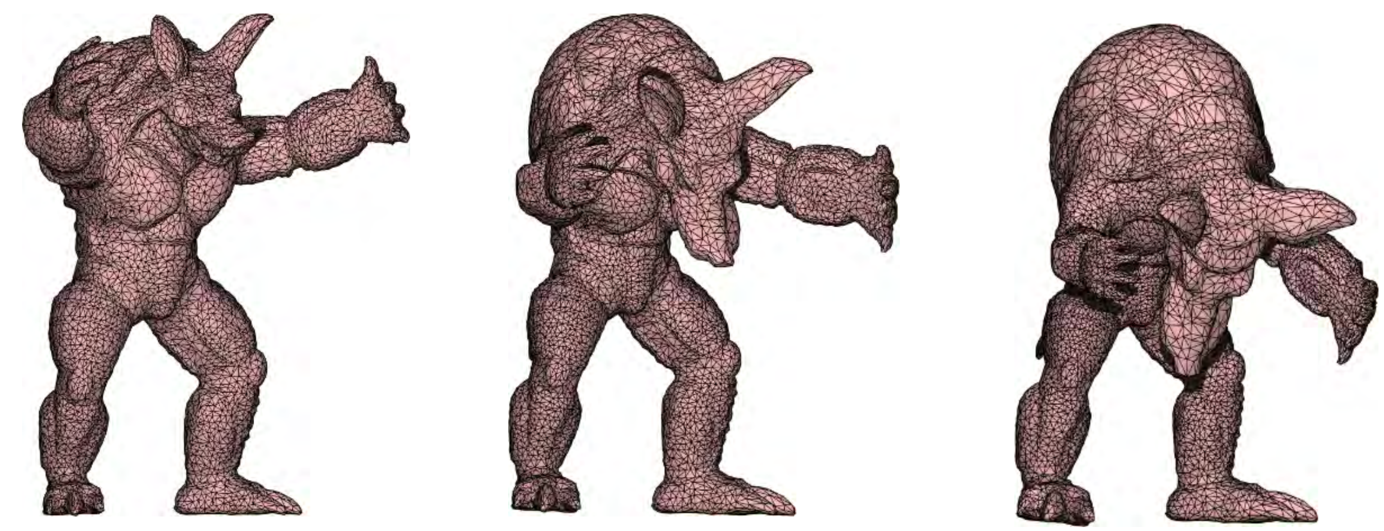
in 3D: conjugation preserving Modulus and phase

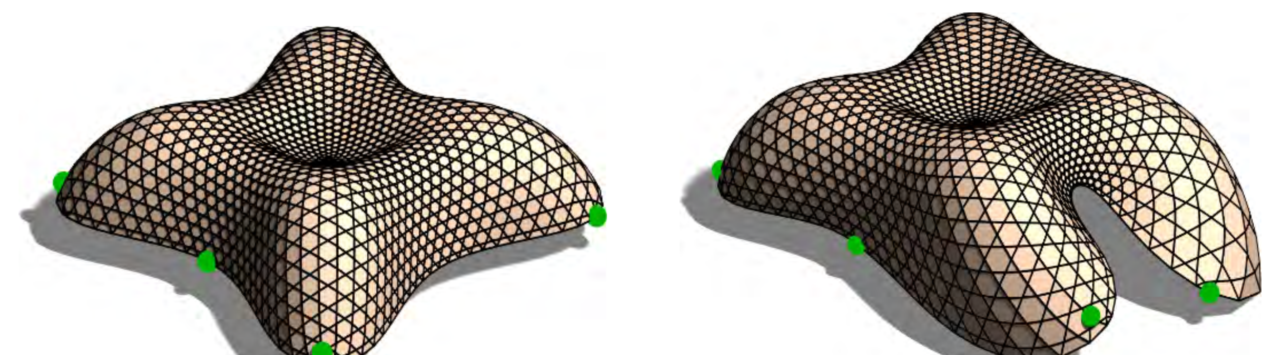
$$cr_w[i,j,k,l] = (cq_i + d)\, cr_q[i,j,k,l]\, (cq_i + d)^{-1}$$

Conformal Deformations

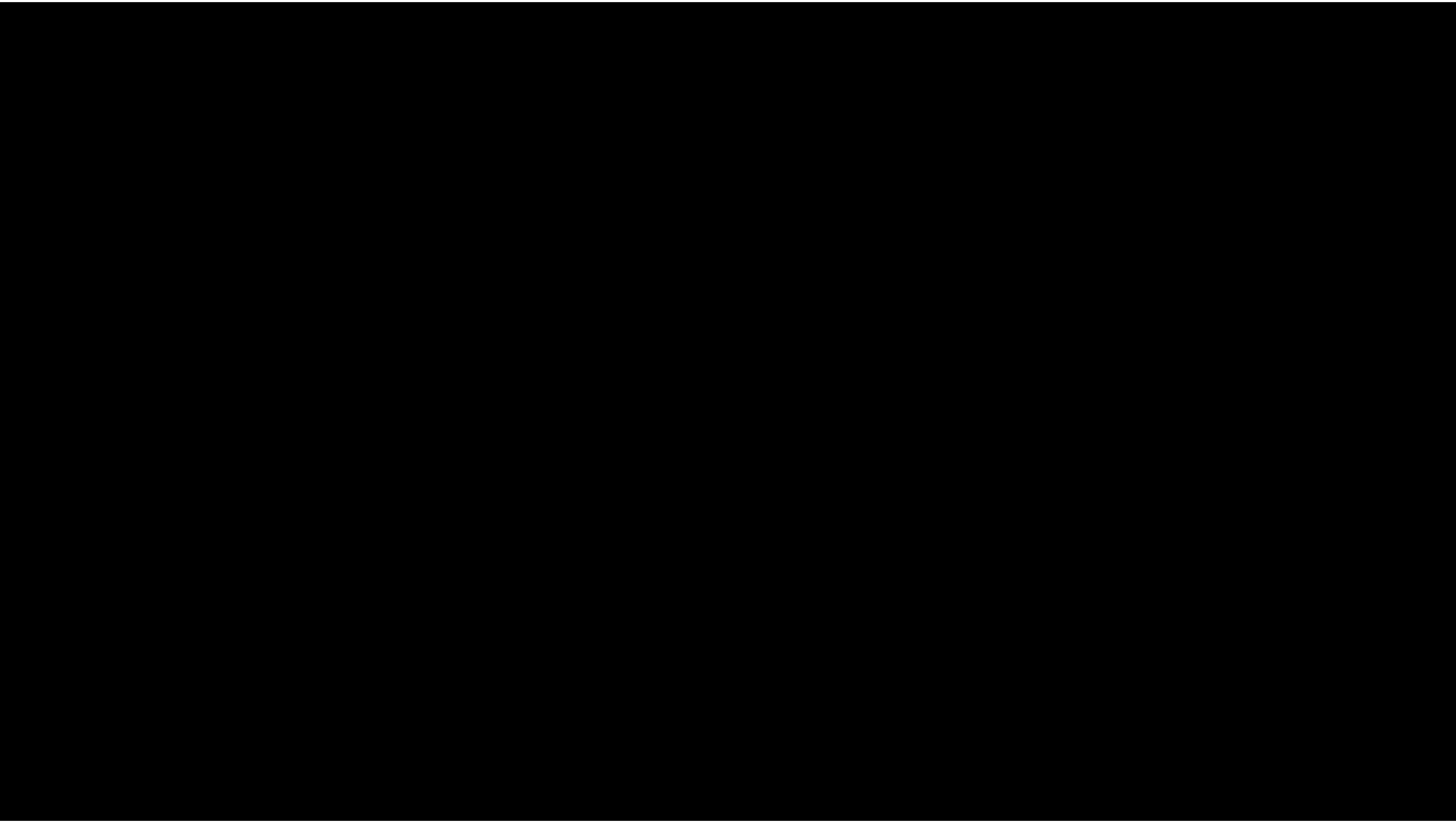
- Positional Constraints
- Unified approach:
    - 2D: complex
    - 3D: quaternions
    - Polygonal & circular meshes





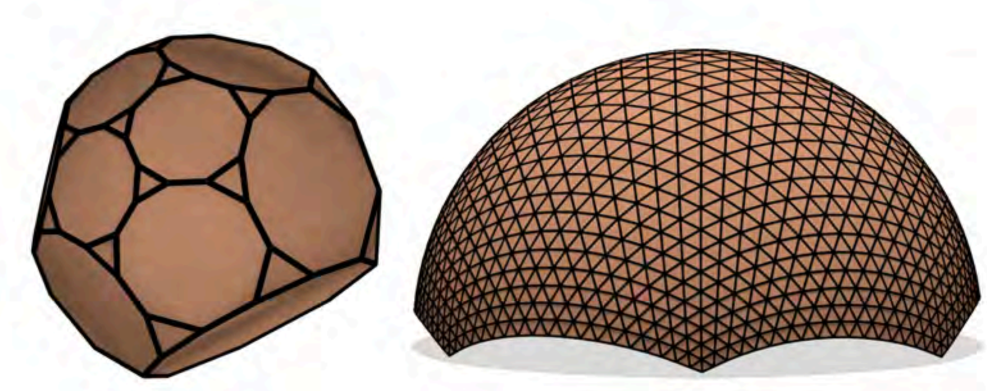


[V., Müller, and Weber 2015]



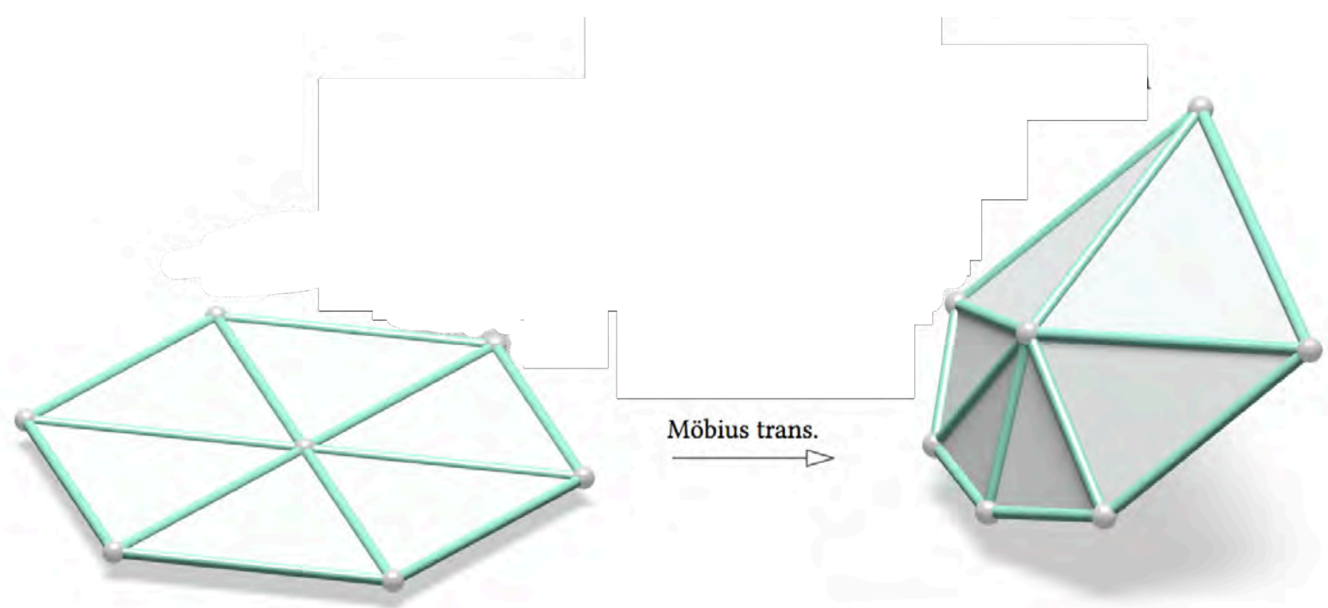
Möbius Regularity

- Every face and 1-ring are regular…
- Up to a Möbius transformation
  - Conformal

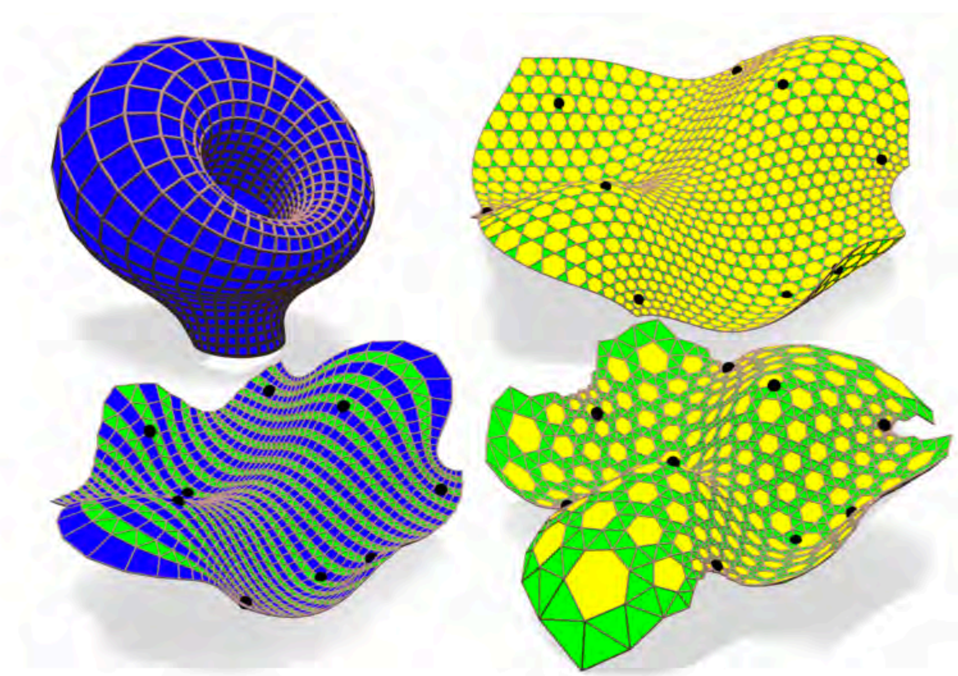


Perfect

- Perfect ring is canonical embedding.

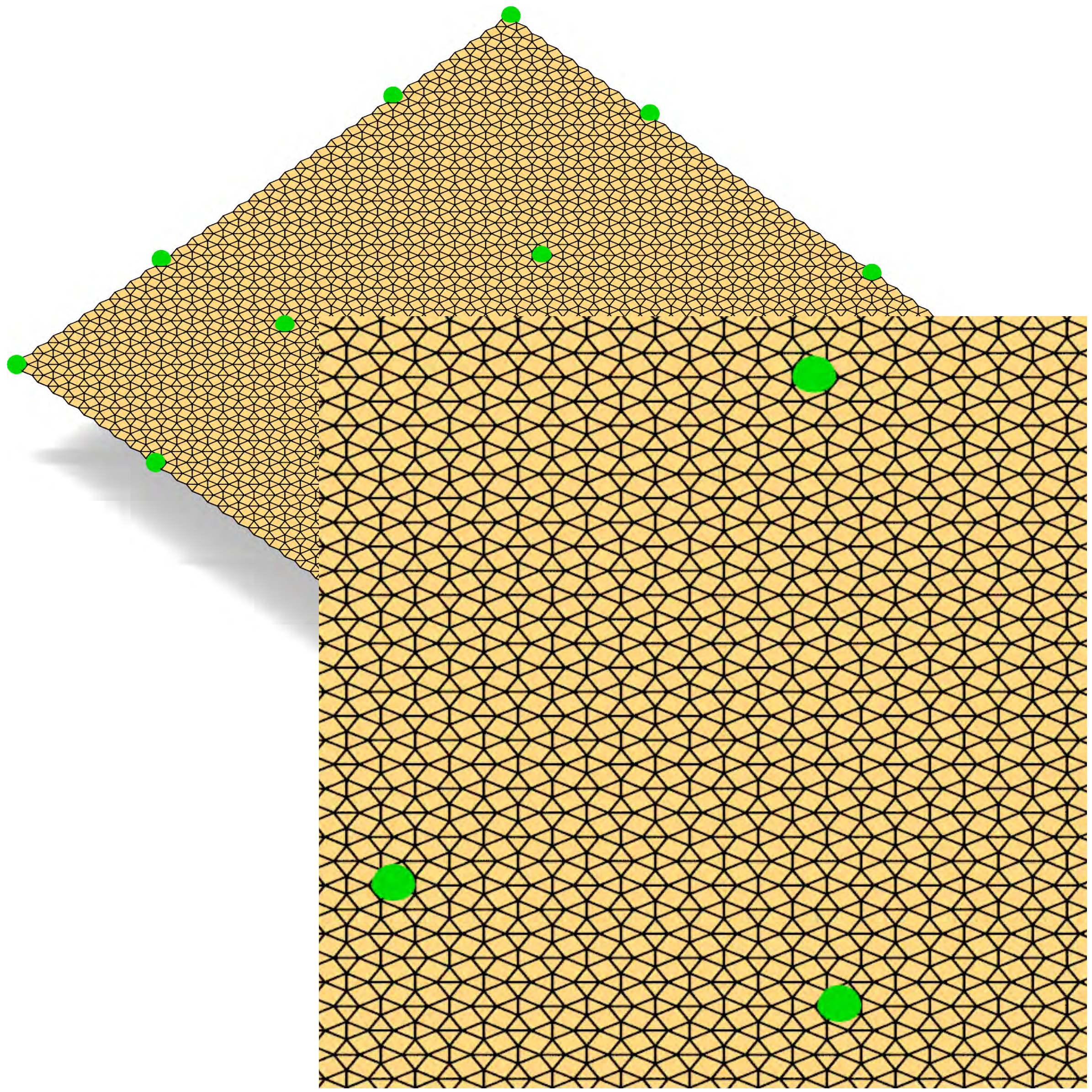


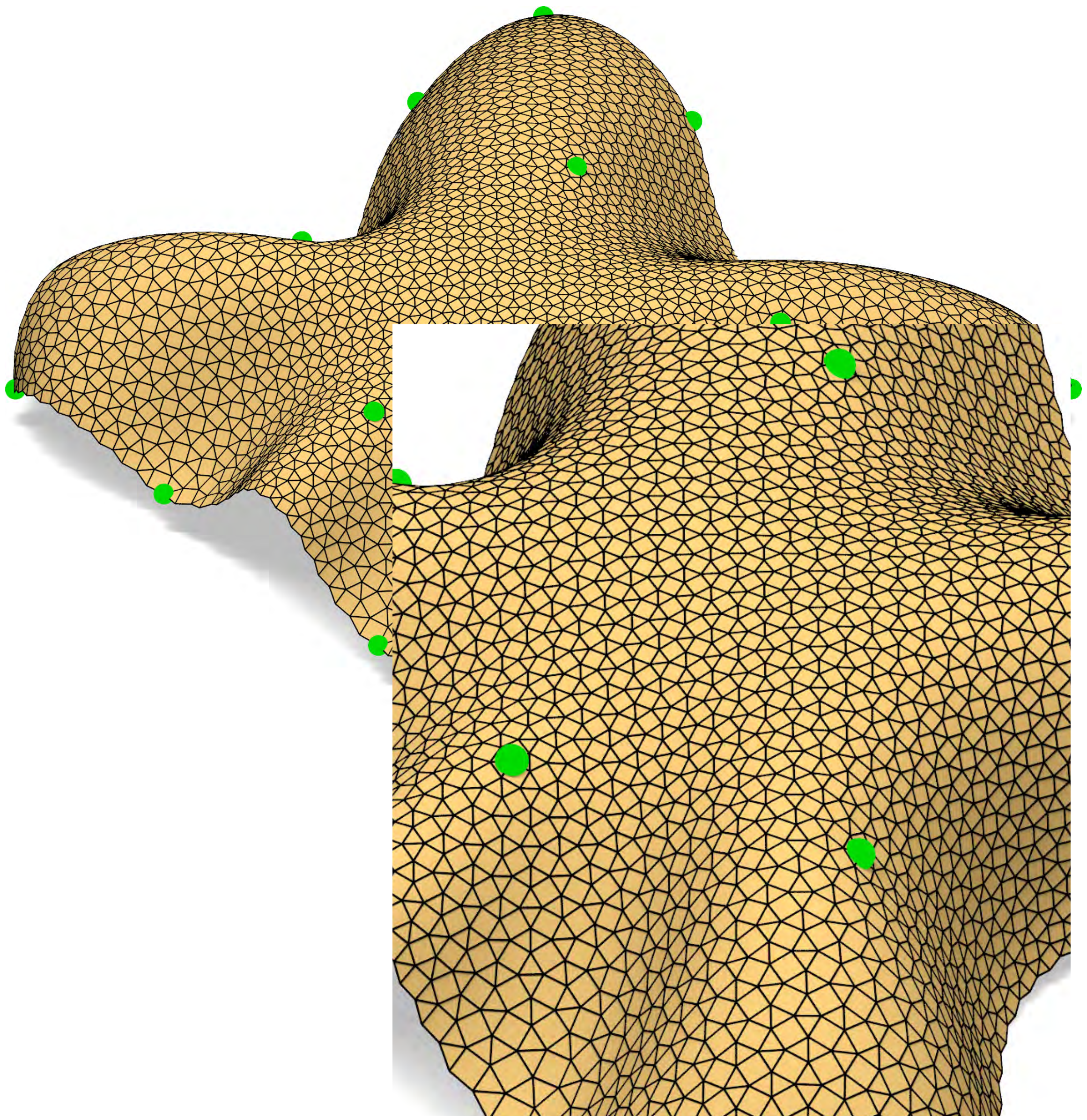
[V., Müller, and Weber 2017]



As-possible

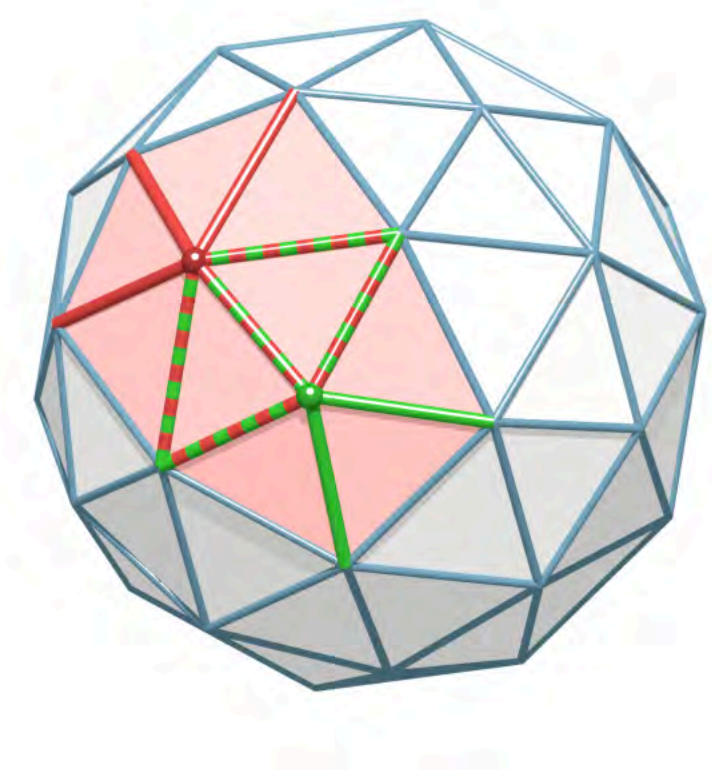
Our Approach: Regular Meshes



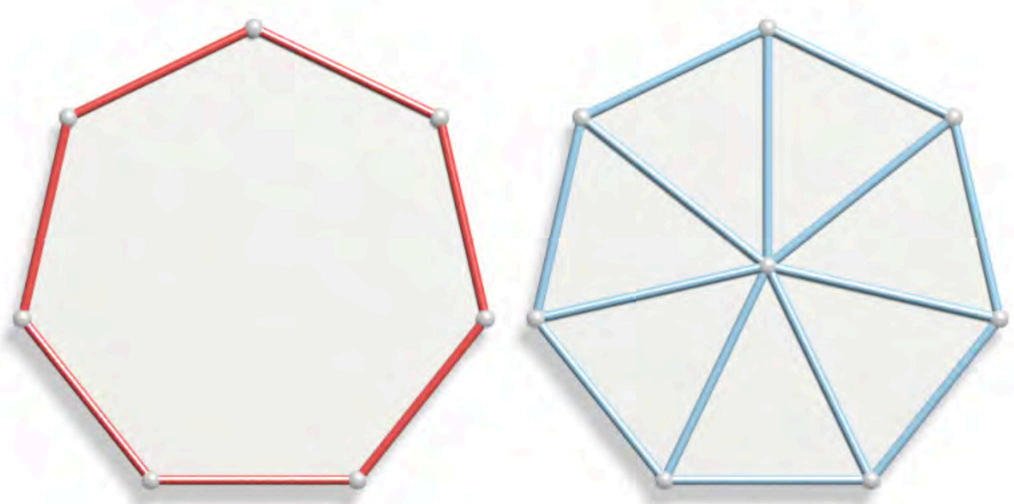


Our Approach: Regular Meshes

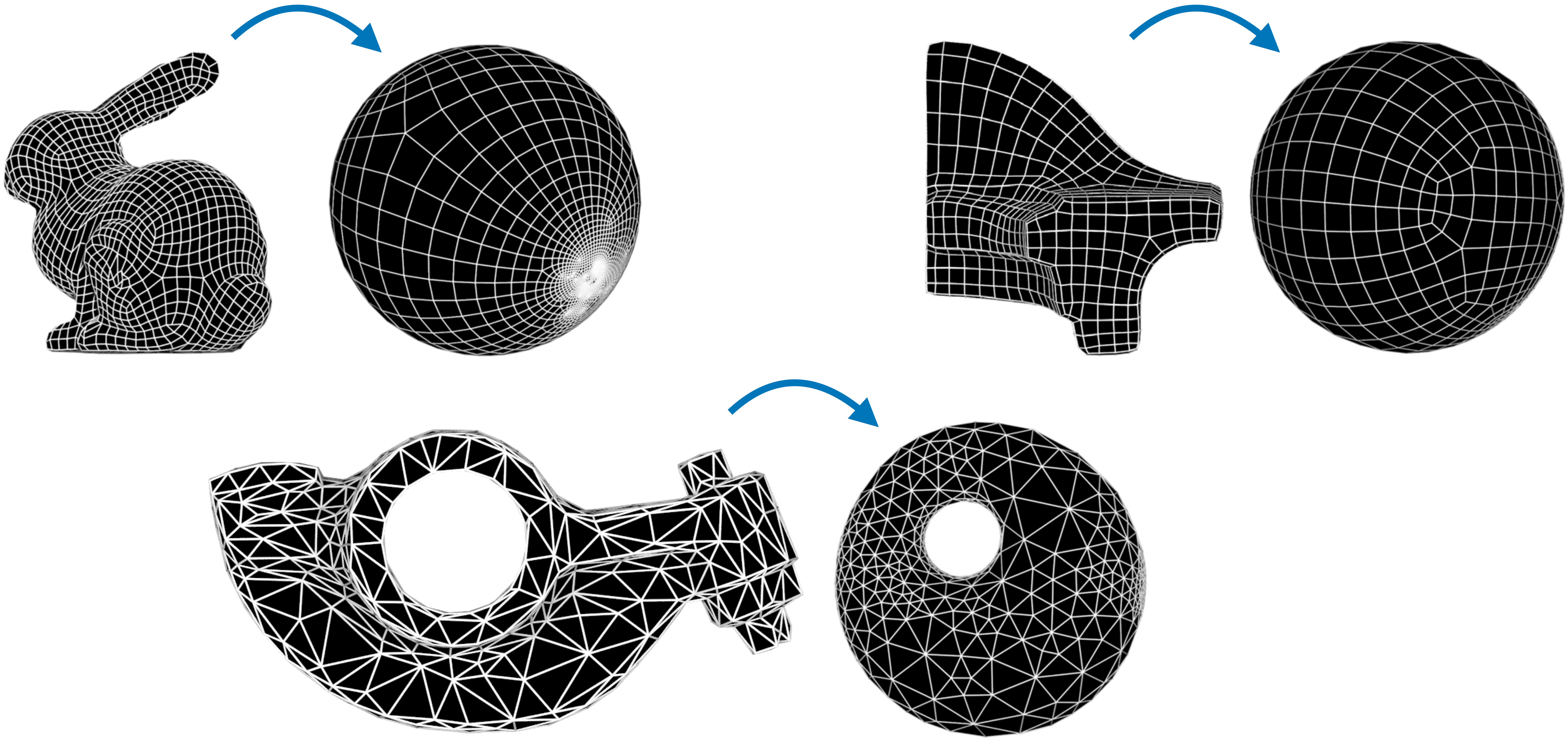
Imperfect Patterns



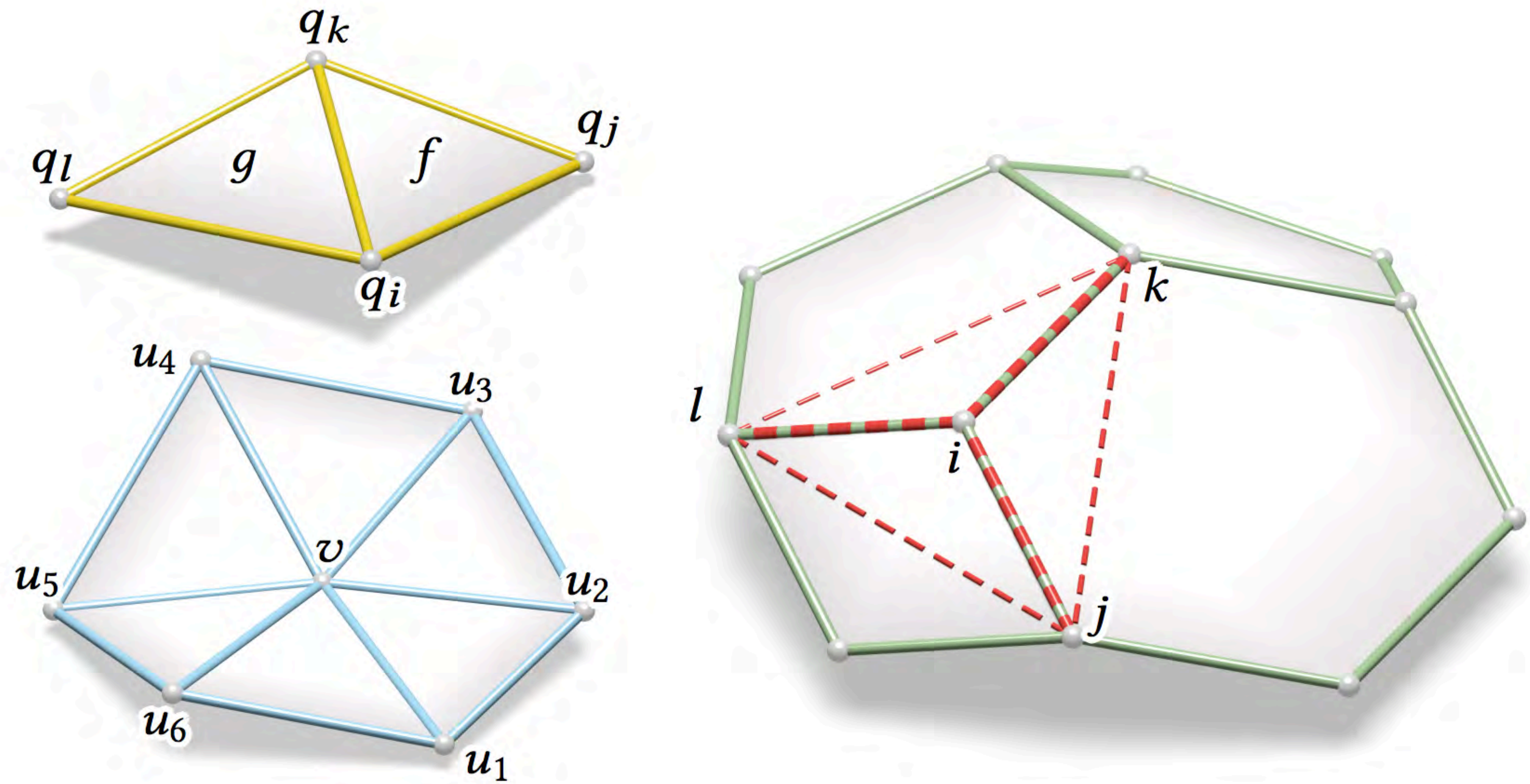
- No perfect solution
- How to do “as-MR-as-possible”?
- Even worse: not all 1-rings canonical.
- Canonicalization:



Imperfect, as Regular as Possible



Conventions



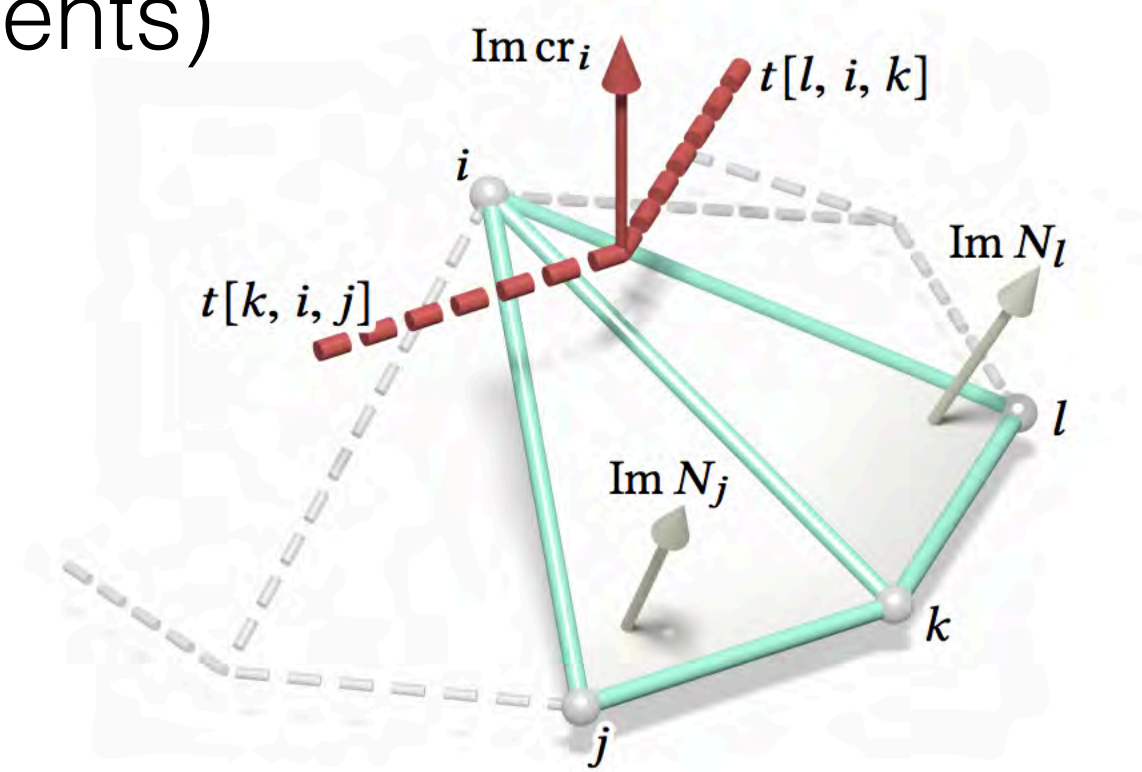
The Corner Tangent

- Oriented tangency to circumcircle at v_i

$$t[k, i, j] = q_{ki}^{-1} + q_{ij}^{-1}$$

- Cross ratio: $cr[i, j, k, l] = t[k, i, j]^{-1} t[k, i, l]$

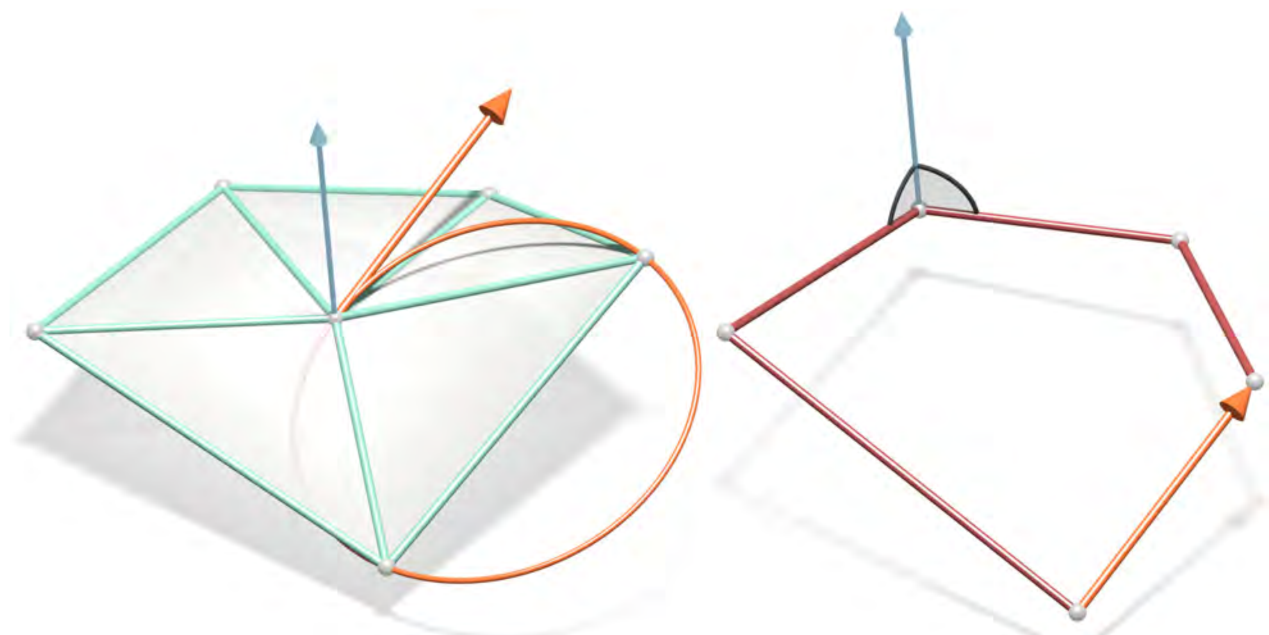
- Geometric Characterization: CR vector = normal to both circles (and their tangents)



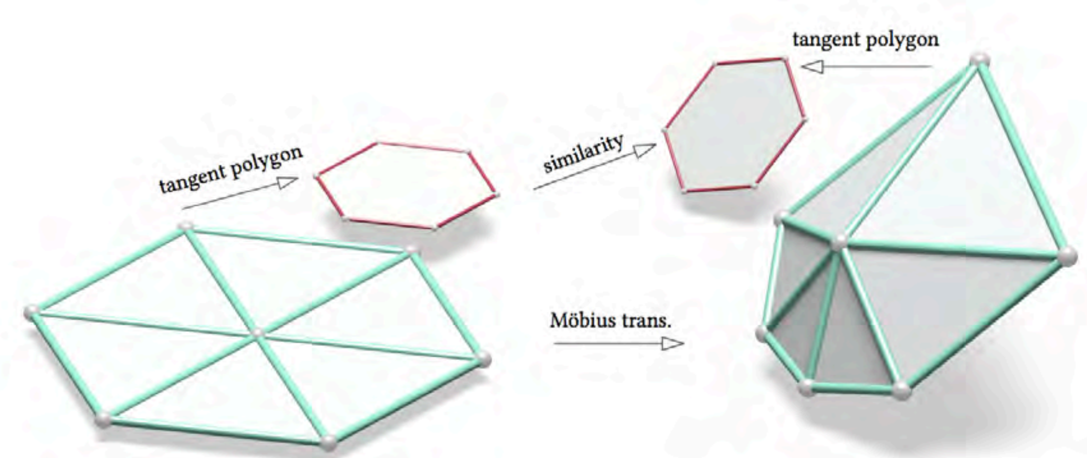
The Tangent Polygon

- Abstract polygon of 1-ring around v.
- **Edges**: corner tangents $t[u_{i-1}, v, u_i]$.
- Closed polygon:

$$\sum_i t[u_{i-1}, v, u_i] = 0$$



- Corner normals: cross ratios.
- Under Möbius transformation: transforms as similarity.

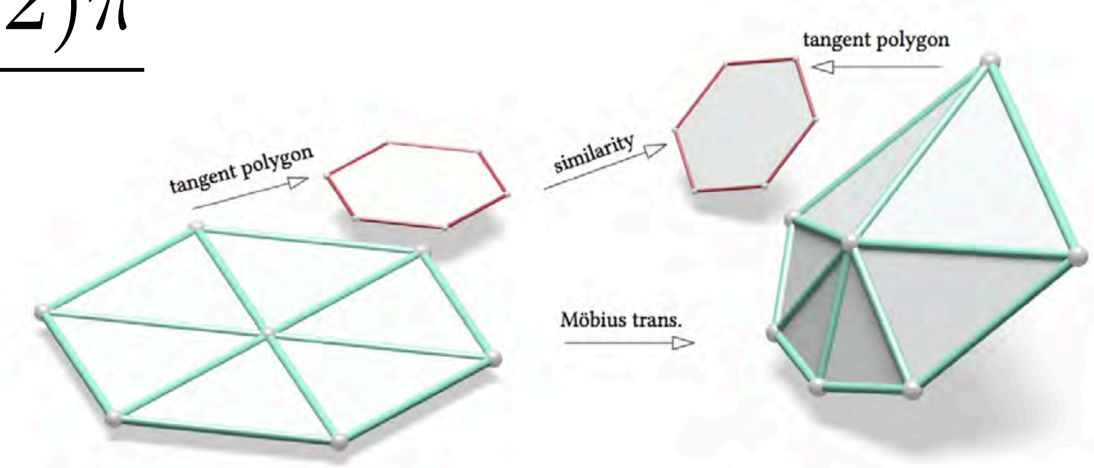


Möbius regularity of Pure Stars

- **Lemma**: tangent polygon of a regular 1-ring (valence n) is regular.
- Möbius-regular rings: the same!
- Practical characterization: all cross-ratios are equal.

$$cr[v, u_{i-1}, u_i, u_{i+1}] = [cos(\phi_n), sin(\phi_n) n_v]$$

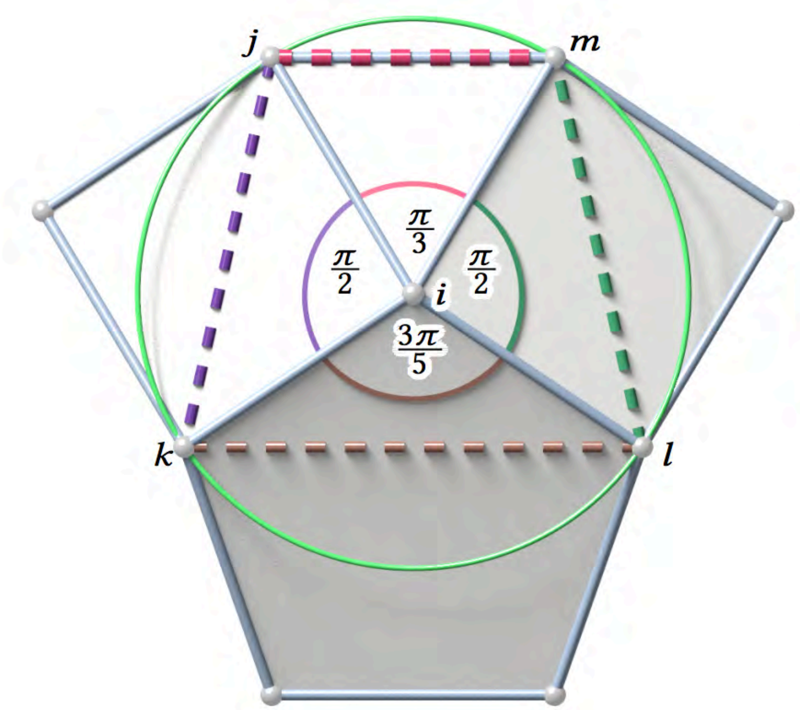
$$\phi_n = \frac{(n-2)\pi}{n}$$

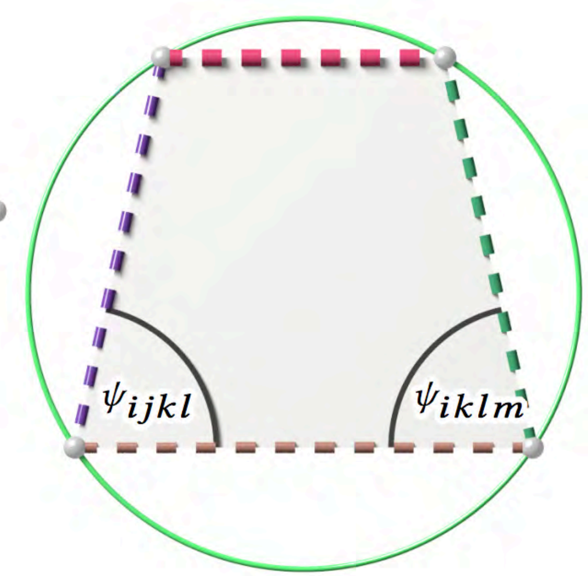


Tangent Polygon for Mixed Stars

- Tangent polygon = Boundary polygon in canonical embedding
- Also: concyclic!
- Custom lengths and phases for cross-ratio

$$cr[v, u_{i-1}, u_i, u_{i+1}] = l_i[cos(\phi_i), sin(\phi_i)n_v]$$





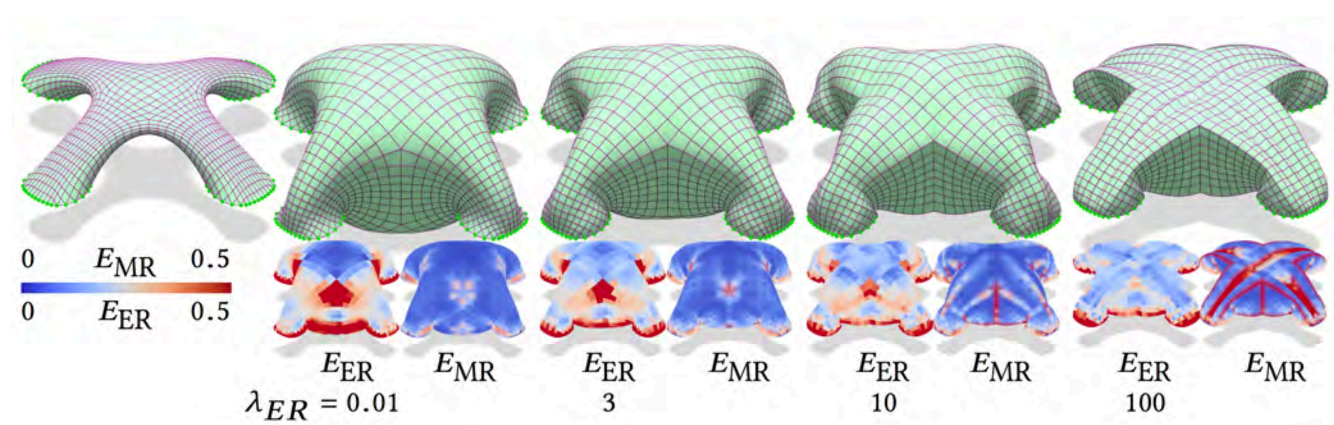
Optimization

- Euclidean Regularity: $$E_{\mathrm{ER}} = \sum_{f\in\mathcal{F}} \sum_{\substack{(ki),(ij)\\ \text{adjacent edges}\in f}} \left| w_{ij} w_{ki}^{-1} - [\cos(\chi_n), -\sin(\chi_n)\mathbf{n}_f] \right|^2$$
- Möbius regularity: $$E_{\mathrm{MR}} = \sum_{f\in\mathcal{F}} \sum_{p=1}^{d} \left| \mathrm{cr}[w_p^f, w_{p+1}^f, w_{p+2}^f, w_{p+3}^f] - [\tfrac{-1}{(1+2\cos(2\pi/d))}, \mathbf{0}] \right|^2 + \sum_{w_i\in\mathcal{V}} \sum_{\text{flap}(ijkl)} \left| \mathrm{cr}[w_i, w_j, w_k, w_l] - l_{ijkl}[\cos\phi_{ijkl}, \sin\phi_{ijkl}\mathbf{n}_i] \right|^2$$

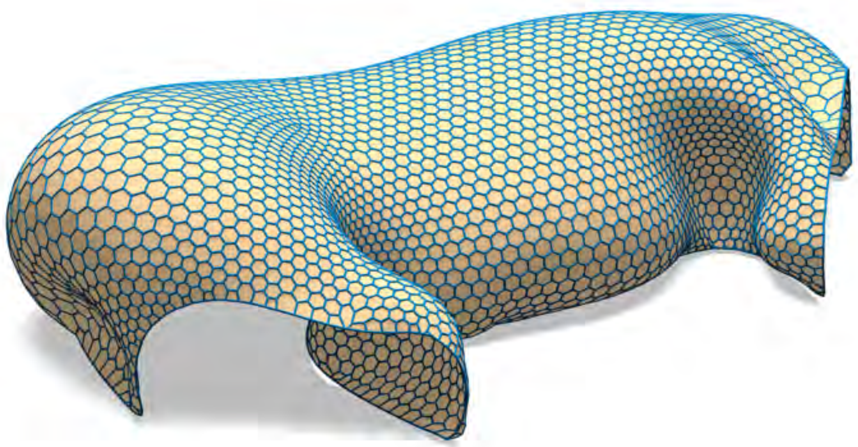
Face Moebius

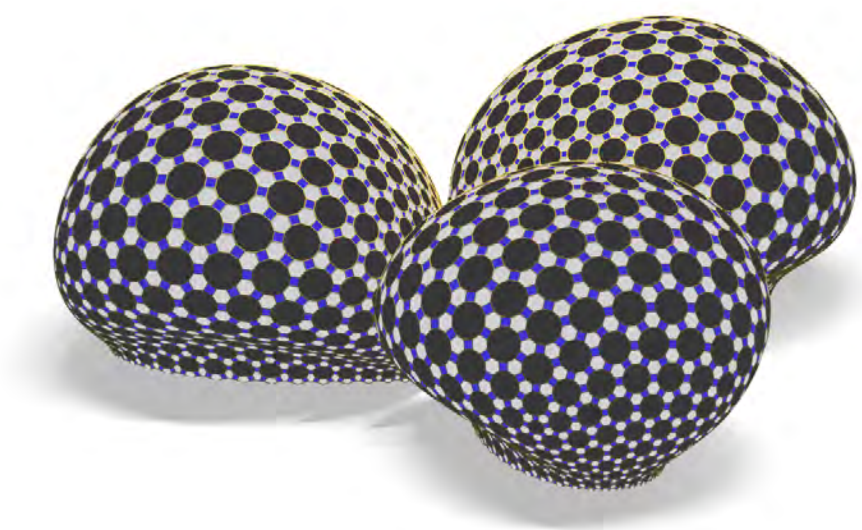
1-ring Moebius

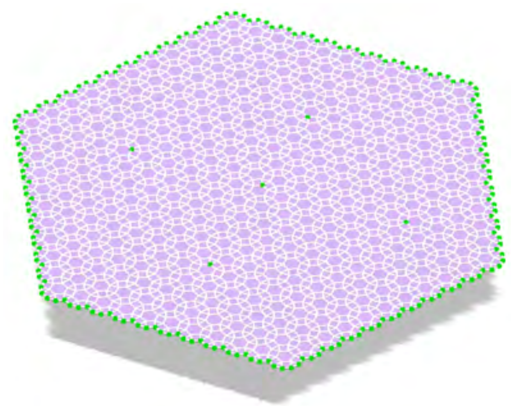
- Total energy: $$E_R = \lambda_{MR} E_{MR} + \lambda_{ER} E_{ER}$$
- Direct Optimization: Levenberg-Marquadt nonlinear least squares.

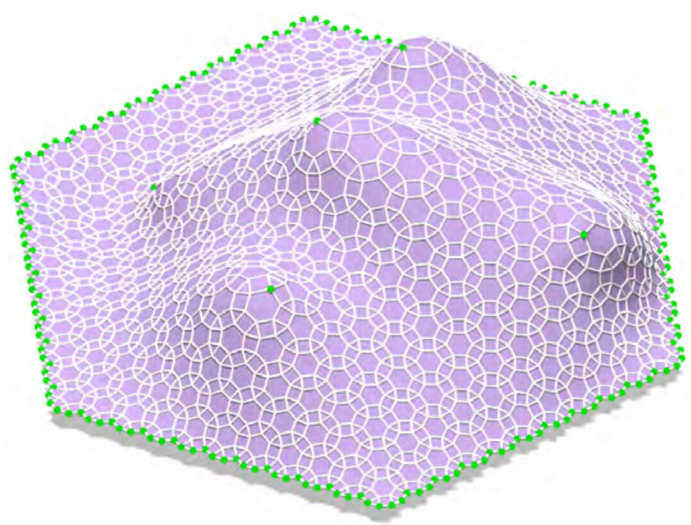


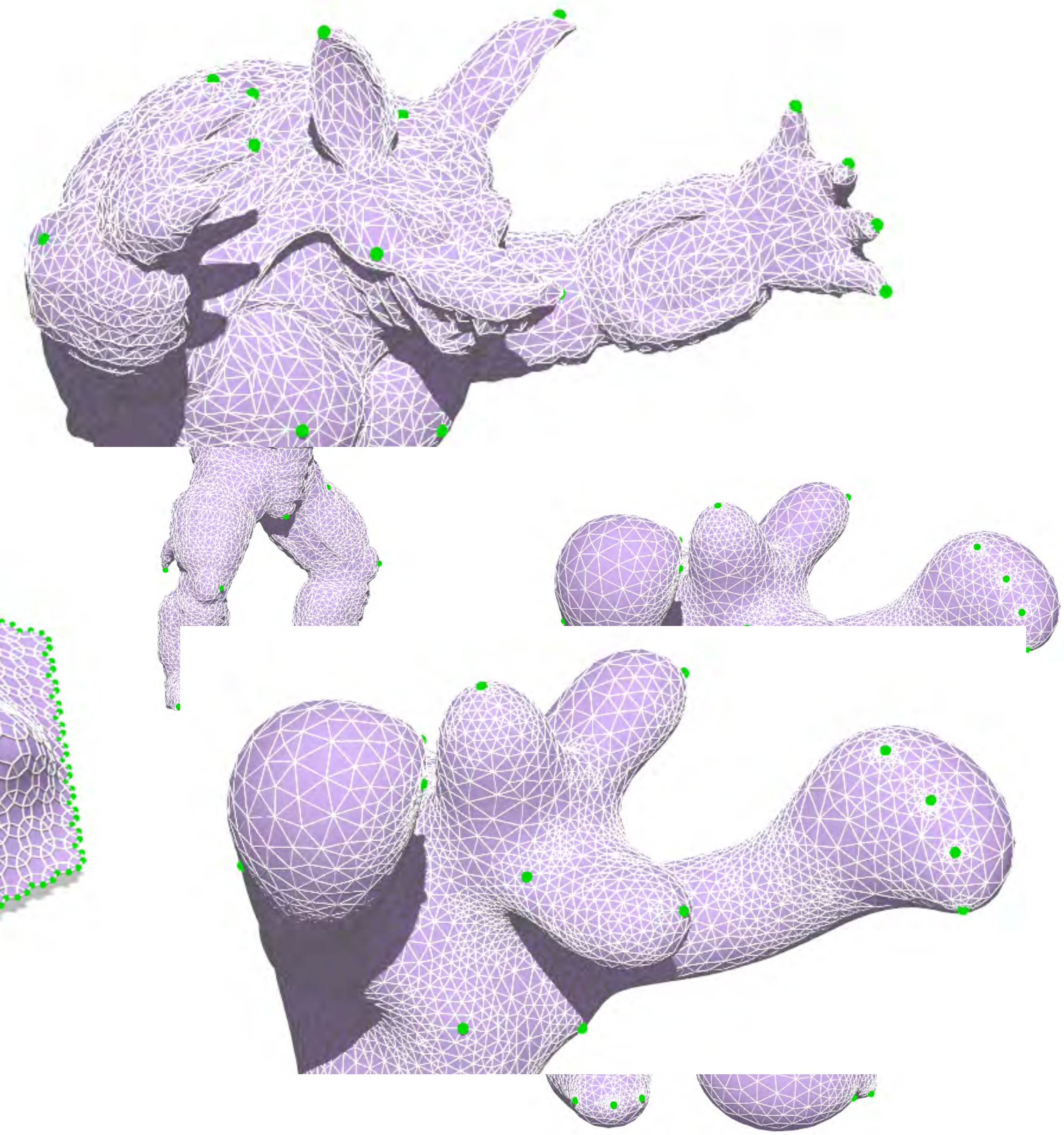
Möbius Regular Meshes











The Vector Part

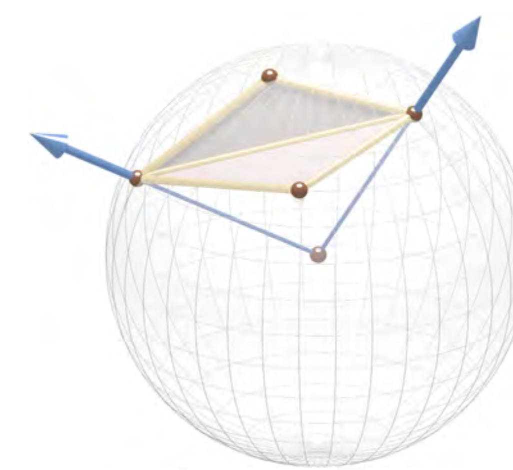
$$q = s\left(\cos\left(\phi\right), \hat{v}\sin\left(\phi\right)\right)$$

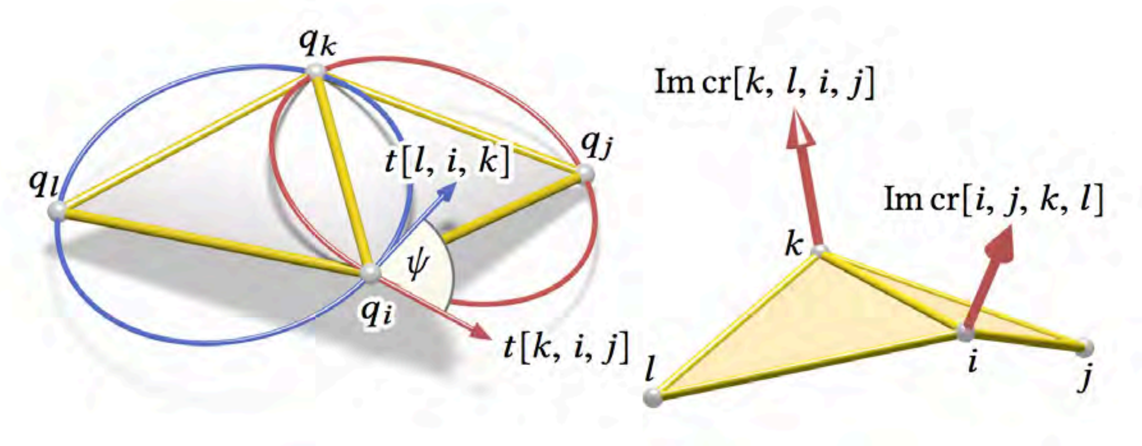
Modulus	Phase	Unit Direction
Metric Conformal	Intersection-angle preserving	???

- Reminder: quaternionic cross ratio = modulus + phase + vector

- What is the *direction*?

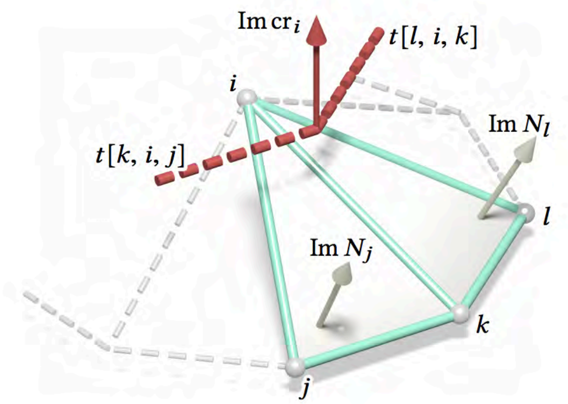
- The radius vector of the mutual sphere



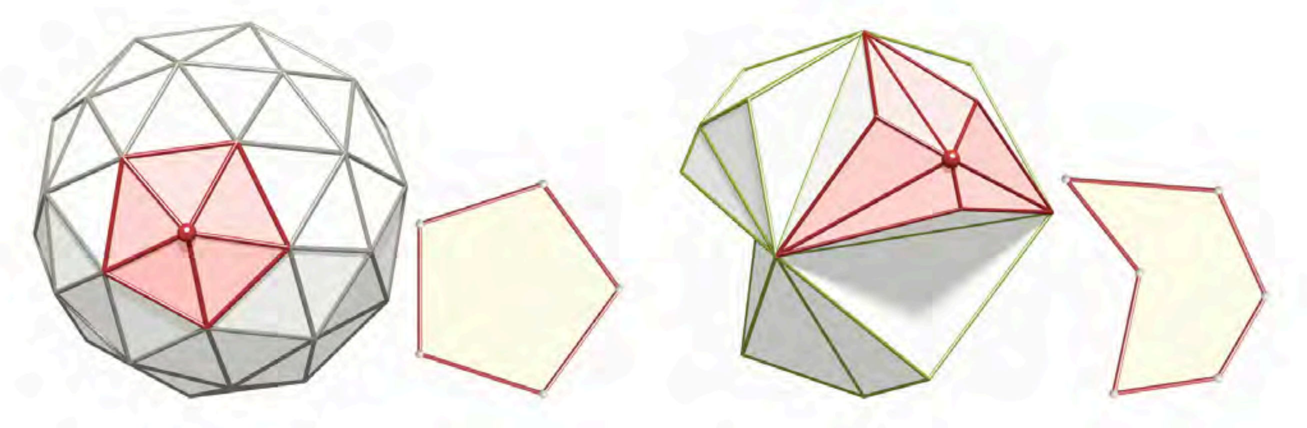


Relation to Willmore Energy

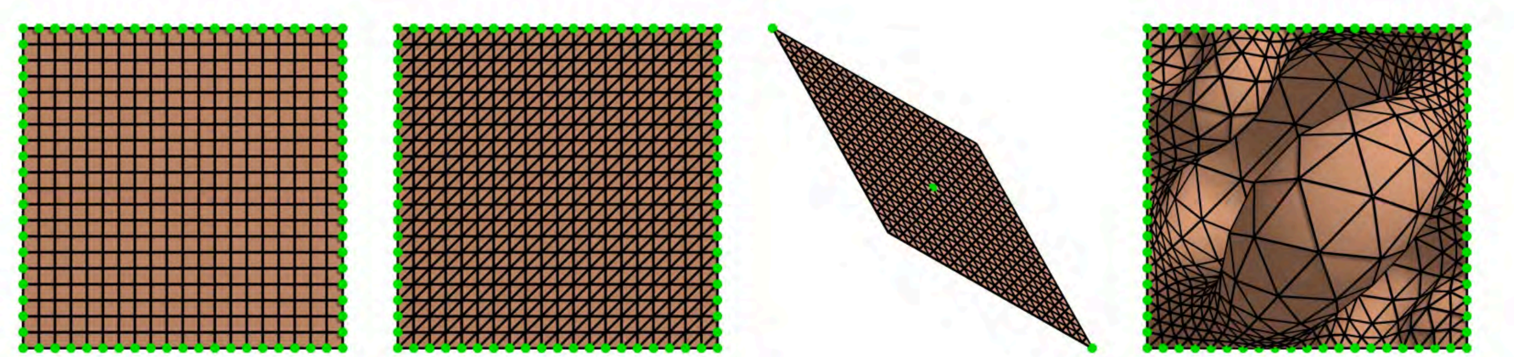
- Willmore energy => inscribed in a sphere
  - Planar tangent polygon.



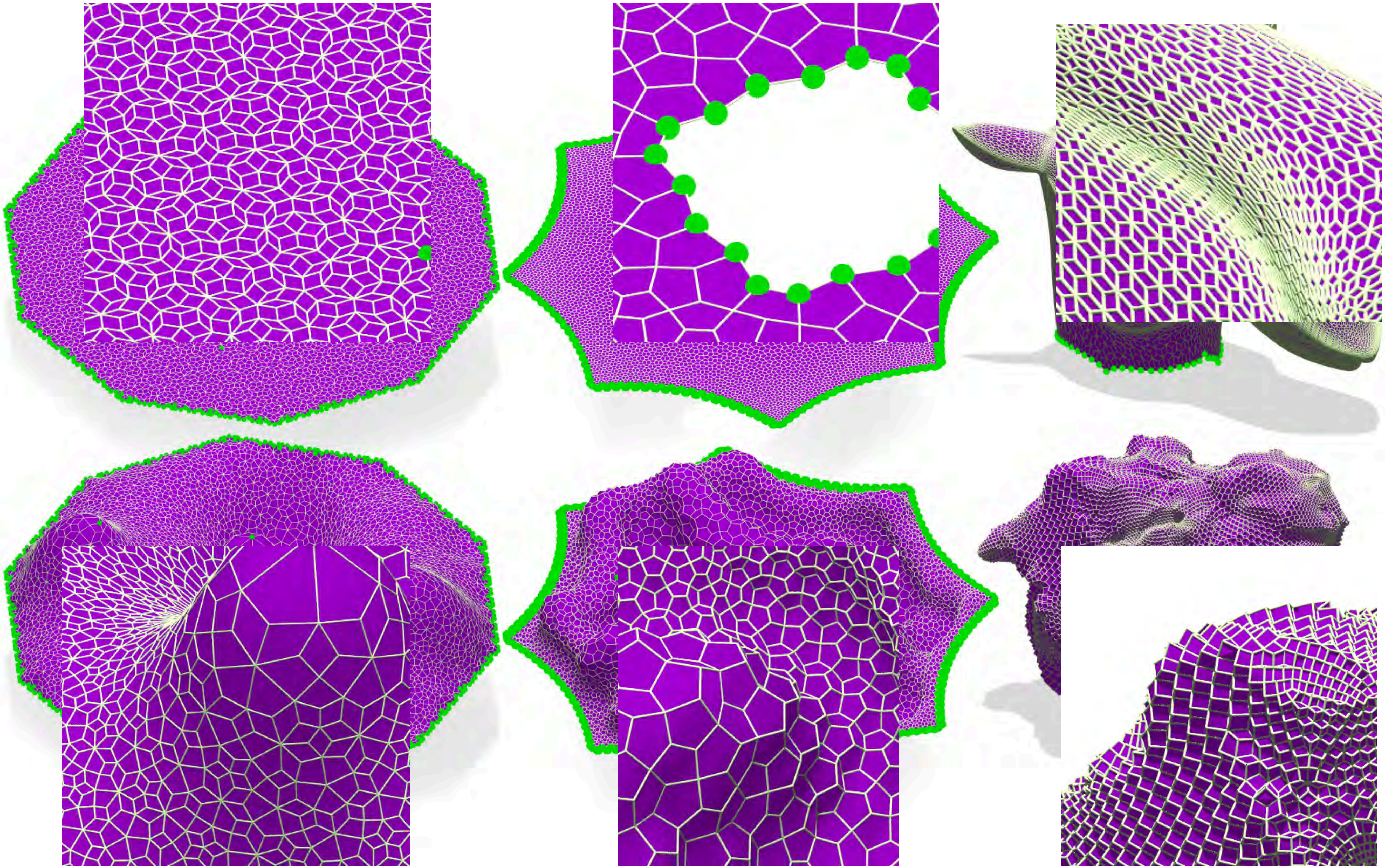
- Perfect Möbius regular => inscribed in a sphere
- BUT
  - Not the converse!



  - as-MR-as-possible: depends on boundary conditions.

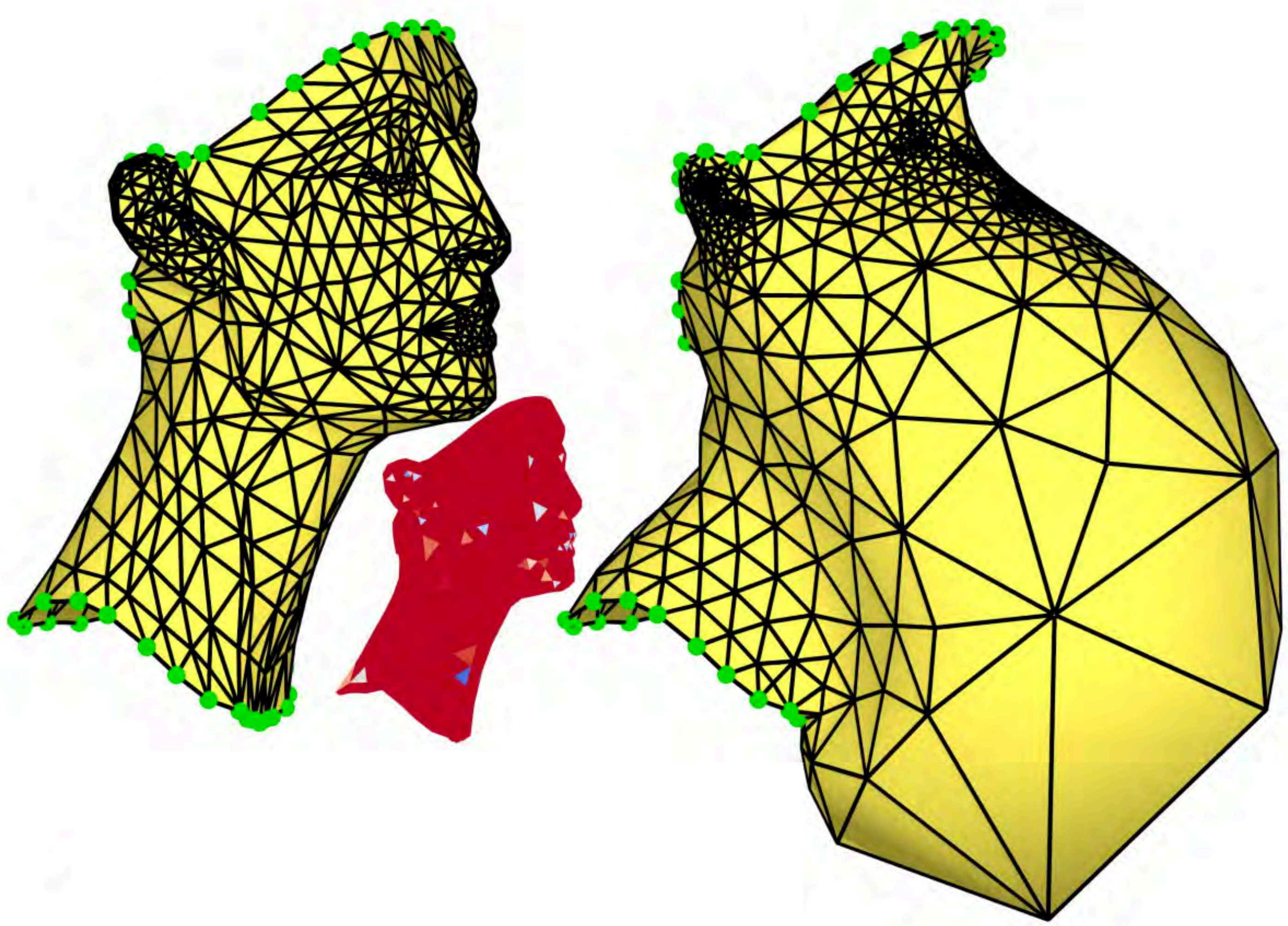


Unconventional Patterns



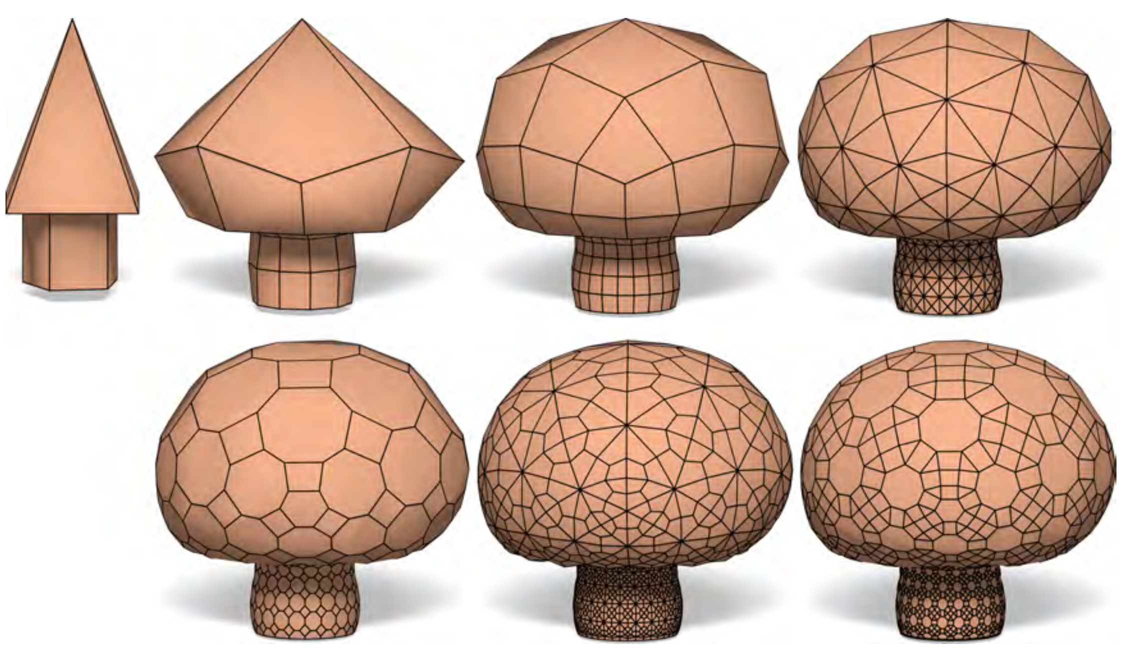
Limitations

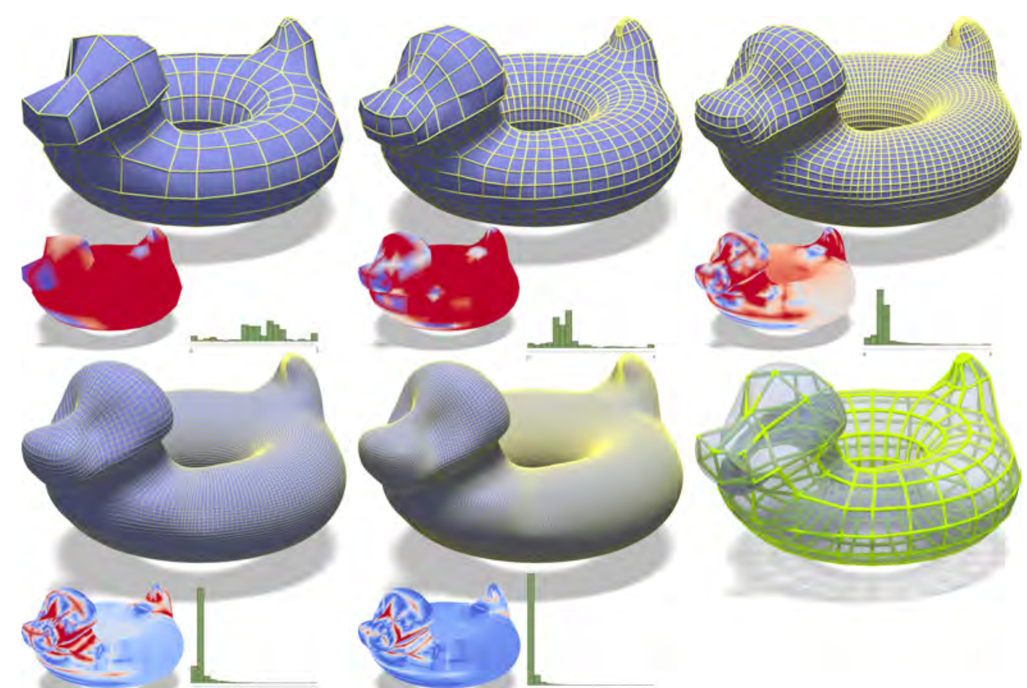
- Möbius inversions
- Nonconvex energy with direct optimization = slow.



Coarse-to-Fine Möbius Editing

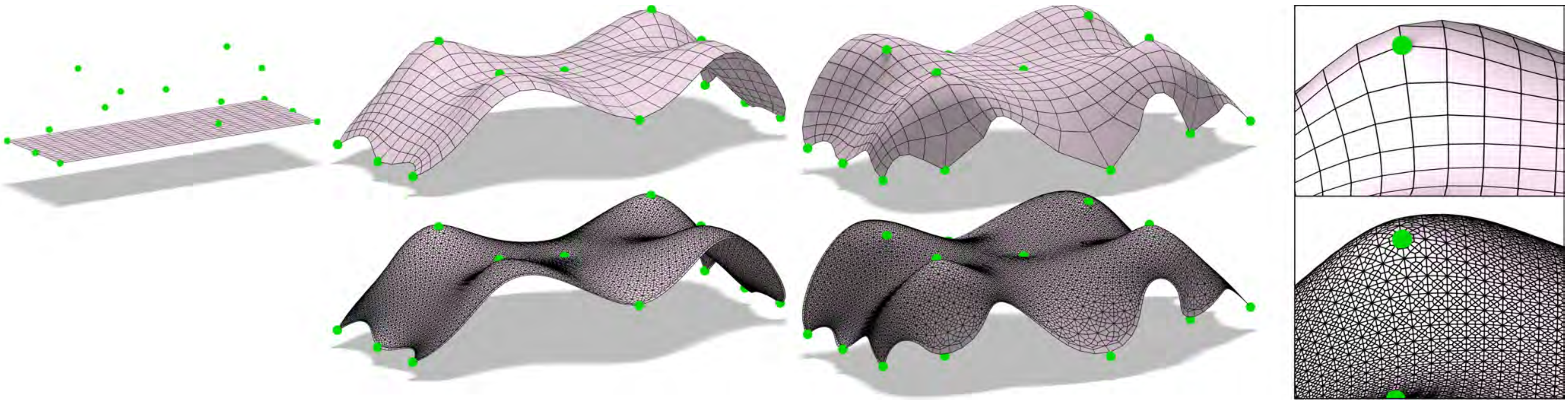
- Trying to optimize for something low-frequency and smooth.
- Possible Solution: use a LOD hierarchy.
- New solution: subdivision operators that commute with Möbius transformations.



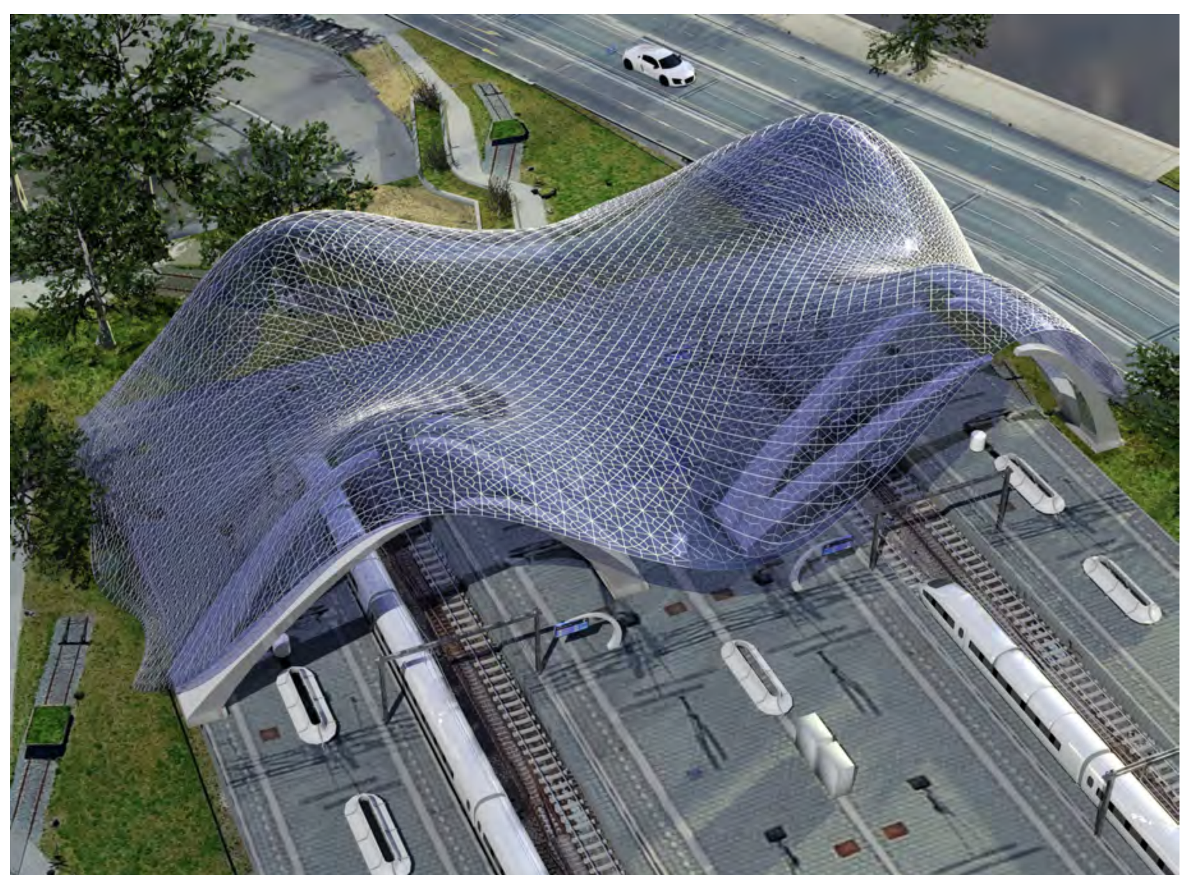


[[V., Müller, and Weber 2018]

Coarse-to-Fine Möbius Editing

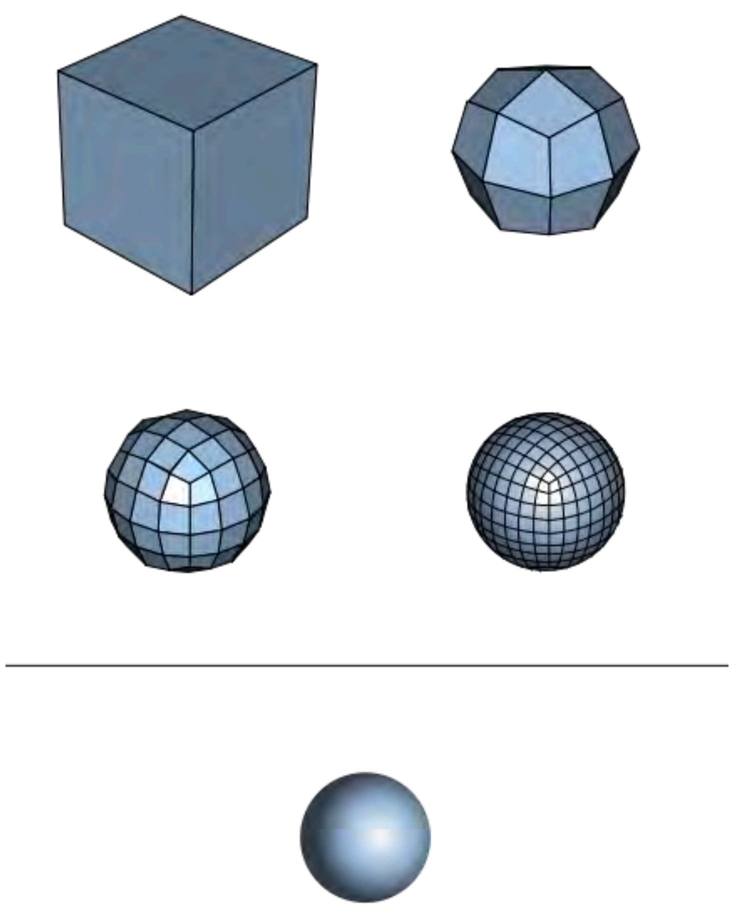


Subdivision + optimization: 1.5secs!



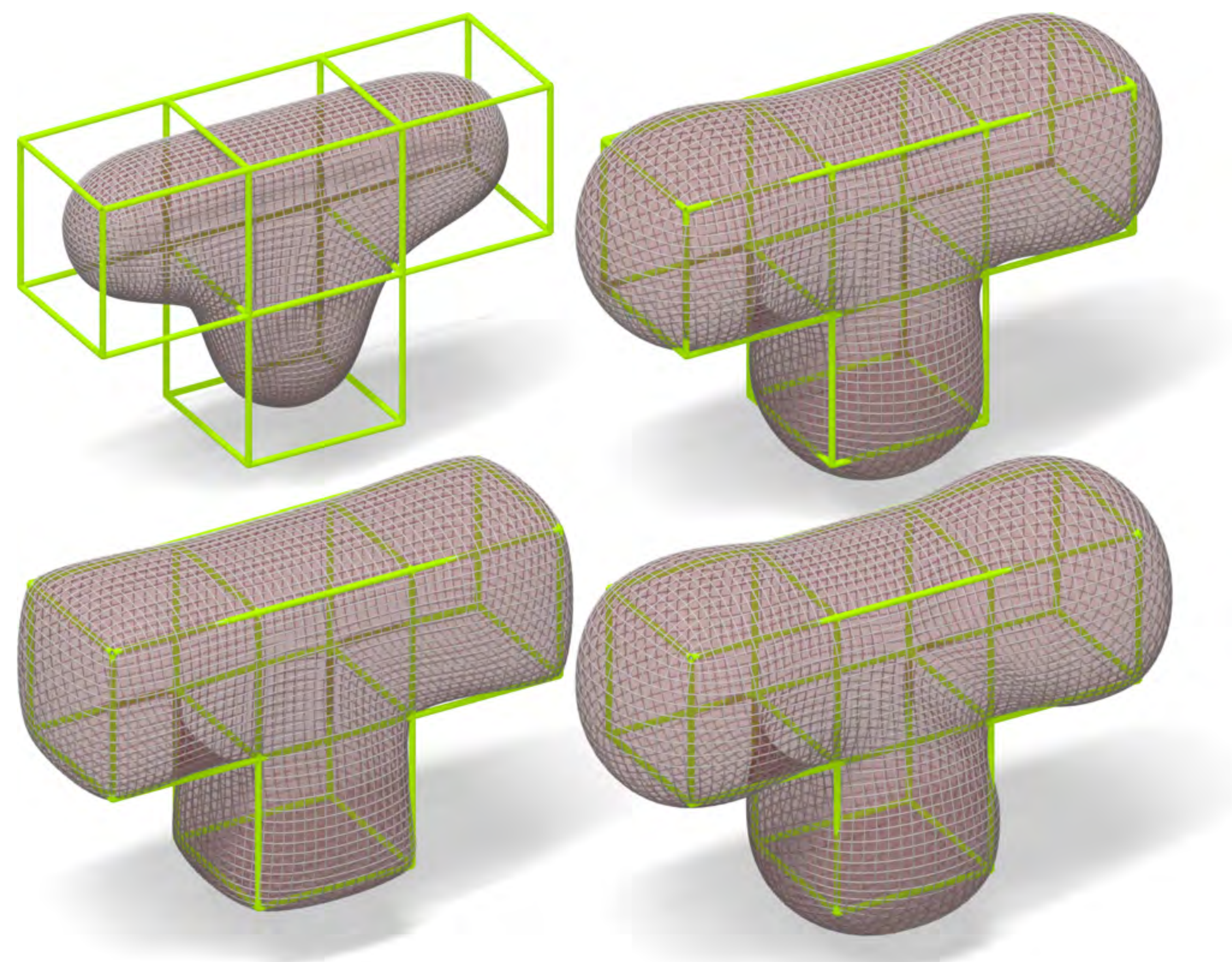
Subdivision surfaces

- Apply (mostly linear and stationary) rules to recursively refine surfaces.



Catmull-Clark

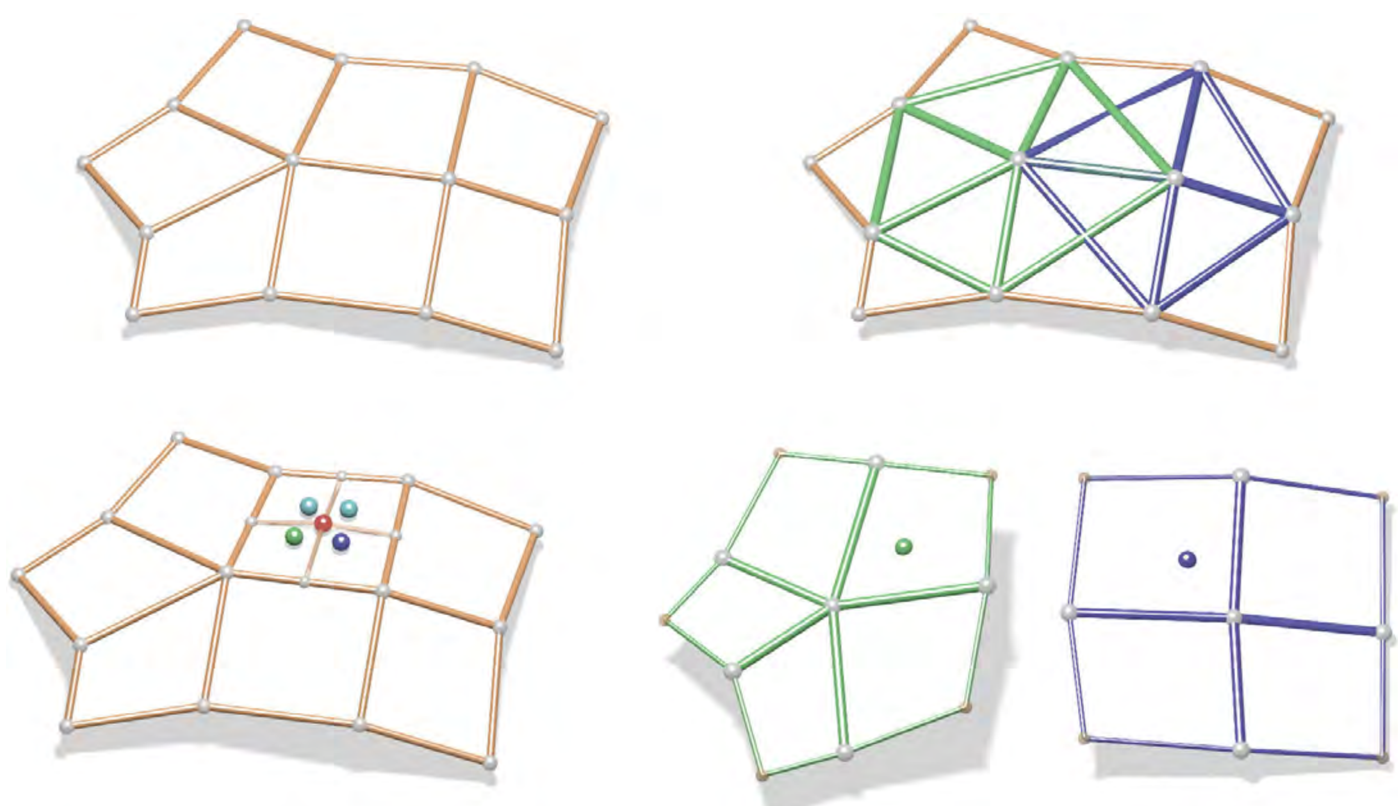
Linear CC   Möbius CC



Linear Kobbelt   Möbius Kobbelt

Coarse-to-Fine Möbius Editing

- Algorithm:
  - Compute canonical forms per 1-ring.
  - Linear subdivision in each form.
  - Transform points back and Blend them.



Canonical Forms

- Generalization of the perfectly symmetric forms to *any* star.

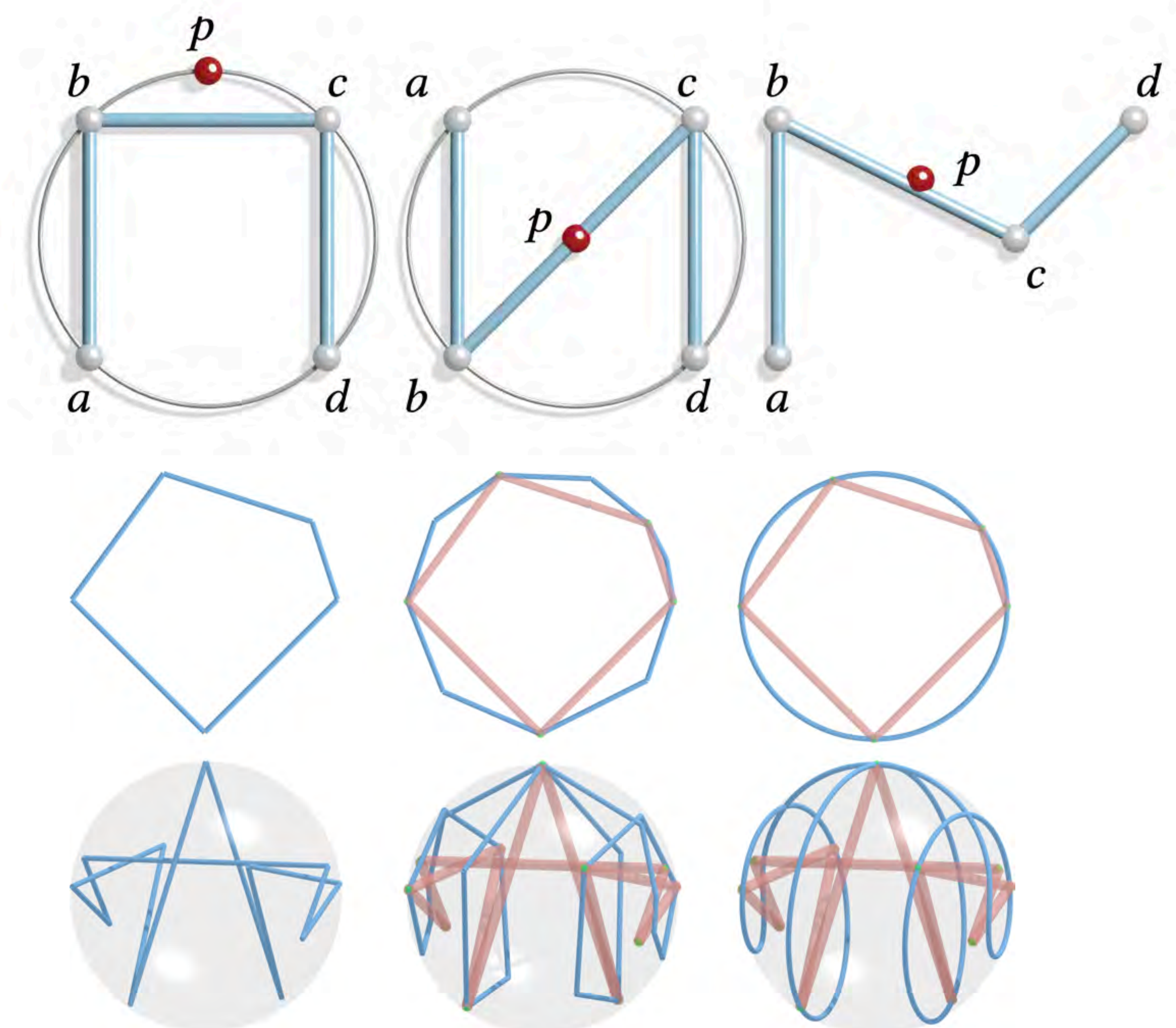
- Using the tangent polygon!

Original Ring | Möbius Trans. to Tangent polygon | Inversion in a center

Blending Points

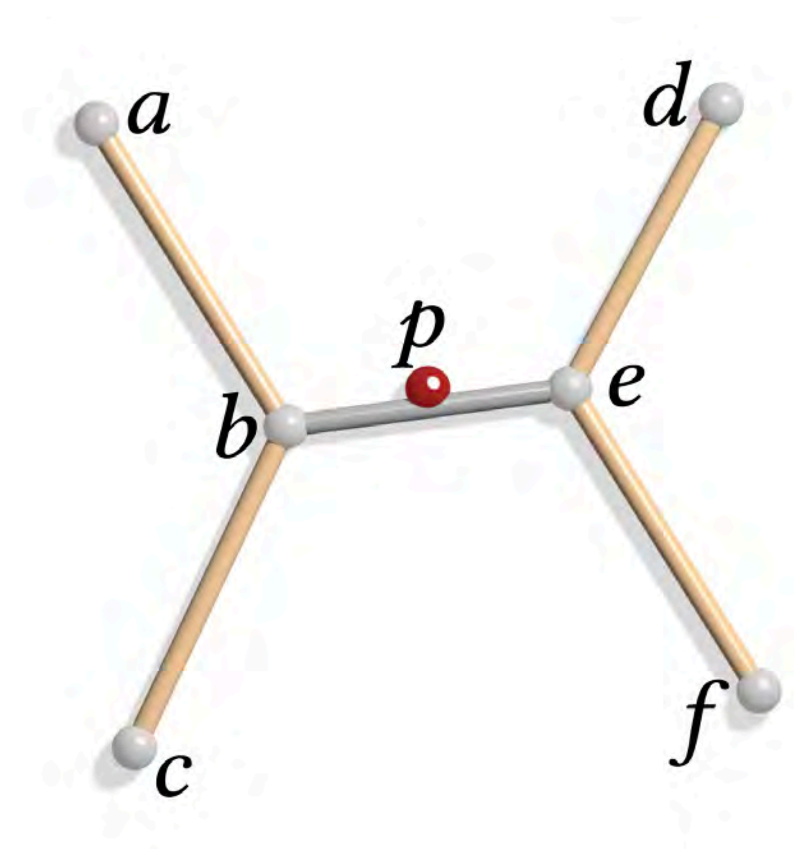
- 4-point scheme:

$$c[c,a,b,p] = -\sqrt{cr[c,a,b,d]}$$



Blending Points

- 6-point scheme:



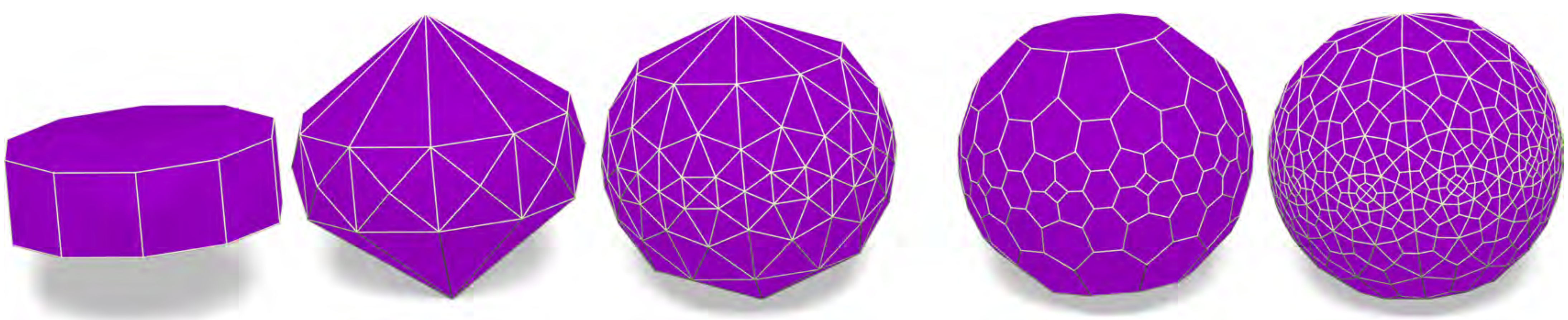
$$cr[e,a,b,p] = -\sqrt{cr[e,a,b,d]}\sqrt{r}$$

$$r = \sqrt{cr[e,a,b,d]}^{-1} \cdot cr[e,a,b,f] \cdot \sqrt{cr[e,b,c,d]}^{-1}$$

Coarse-to-Fine Möbius Editing

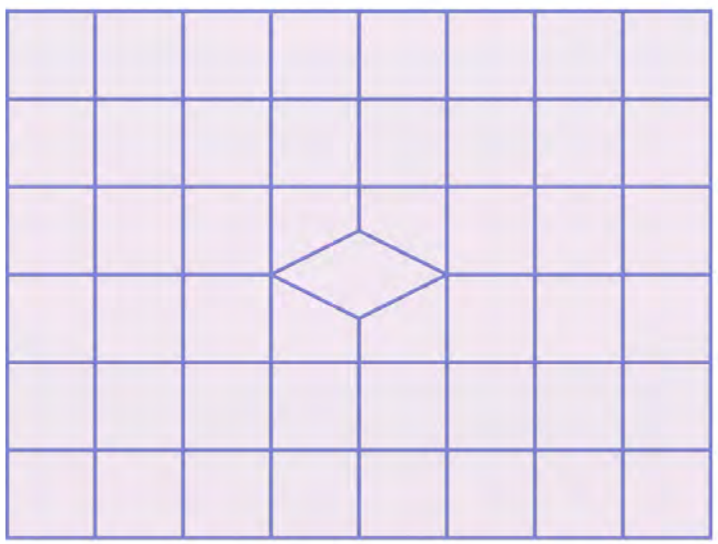
- Linear subdivision preserves lines, planes and Euclidean regularity =>

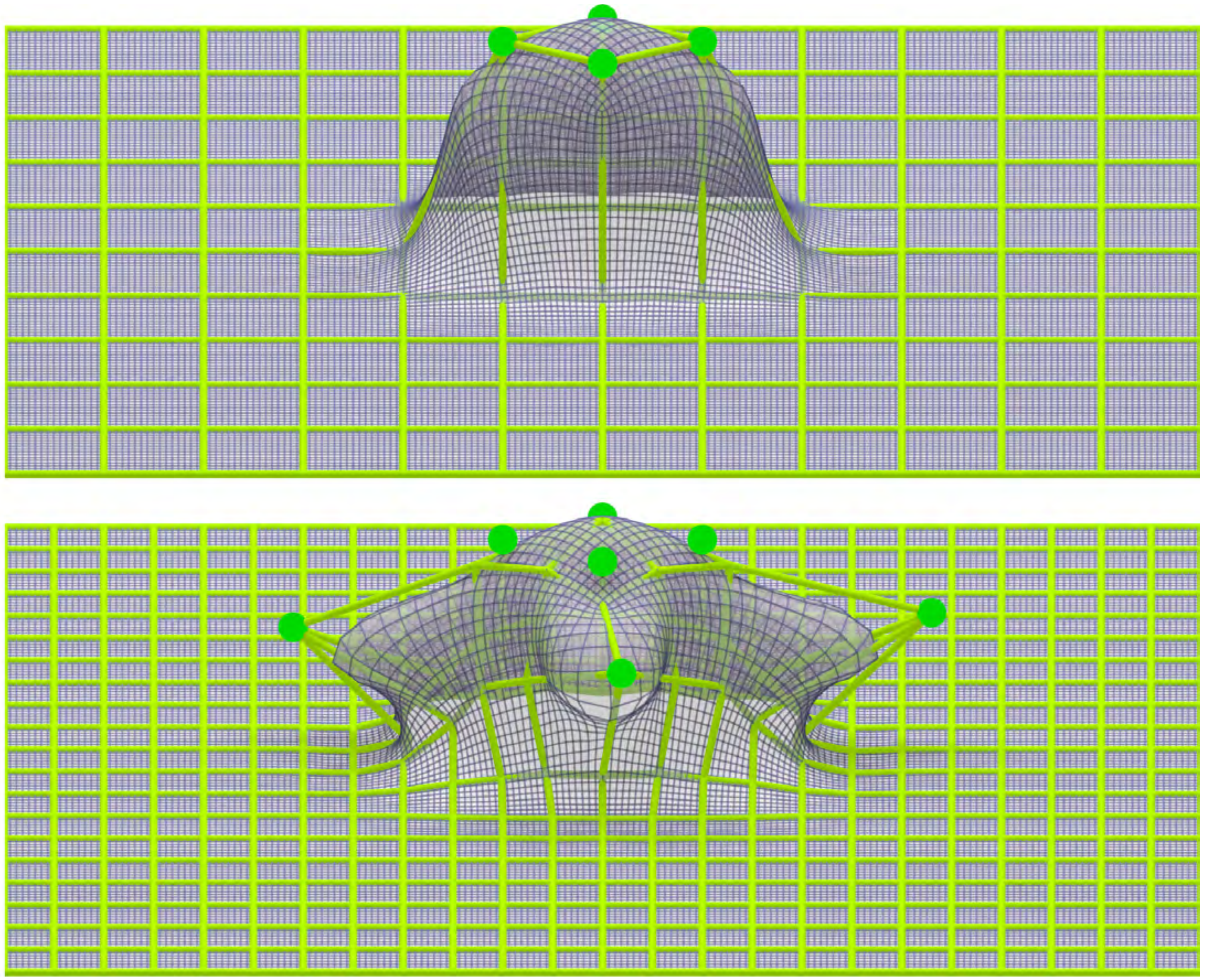
- Möbius scheme preserves spheres, circles, and Möbius regularity.



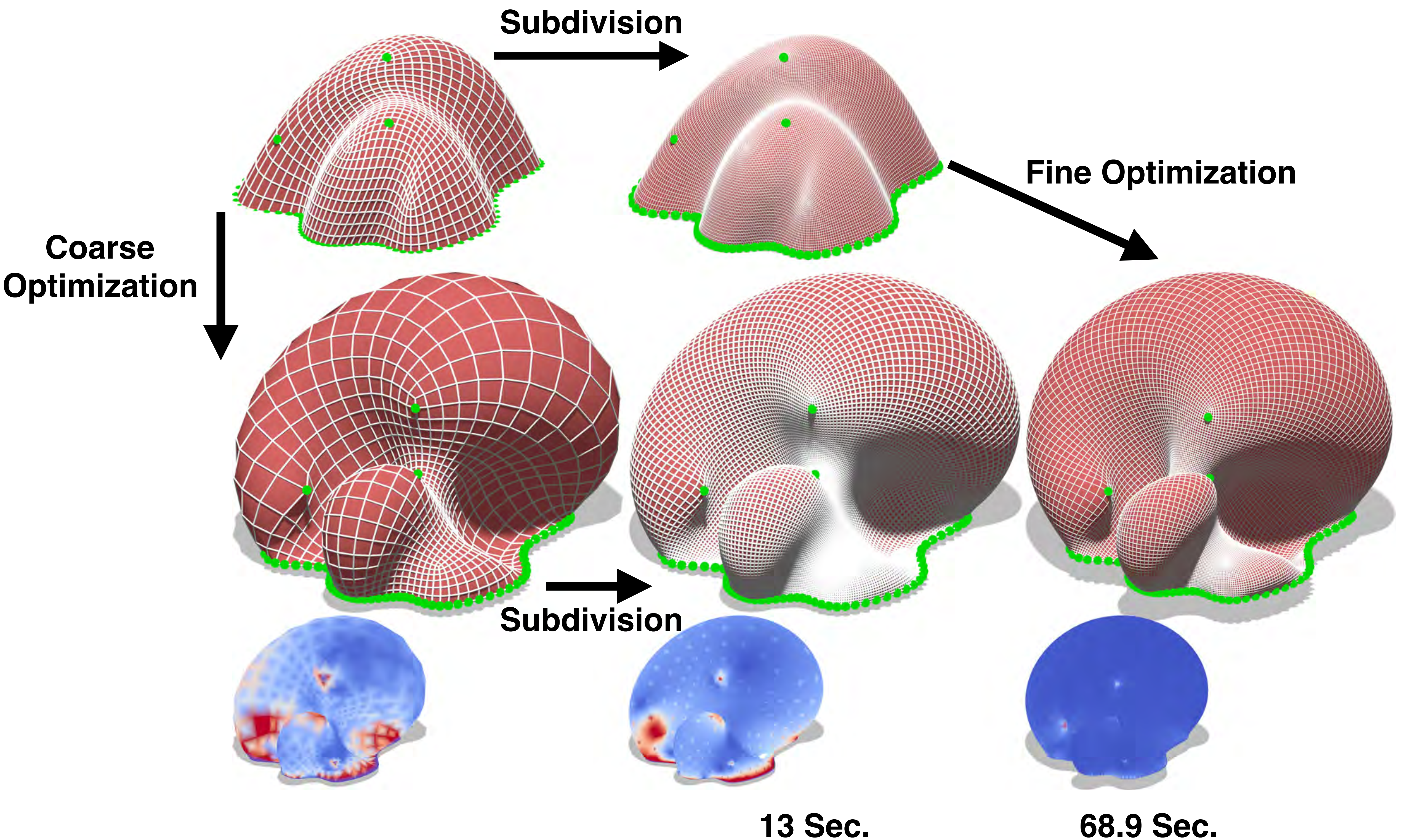
Coarse-to-Fine Möbius Editing

Direct Editing

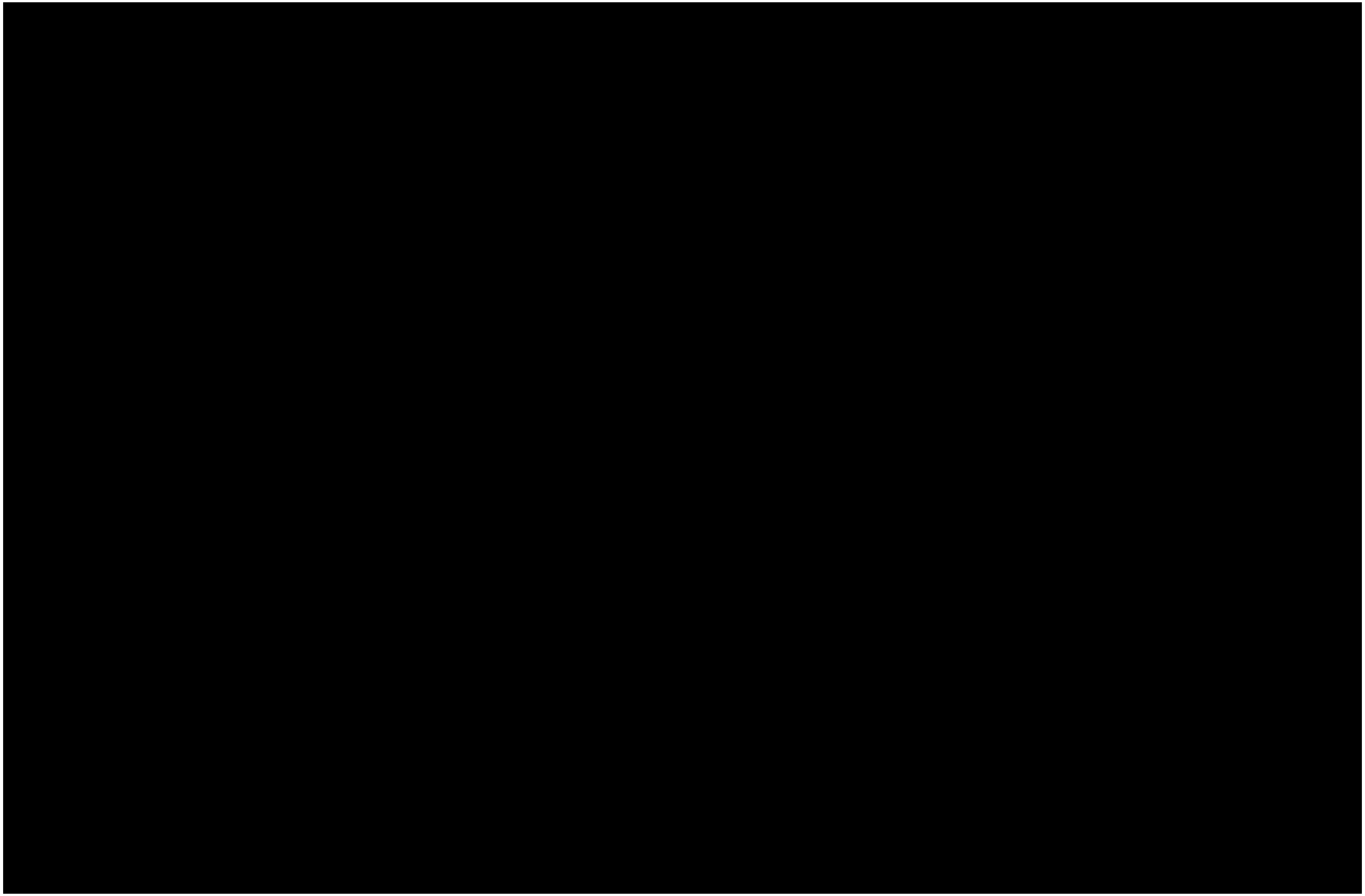




Coarse-to-Fine Möbius Editing

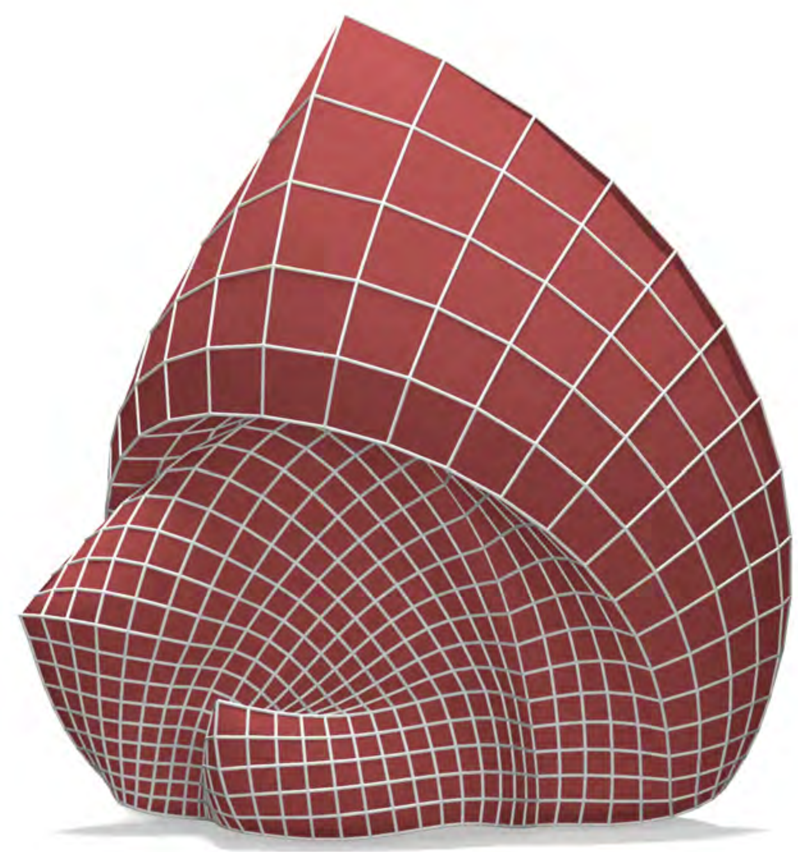


Coarse-to-Fine Möbius Editing



Future Prospects

- Fabrication & other constraints
- Parameterization
- Möbius calculus



References

Code will soon be available online through libhedra:
https://avaxman.github.io/libhedra/

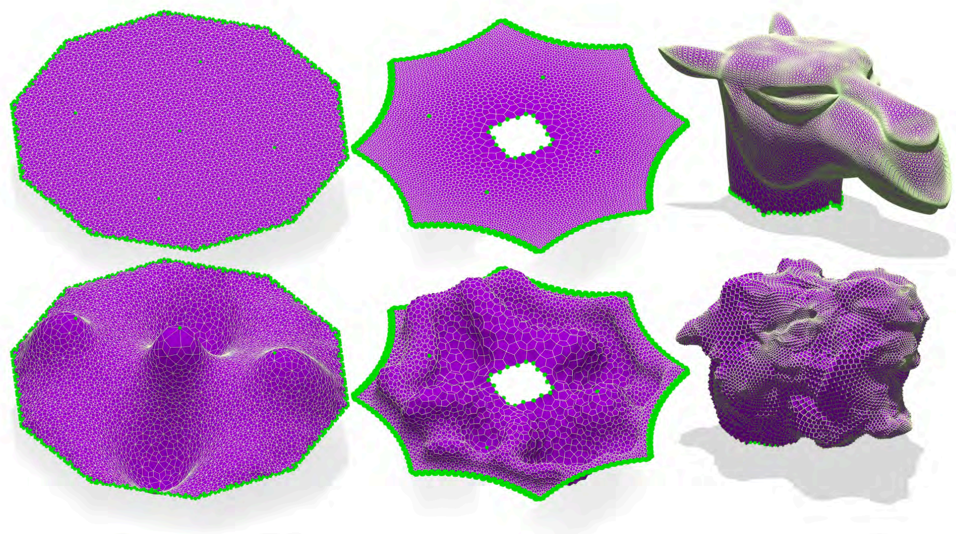
Conformal mesh deformations with Möbius transformations, Amir Vaxman, Christian Müller, Ofir Weber, ACM Transactions on Graphics (TOG) 34 (4), 2015.

Regular Meshes from Polygonal Patterns, Amir Vaxman, Christian Müller, and Ofir Weber, ACM Transactions on Graphics (Proc. SIGGRAPH), 36(4), 2017.

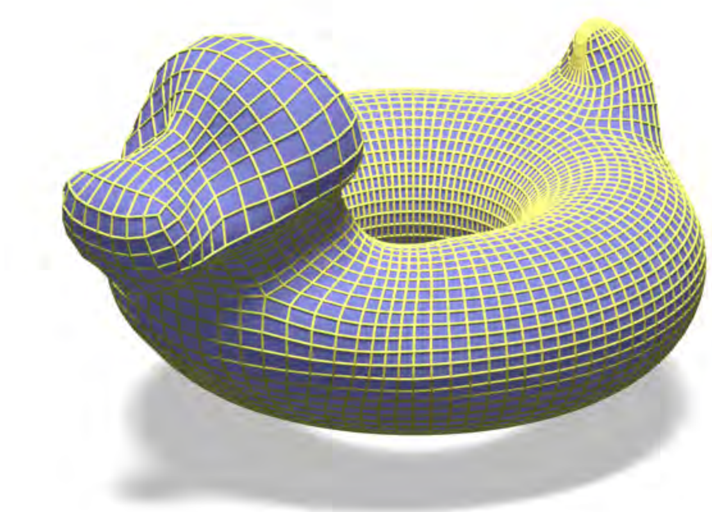
Canonical Möbius Subdivision, Amir Vaxman, Christian Müller, and Ofir Weber, ACM Transactions on Graphics (Proc. SIGGRAPH ASIA), 37(6), 2018.

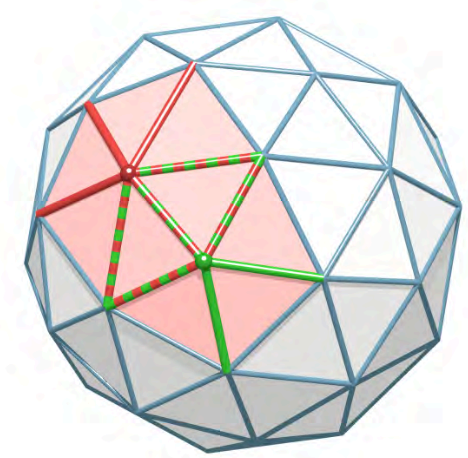
Thanks:
Ron vanderfeesten
Udo Hertrich-Jeromin
Zohar Levi
Helmut Pottmann

Funding:
FWF Lise-Meitner grant M1618-N25
FWF grant P23735-N13, I 706-N26
Israel Science Foundation, grants 1869/15 and 2102/15
NVIDIA corp.



Questions?





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