A Tutorial on How to Simulate Versatile Physical Materials with Particle-based Methods

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中科院软件所

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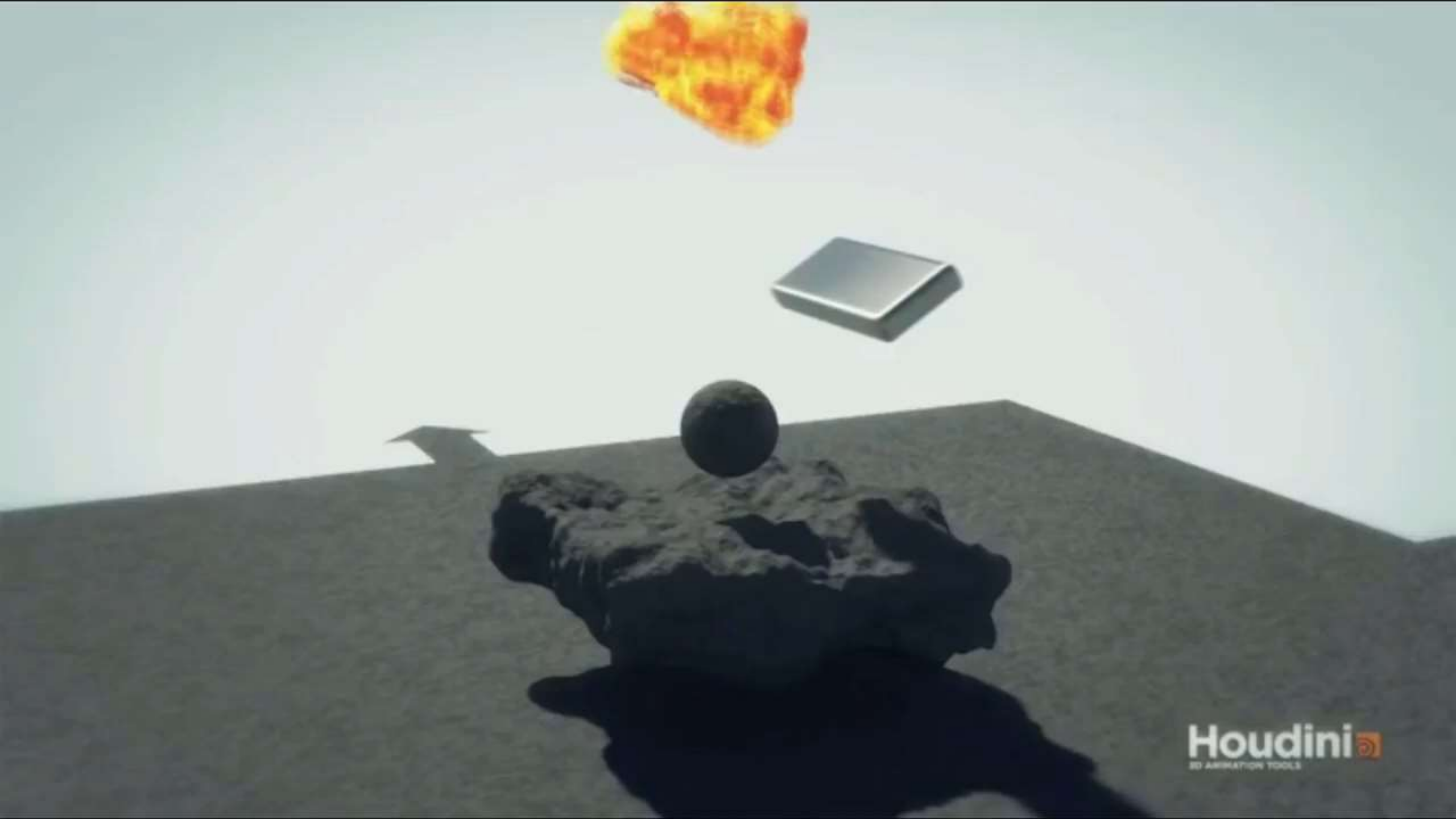
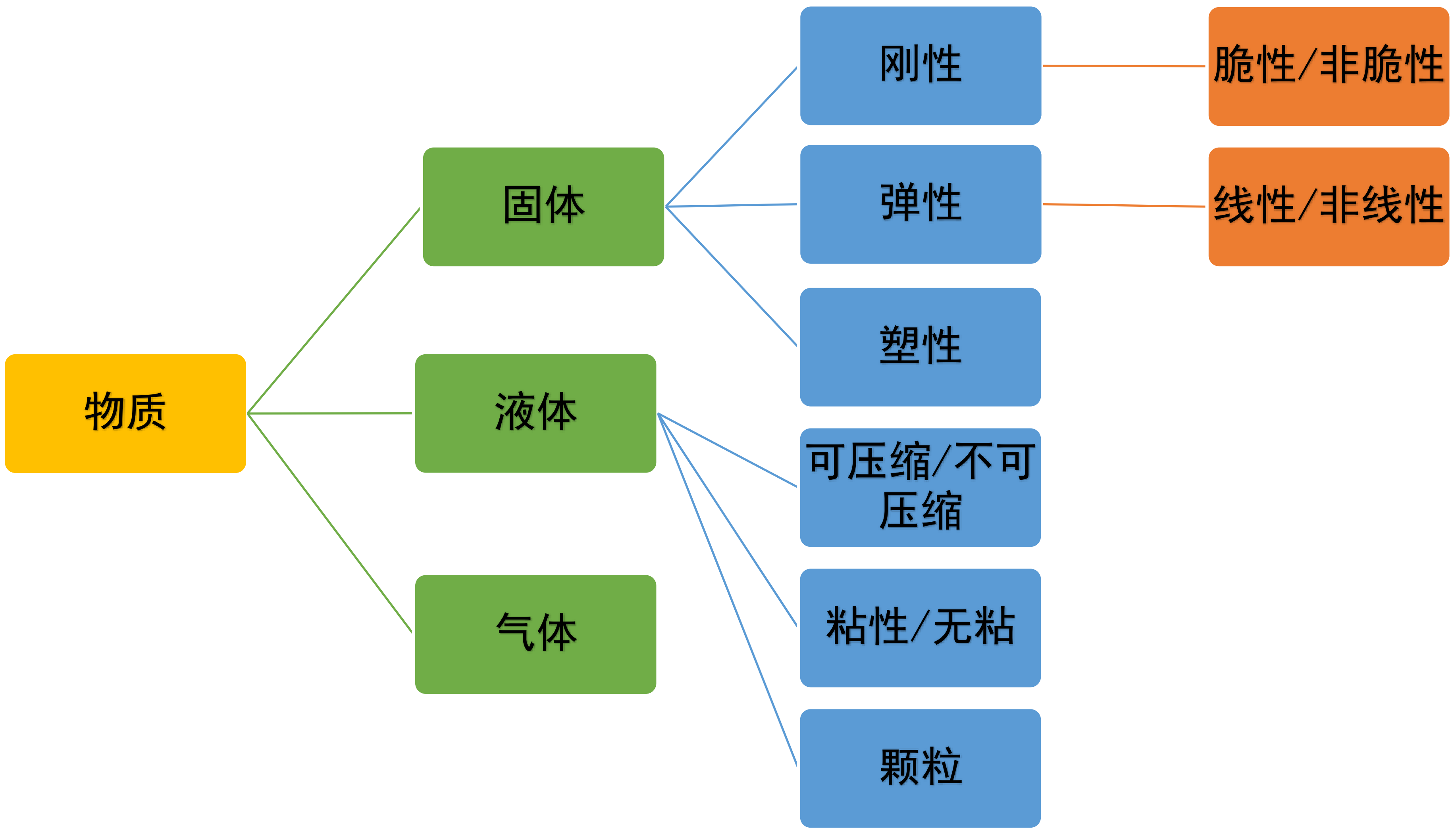
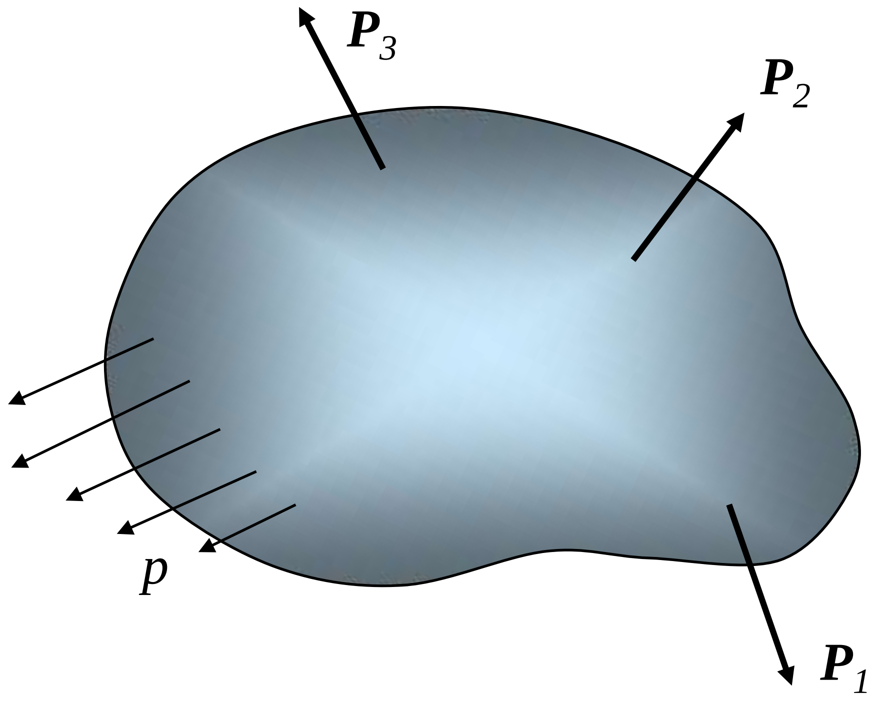
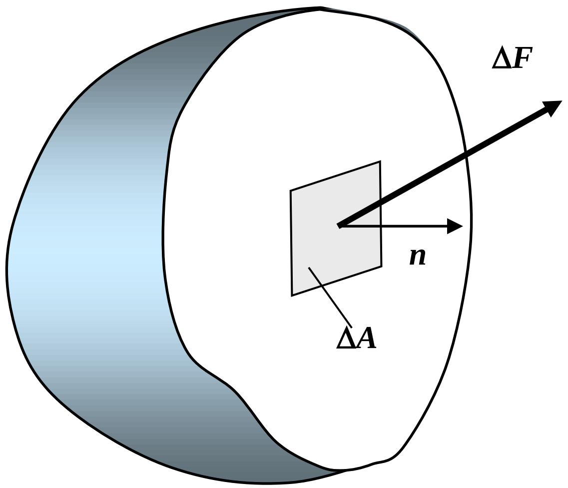

Houdini

Materials in Real World



Stress

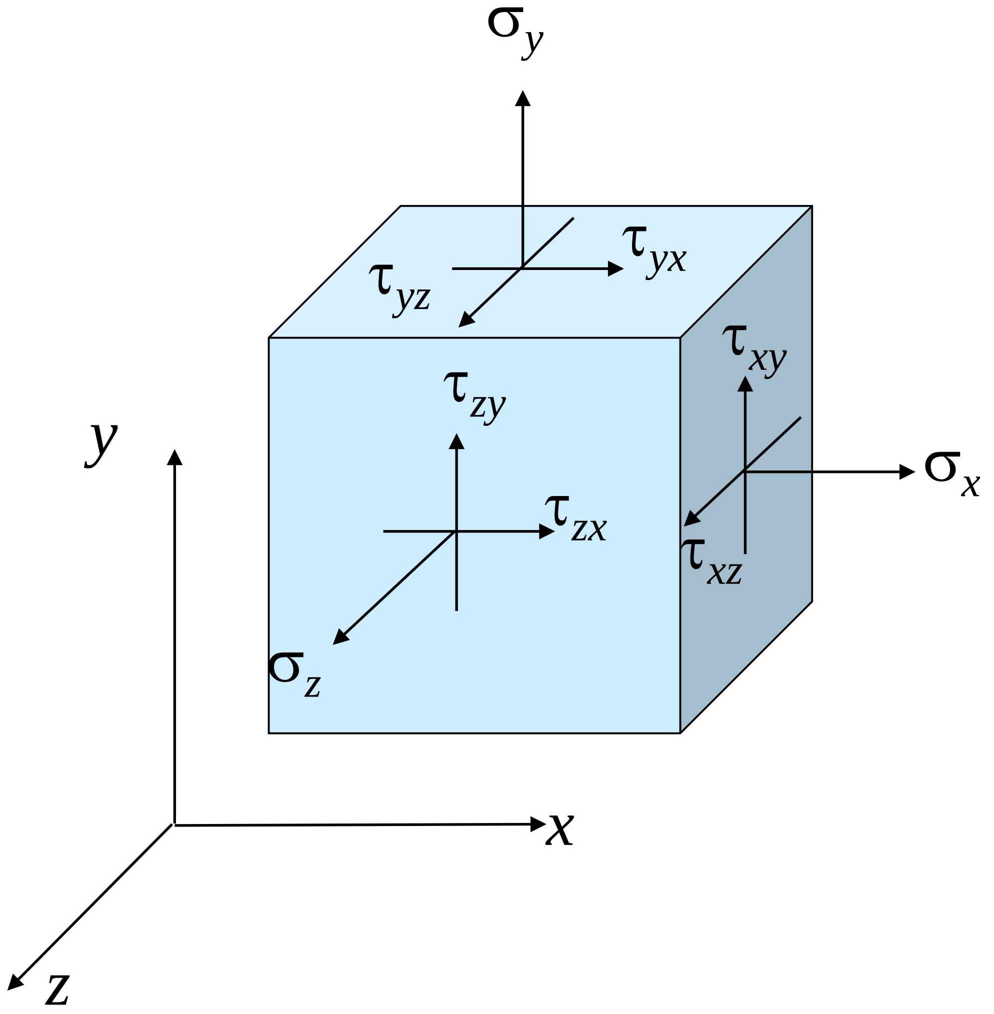




$$T^{n}(x, n) = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

Externally Loaded Body **Sectioned Body** **Stress**

Stress Tensor



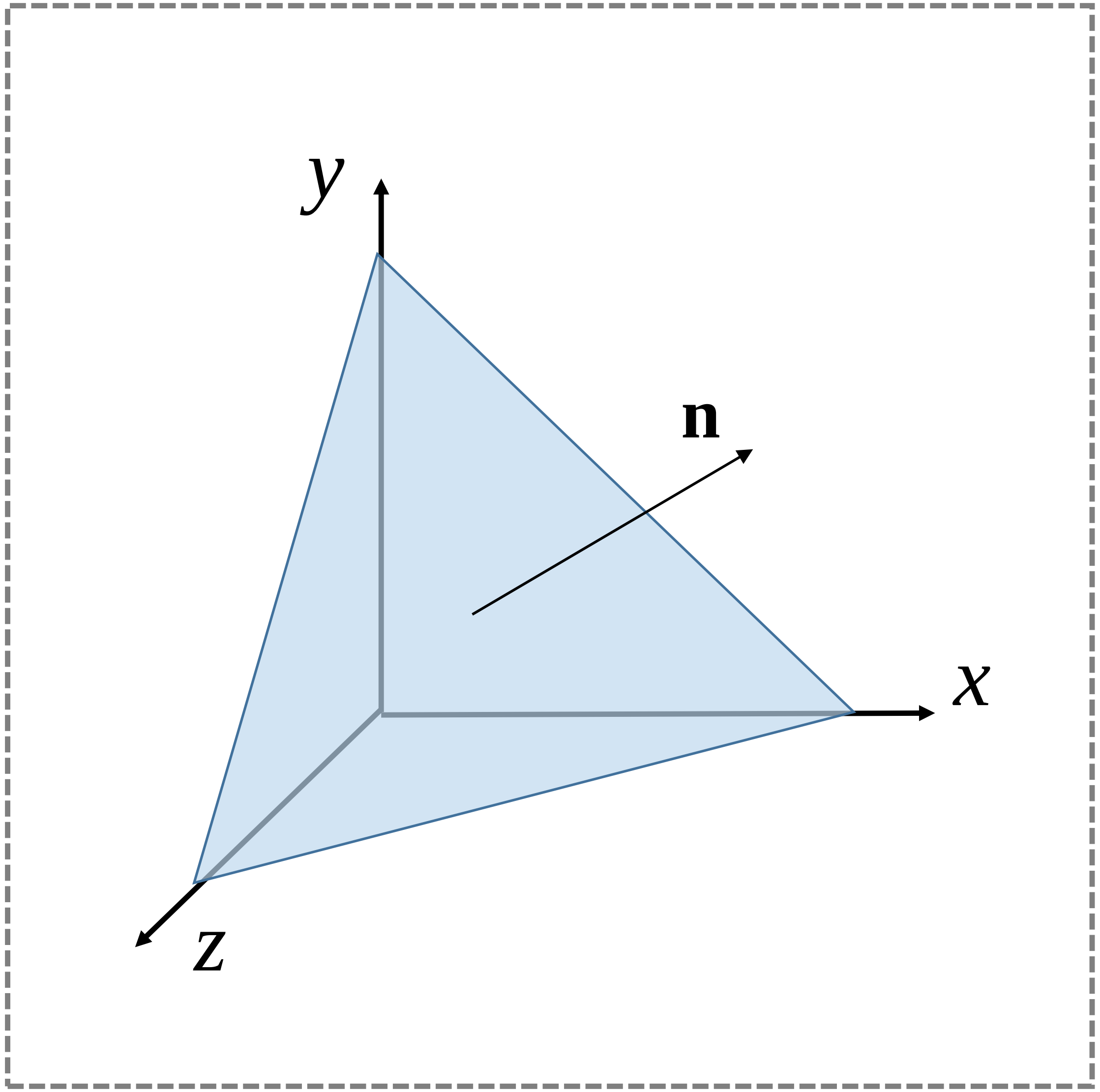
$$\boldsymbol{T}^{n}(\boldsymbol{x},\boldsymbol{n}=\boldsymbol{e}_1)=\sigma_x\boldsymbol{e}_1+\tau_{xy}\boldsymbol{e}_2+\tau_{xz}\boldsymbol{e}_3$$

$$\boldsymbol{T}^{n}(\boldsymbol{x},\boldsymbol{n}=\boldsymbol{e}_2)=\tau_{yx}\boldsymbol{e}_1+\sigma_y\boldsymbol{e}_2+\tau_{yz}\boldsymbol{e}_3$$

$$\boldsymbol{T}^{n}(\boldsymbol{x},\boldsymbol{n}=\boldsymbol{e}_3)=\tau_{zx}\boldsymbol{e}_1+\tau_{zy}\boldsymbol{e}_2+\sigma_z\boldsymbol{e}_3$$

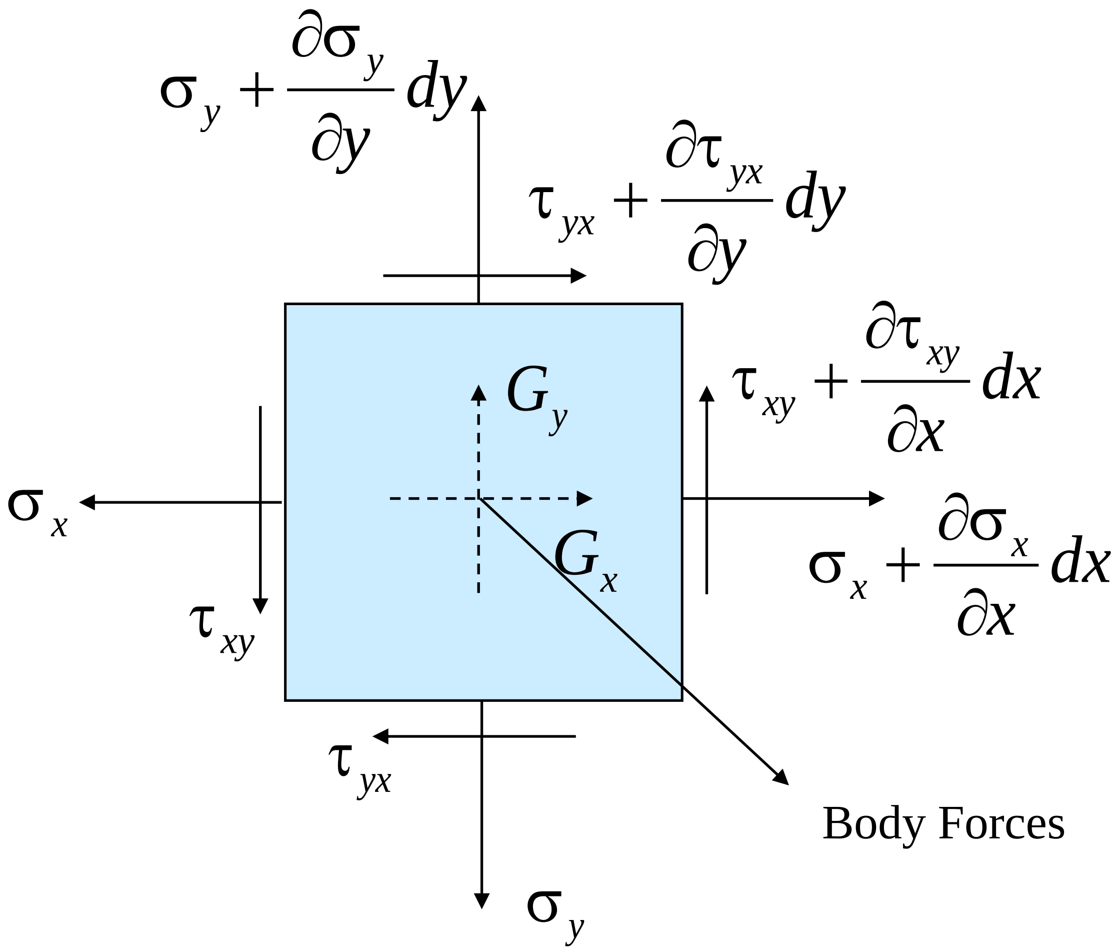
$$\boldsymbol{\sigma}=[\boldsymbol{\sigma}]=\begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_y & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_z\end{bmatrix}$$

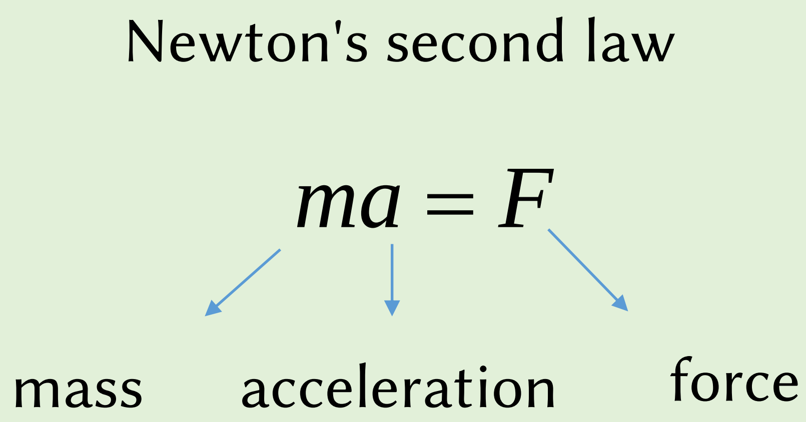
Stress Tensor



$$
\begin{aligned}
\mathbf{T}^{\mathrm{n}} = \sigma\mathbf{n} &= (\sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z)\mathbf{e_1} \\
&+ (\tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z)\mathbf{e_2} \\
&+ (\tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z)\mathbf{e_3}
\end{aligned}
$$

Equilibrium Equations





$$\rho \frac{dv_x}{dt} = \sum F_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + G_x$$

$$\rho \frac{dv_y}{dt} = \sum F_y = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + G_y$$

Equilibrium Equations

$$\nabla \cdot = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

Stress Tensor

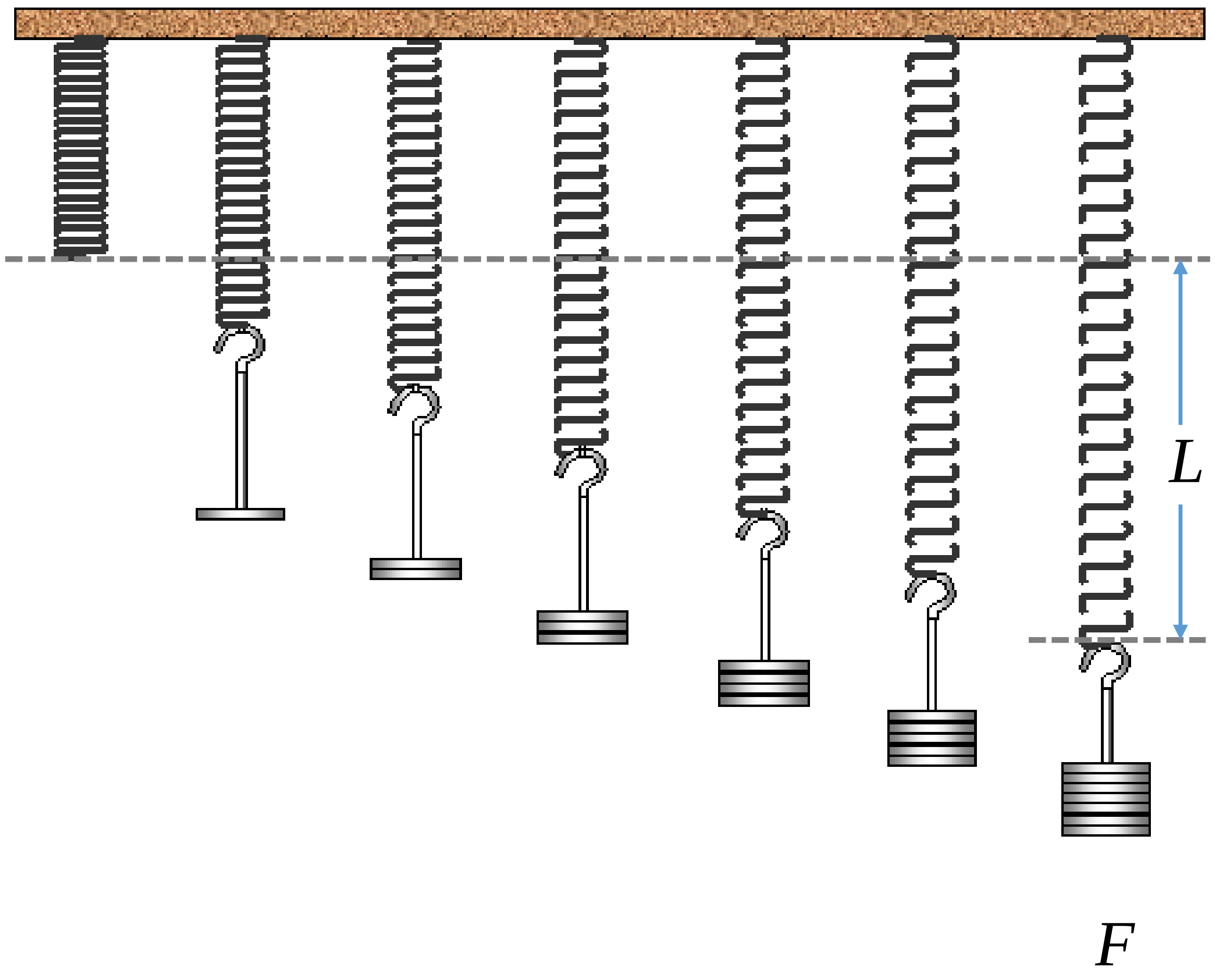
$$\rho \frac{d\mathbf{v}}{dt} = \nabla \cdot \sigma + g$$

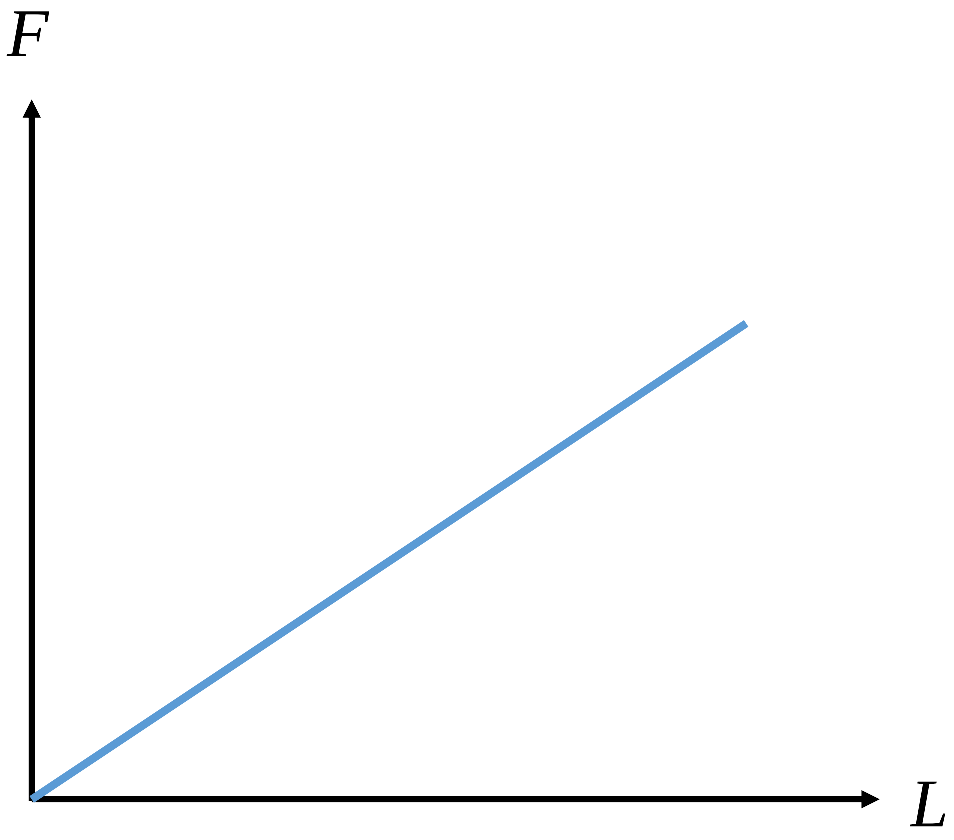
Acceleration

Internal force
(Pressure, Viscosity,
Elasticity, etc)

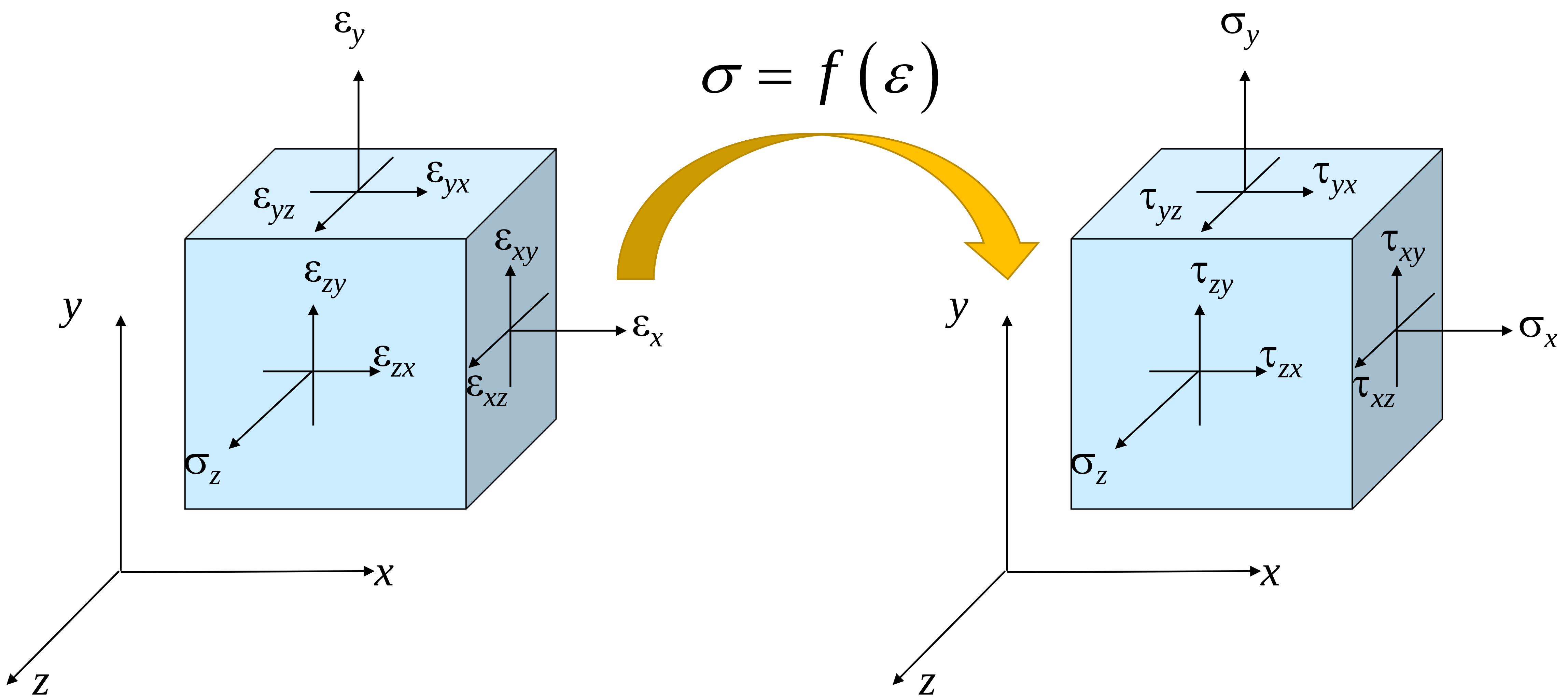
External force
(Gravity, Buoyancy,
Surface tension, etc.)

Constitutive Relation

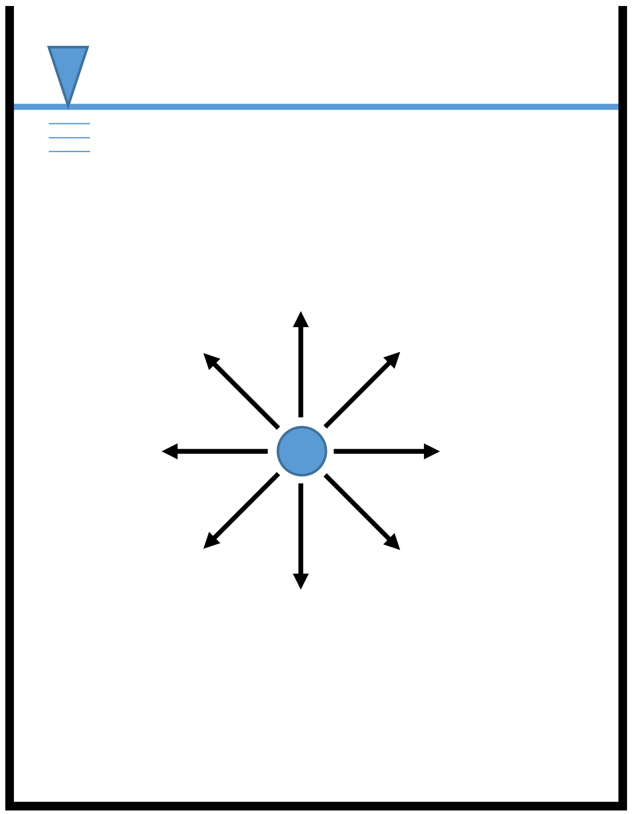




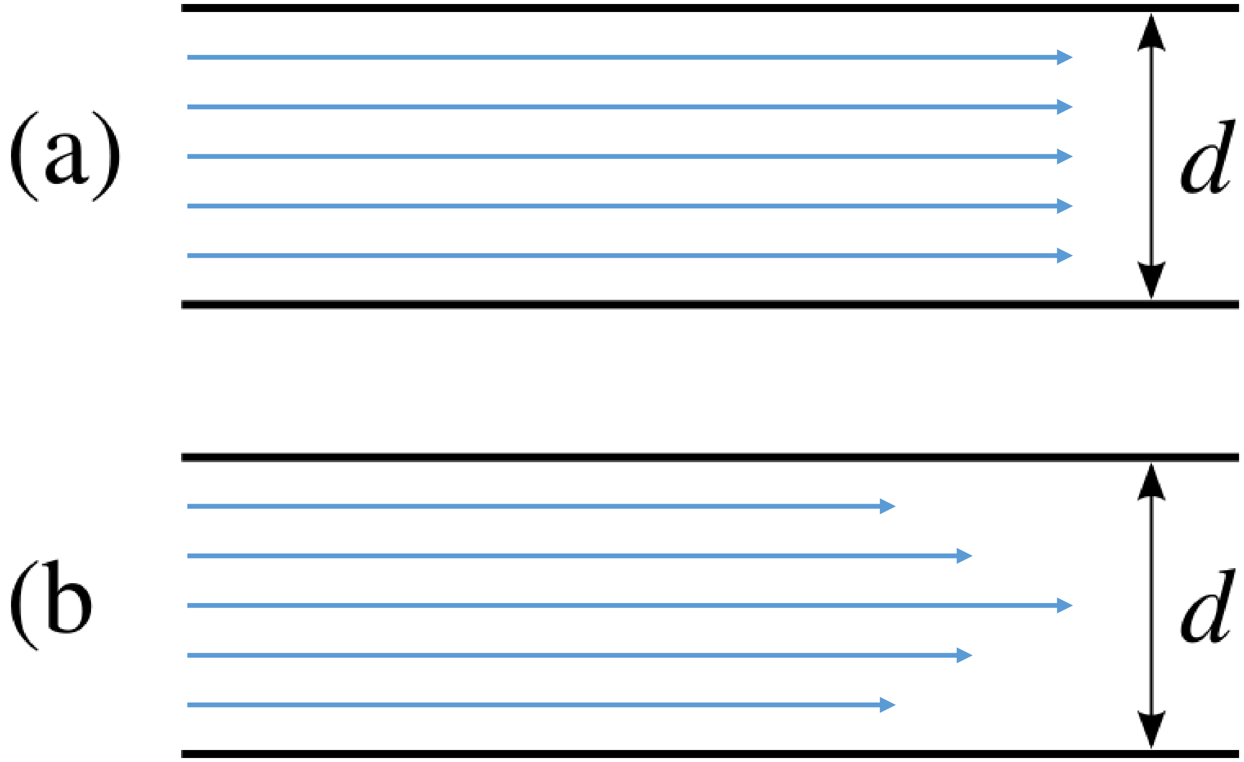
Constitutive Relation



Fluid



Hydrostatic water



Laminar flow

Fluid

Viscosity coefficient

$$\sigma = -p\mathbf{I} + \mu\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right)$$

Hydrostatic pressure

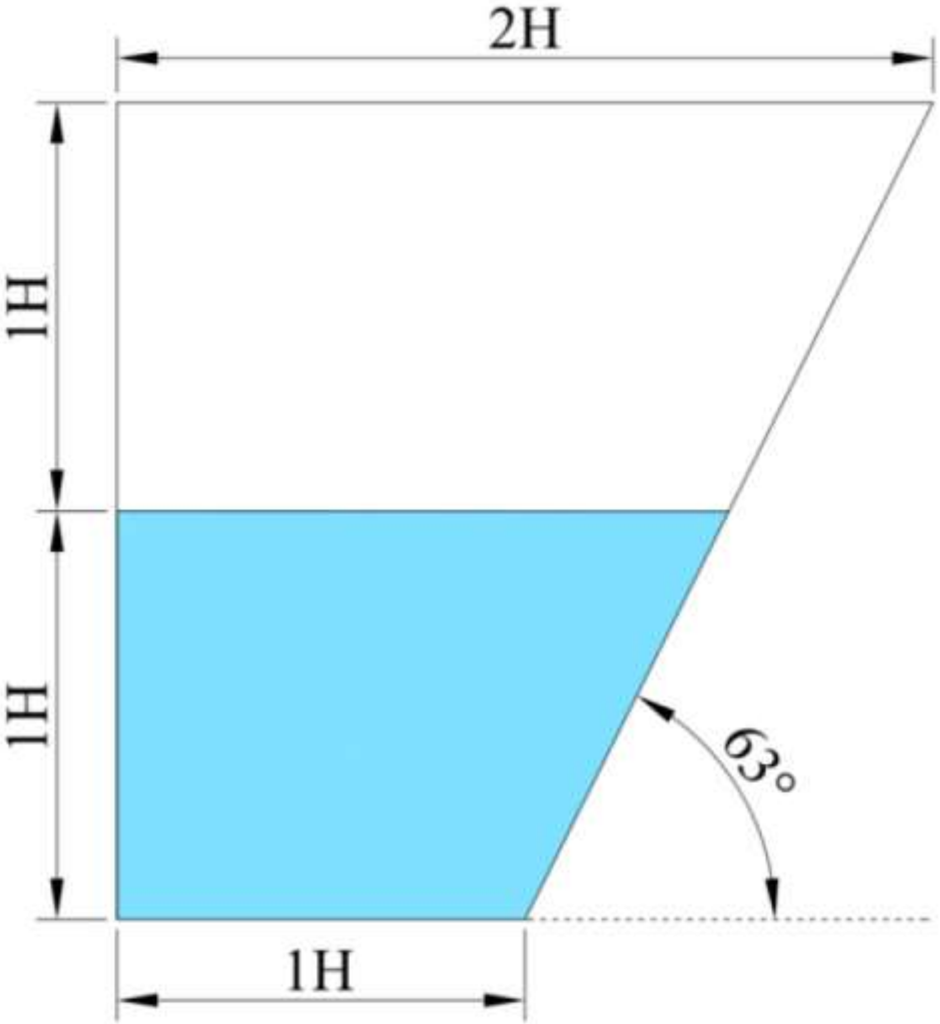
Identity matrix

$$\nabla\mathbf{v} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix}$$

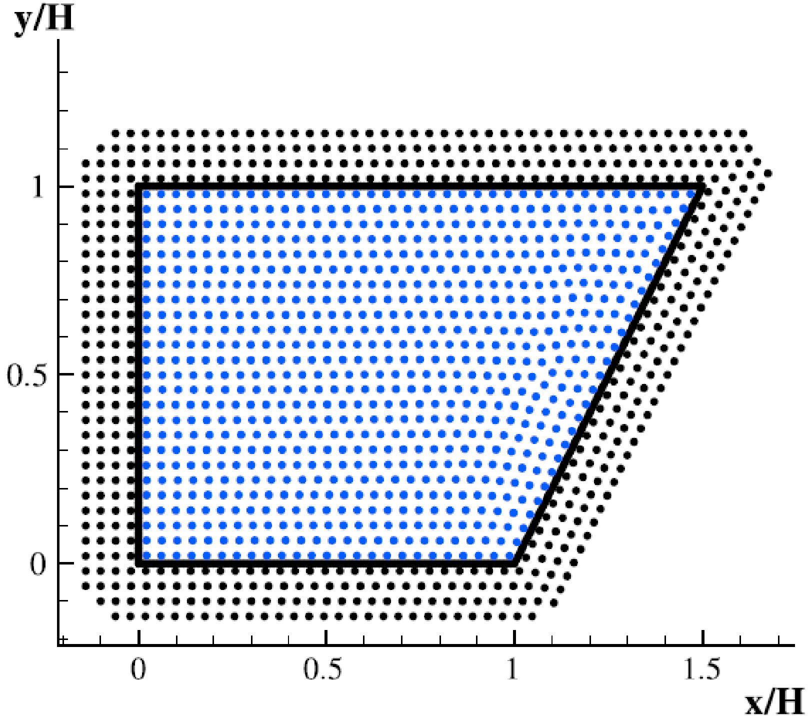
Navier-Stokes Equation

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla p + \mu \nabla \cdot \left(\nabla \mathbf{v} + \nabla \mathbf{v}^T \right) + g$$

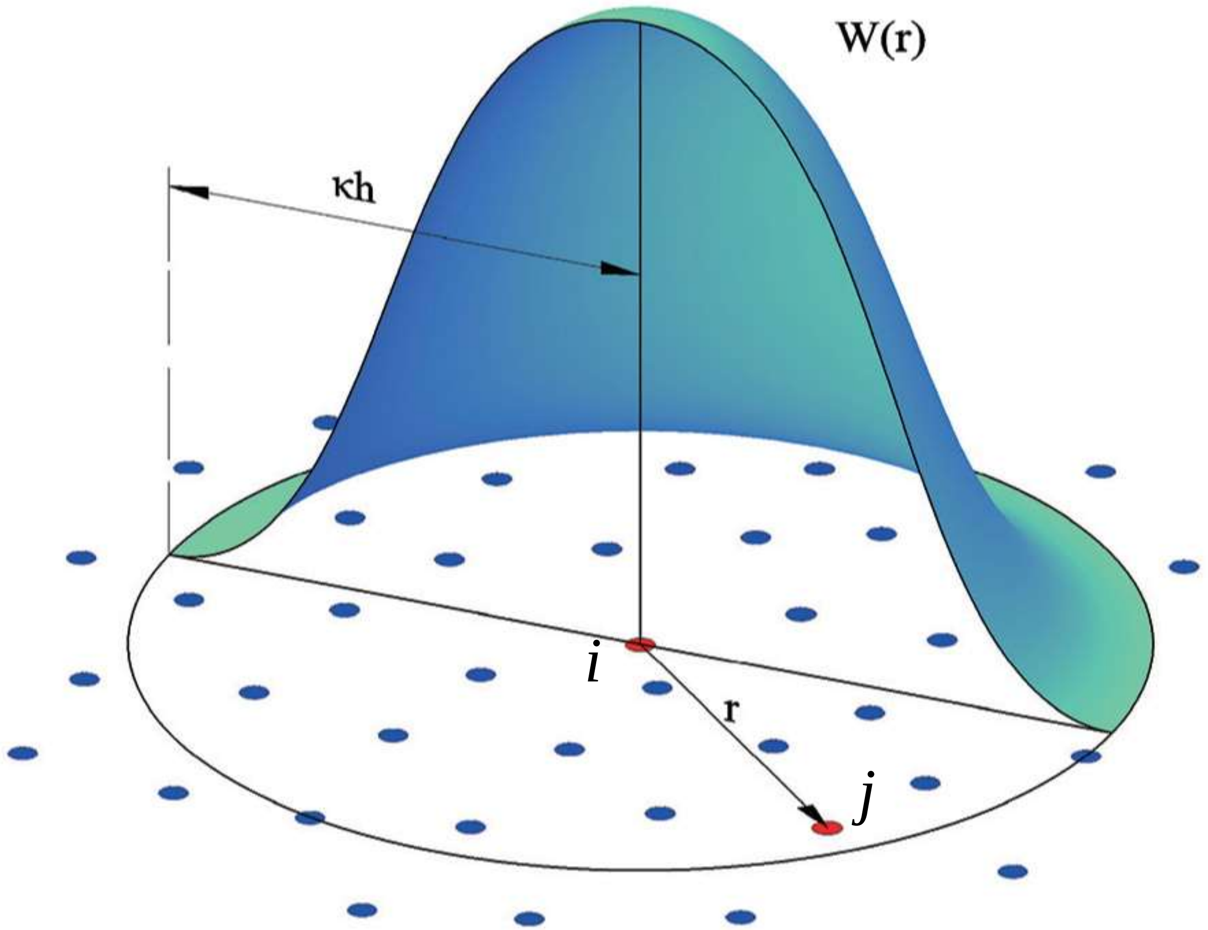
Smoothed Particle Hydrodynamics(SPH)



Particle discretization



Smoothed Particle Hydrodynamics(SPH)



Independent: $m_i, \mathbf{x}_i, \mathbf{v}_i, V_i$

Dependent: $\rho_i, \ p_i$

Smoothed Particle Hydrodynamics(SPH)

$$\langle A \rangle_i \approx \sum_j \frac{m_j}{\rho_j} A_j W(r_{ij})$$

$$\langle \nabla A \rangle_i \approx \sum_j \frac{m_j}{\rho_j} A_j \nabla W(r_{ij})$$

$$\langle \nabla^2 A \rangle_i \approx \sum_j \frac{m_j}{\rho_j} A_j \nabla^2 W(r_{ij})$$

→

$$\langle \rho \rangle_i \approx \sum_j \frac{m_j}{\rho_j} \rho_j W(r_{ij}) \approx \sum_j m_j W(r_{ij})$$

∇p,

$\nabla \cdot \left(\nabla \mathbf{v} + \nabla \mathbf{v}^T \right)$,

…

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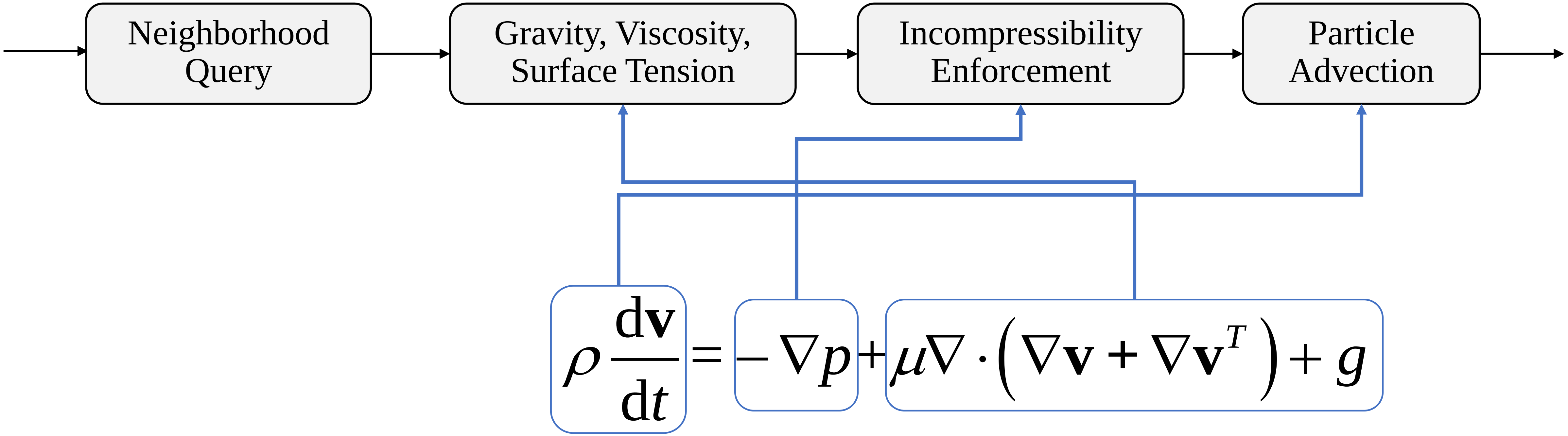
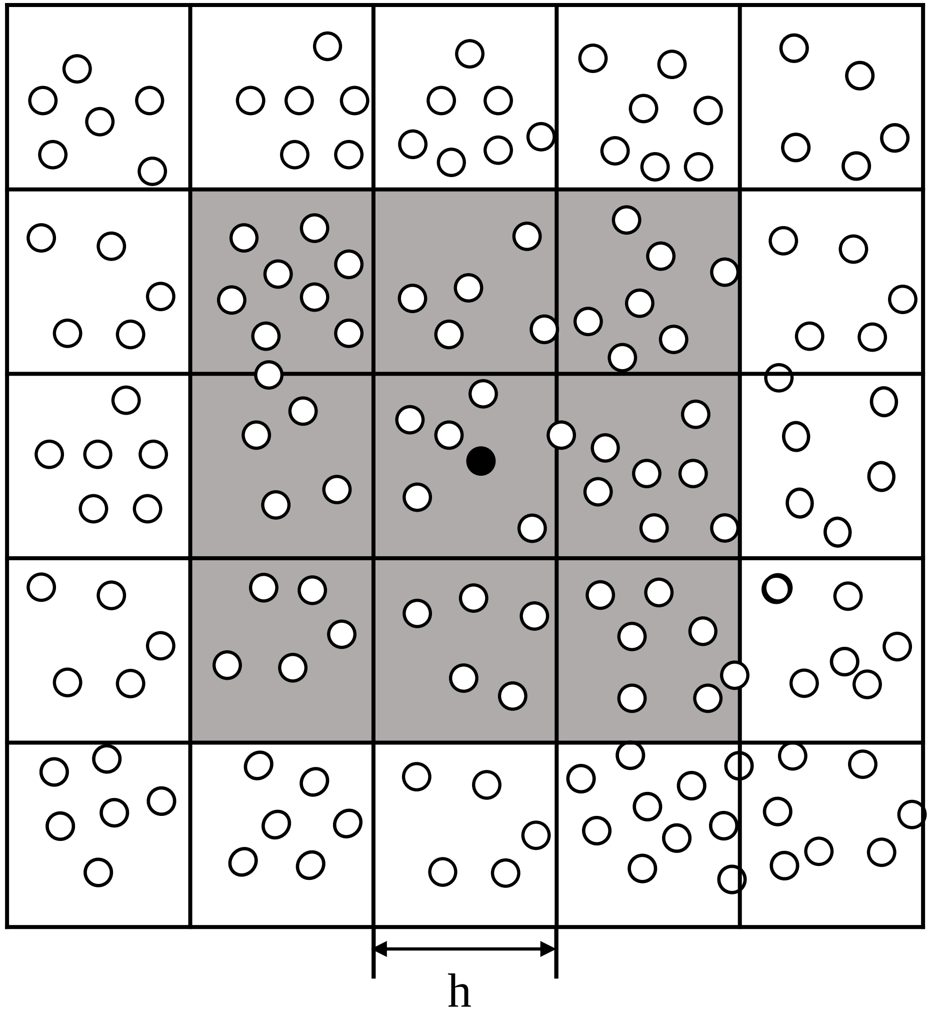
Framework

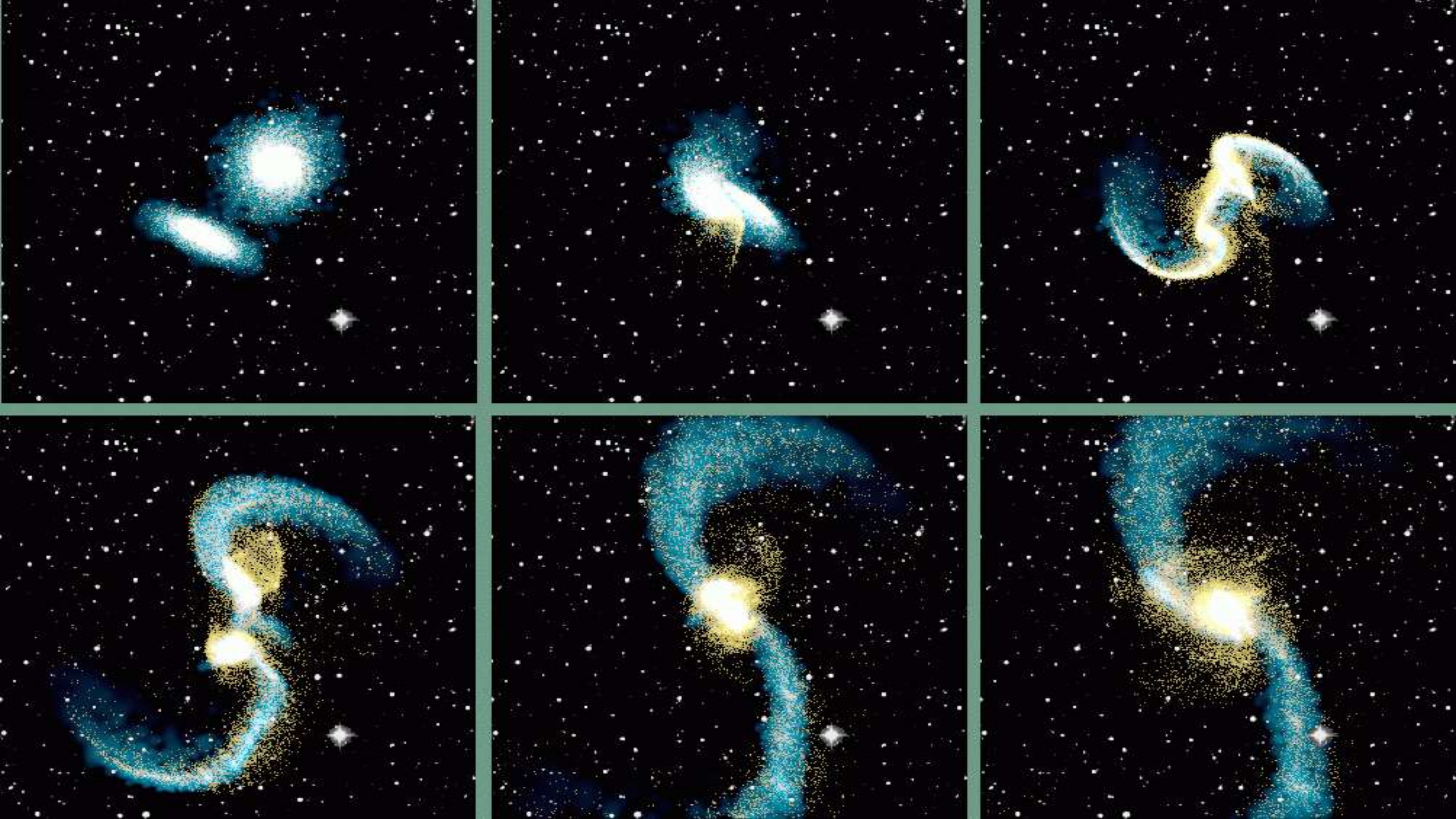


$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla p + \mu \nabla \cdot \left(\nabla \mathbf{v} + \nabla \mathbf{v}^T \right) + g$$

Neighborhood Query

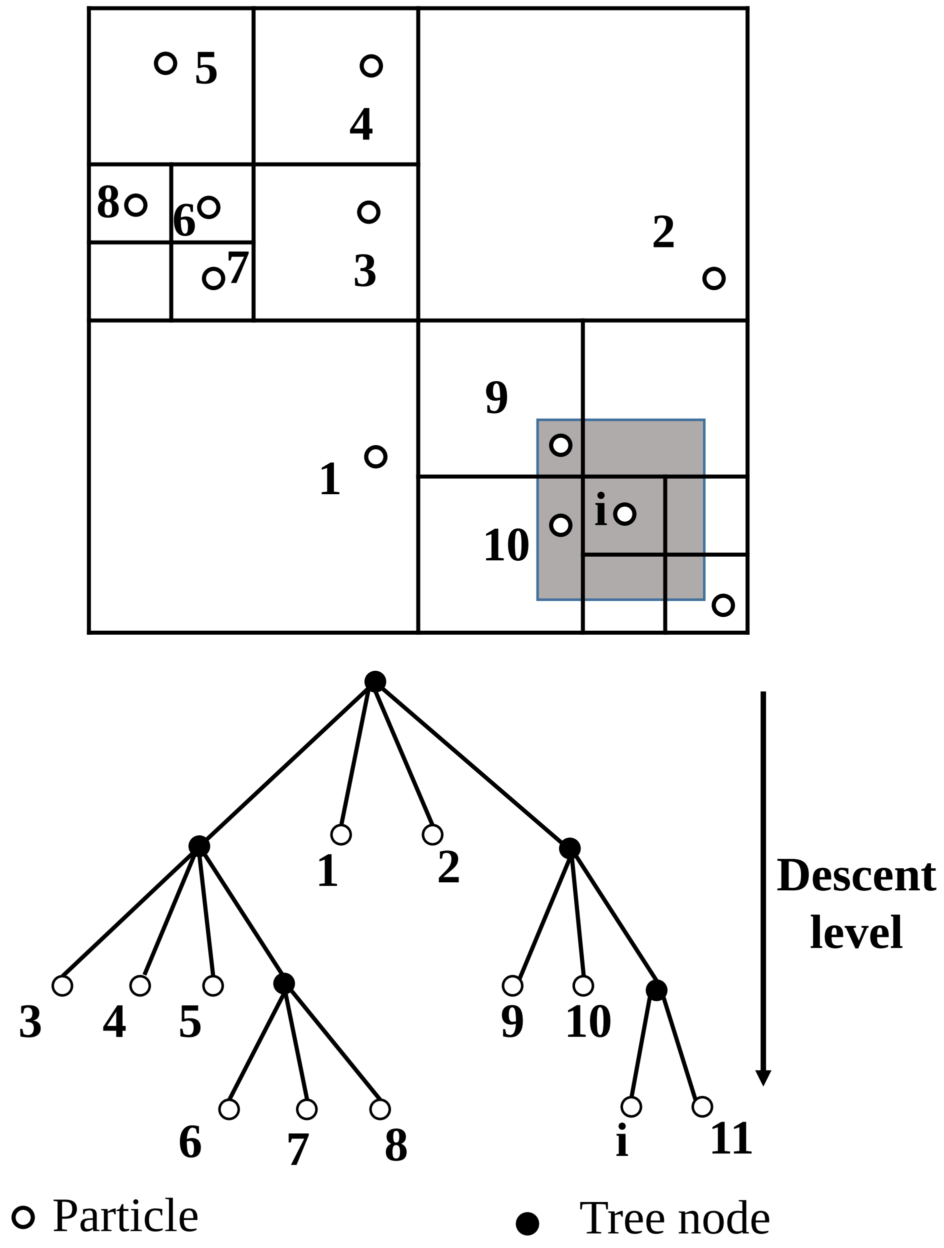
- Uniform Grid
  - E.g., [Mueller03, Harada07, Green08, Goswami11, Ihmsen11, Macklin13]
  - Generally preferred - construction in O(n), access in O(1)
  - Friendly with GPU implementation
- Disadvantage
  - Not quite suitable for sparse particle distribution





Neighborhood Query

- Hierarchical structures
  - E.g., [Vermuri98, Keiser06, Adams07]
- Disadvantage
  - Less efficient - construction in O(n log n), access in O(log n)
  - Not quite friendly with GPU implementation



Viscous force

$$\mathbf{f}^{vis} = \mu\nabla\cdot\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right)$$

If incompressible, $\nabla\cdot\mathbf{v} = \mathbf{0}$

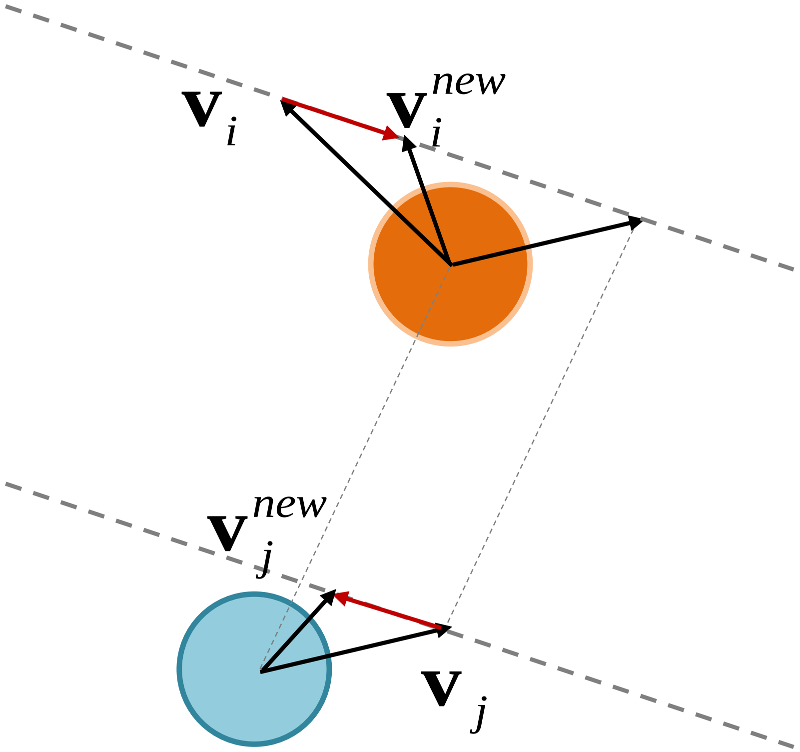
$$\mathbf{f}^{vis} = \mu\nabla^2\mathbf{v}$$

$$\langle\nabla^2 A\rangle_i \approx \sum_j \frac{m_j}{\rho_j} A_j \nabla^2 W(r_{ij})$$

$$\mathbf{f}^{vis} \approx \mu\sum_j \frac{m_j}{\rho_j}\mathbf{v}_j \nabla^2 W(r_{ij})$$

Momentum conservative

$$\mathbf{f}^{vis} \approx \mu\sum_j \frac{m_j}{\rho_j}\left(\mathbf{v}_j - \mathbf{v}_i\right)\nabla^2 W(r_{ij})$$



Pressure force

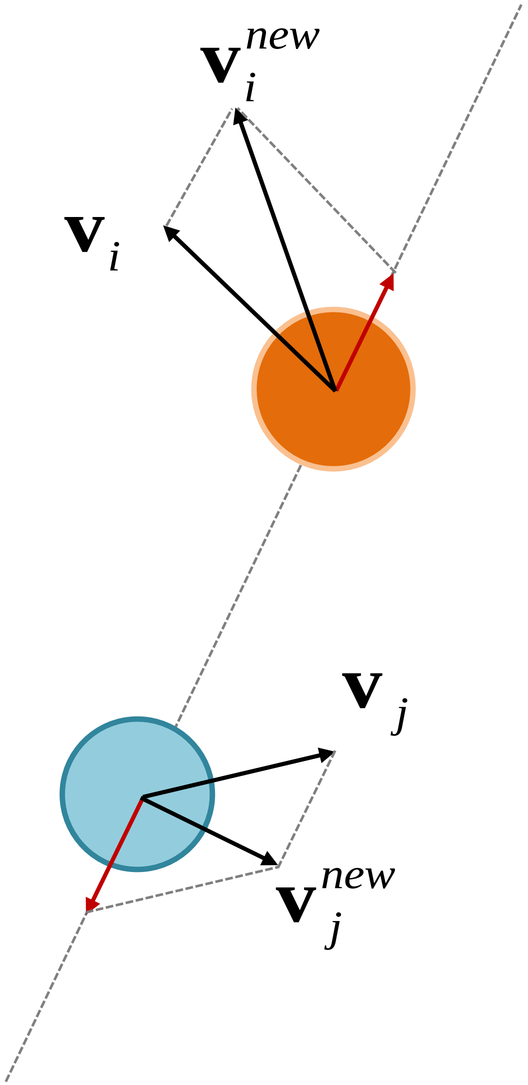
$$\mathbf{f}^{p} = -\nabla p$$

$$\langle \nabla A \rangle_i \approx \sum_j \frac{m_j}{\rho_j} A_j \nabla W(r_{ij})$$

$$\mathbf{f}^{p} = -\sum_j \frac{m_j}{\rho_j} p_j \nabla W\left(r_{ij}\right)$$

Momentum conservative

$$\mathbf{f}^{p} = -\sum_j \frac{m_j}{\rho_j} \left(p_i + p_j\right) \nabla W\left(r_{ij}\right)$$



How to calculate pressure

- Non-iterative state-equation based
  - Compressible [Müller03]
  - Weakly-compressible [Becker07]

$$p_i = c_s^2(\rho_i - \rho_0)$$

- Iterative state-equation based
  - PCISPH [Solenthaler09]
  - Local Poisson SPH [He12]
  - PBF [Macklin13]

$$\rho_i = \rho_0$$

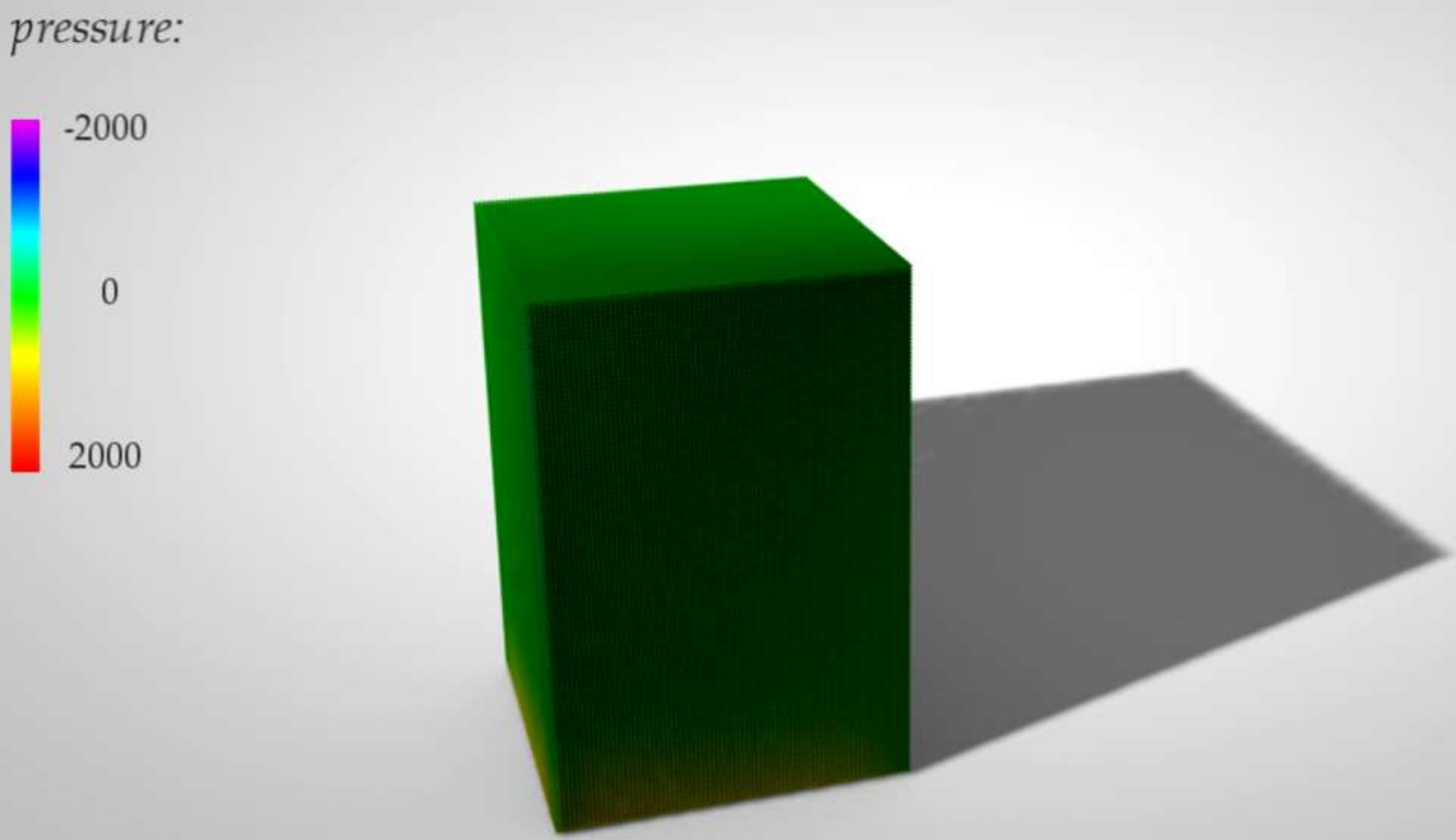
Review: [Ihmsen et al. 2012]

- Pressure projection
  - Divergence free [Cummins99]
  - Density invariant [Shao03]
  - Staggered SPH[He12]
  - IISPH [Ihmsen13]
  - Divergence-free[Bender15]

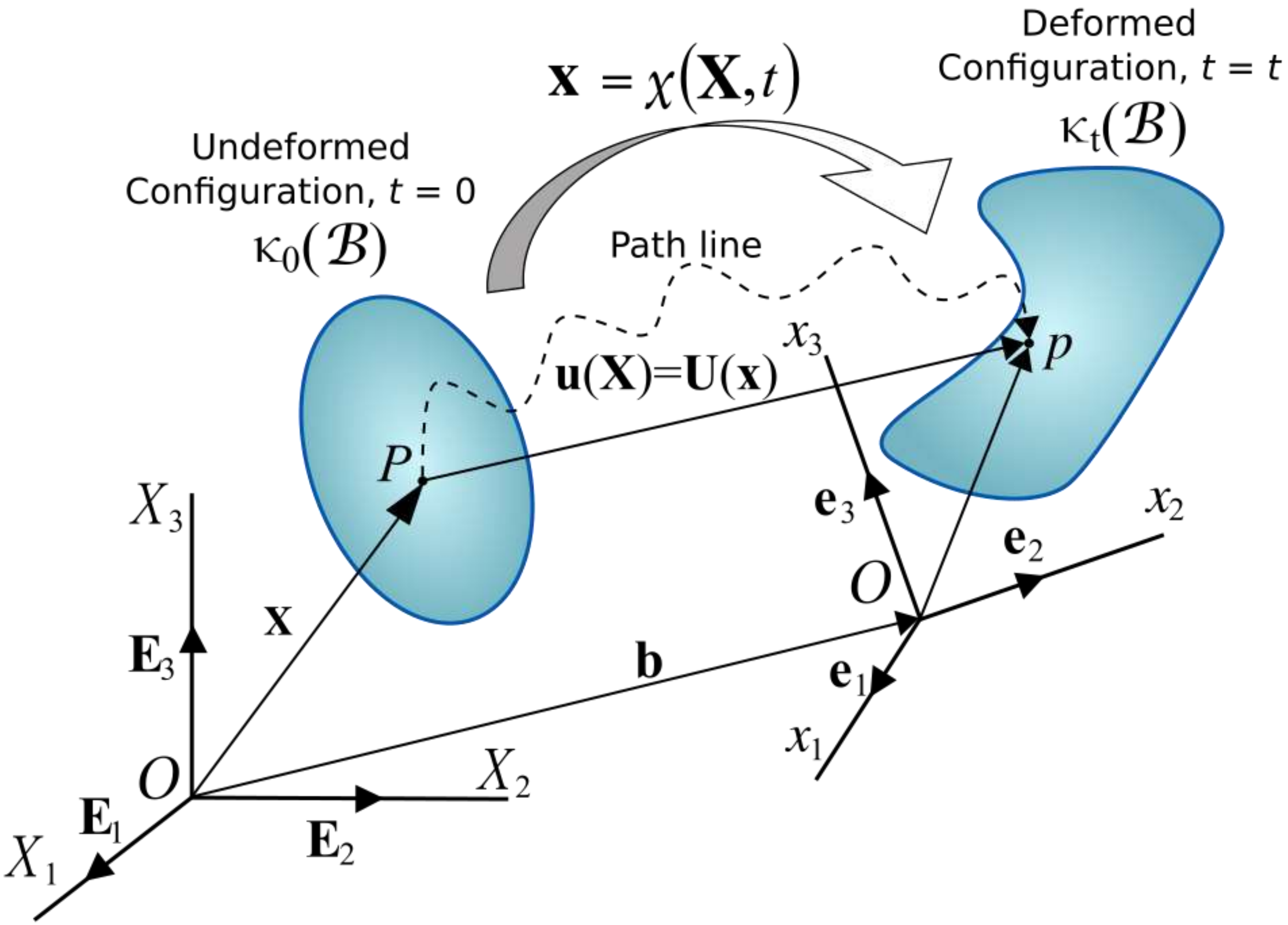
$$\nabla \cdot \left(\frac{\nabla p}{\rho} \right) = \frac{\nabla \cdot \mathbf{v}^*}{\Delta t} \Big/ \frac{\rho_0 - \rho}{\rho_0 \Delta t^2}$$

Review : [Gotoh et al. 2016]

$\kappa = 0$

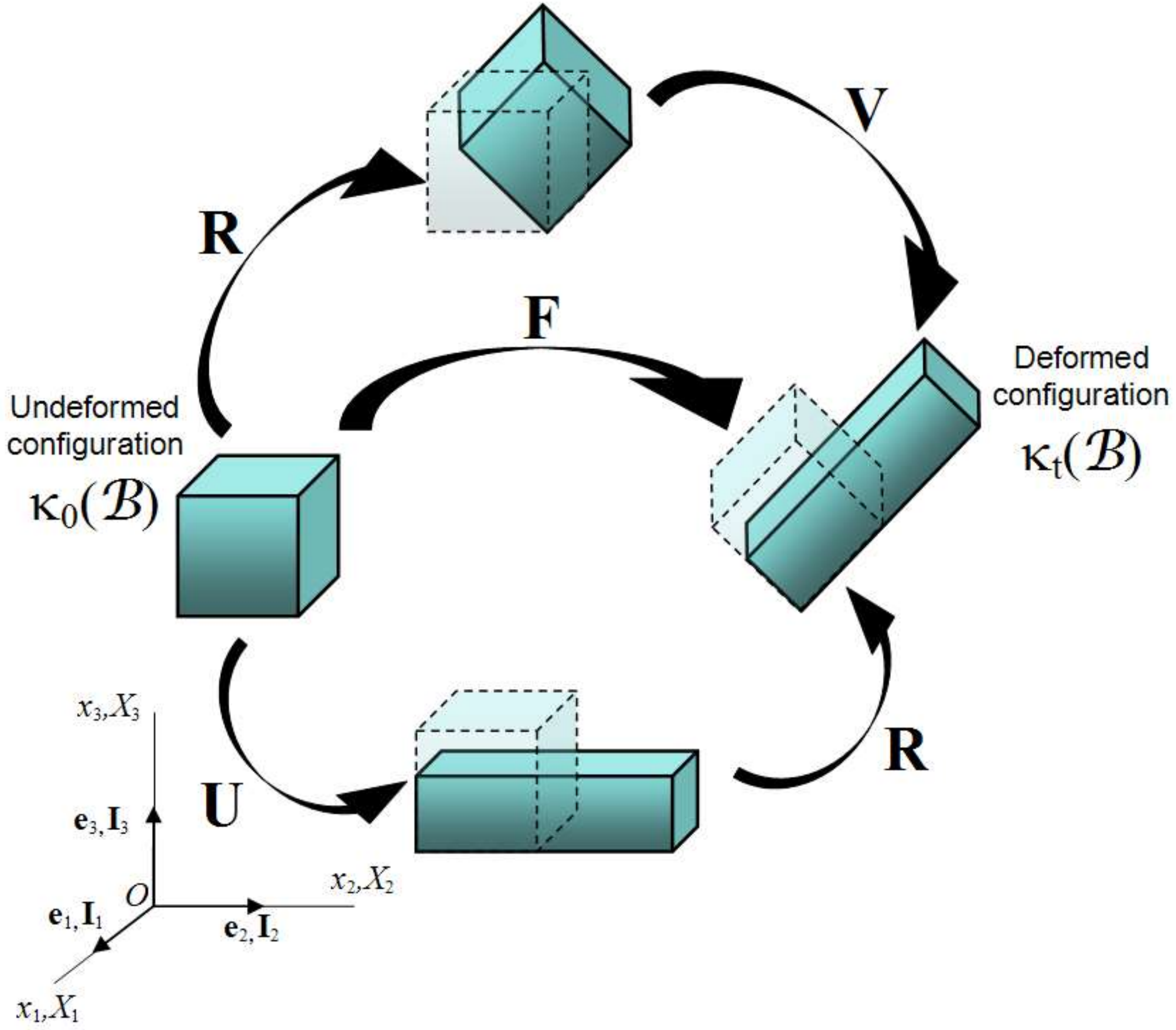


Elastic Material



$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$$

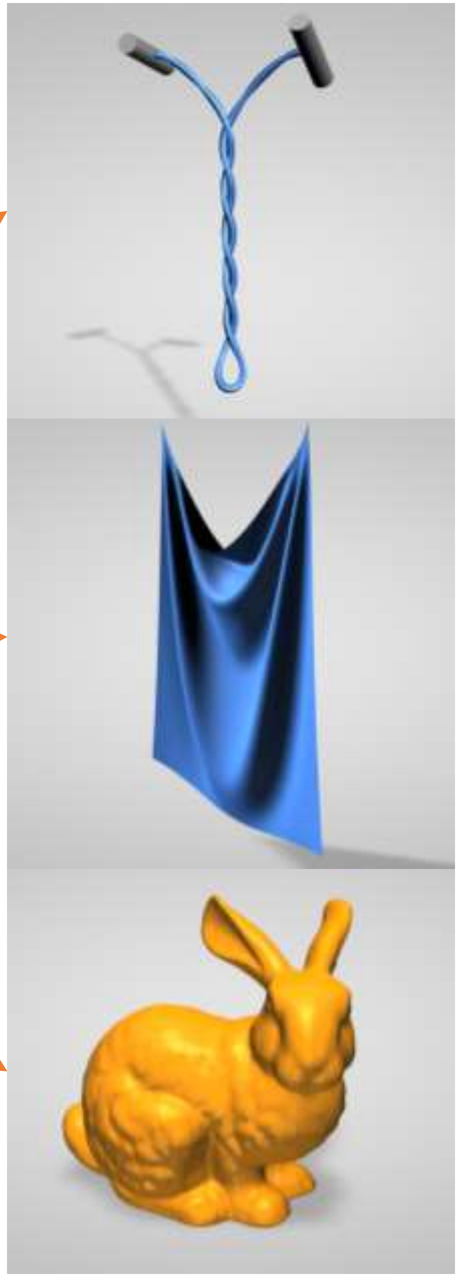
Elastic Material



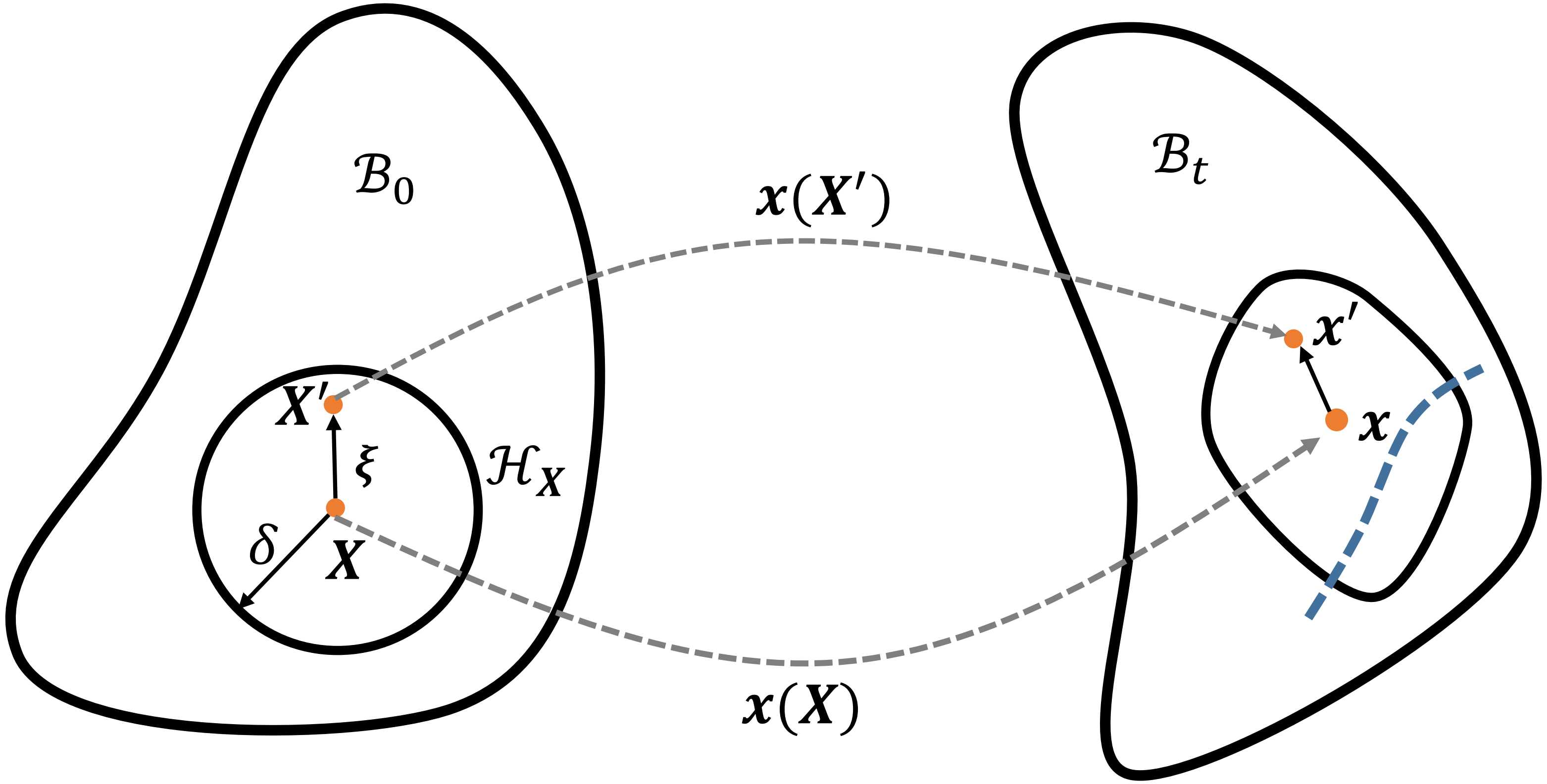
$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}$$

Problem: Diverse Dimensions

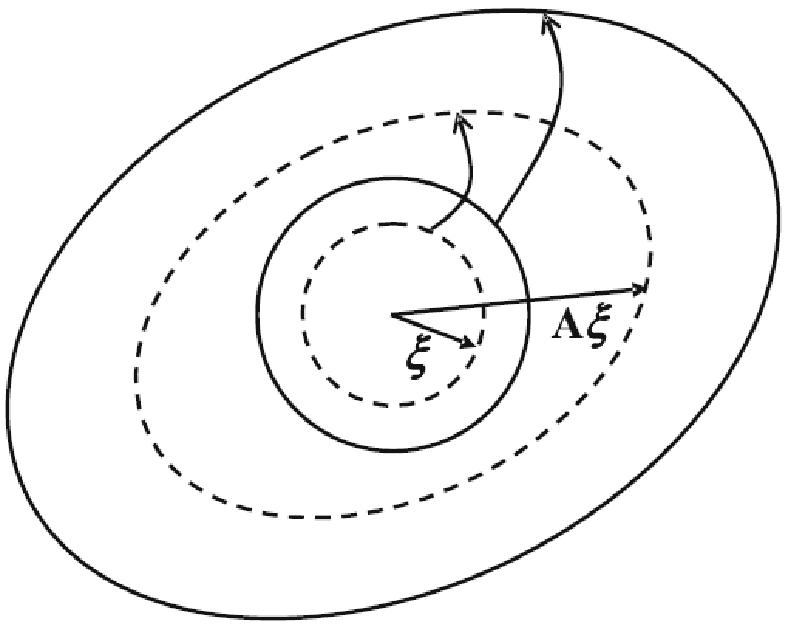
$$\mathbf{F}=\begin{bmatrix} 1+\frac{\partial u_1}{\partial X} & \frac{\partial u_1}{\partial Y} & \frac{\partial u_1}{\partial Z} \\ \frac{\partial u_2}{\partial X} & 1+\frac{\partial u_2}{\partial Y} & \frac{\partial u_2}{\partial Z} \\ \frac{\partial u_3}{\partial X} & \frac{\partial u_3}{\partial Y} & 1+\frac{\partial u_3}{\partial Z} \end{bmatrix}$$



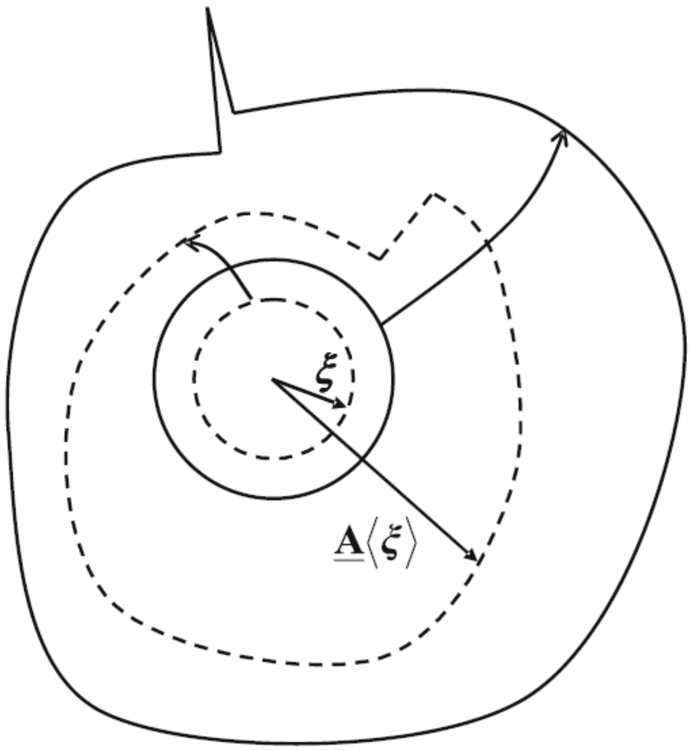
Problem: Discontinuity



Peridynamics vs Classical Model



Classical Model



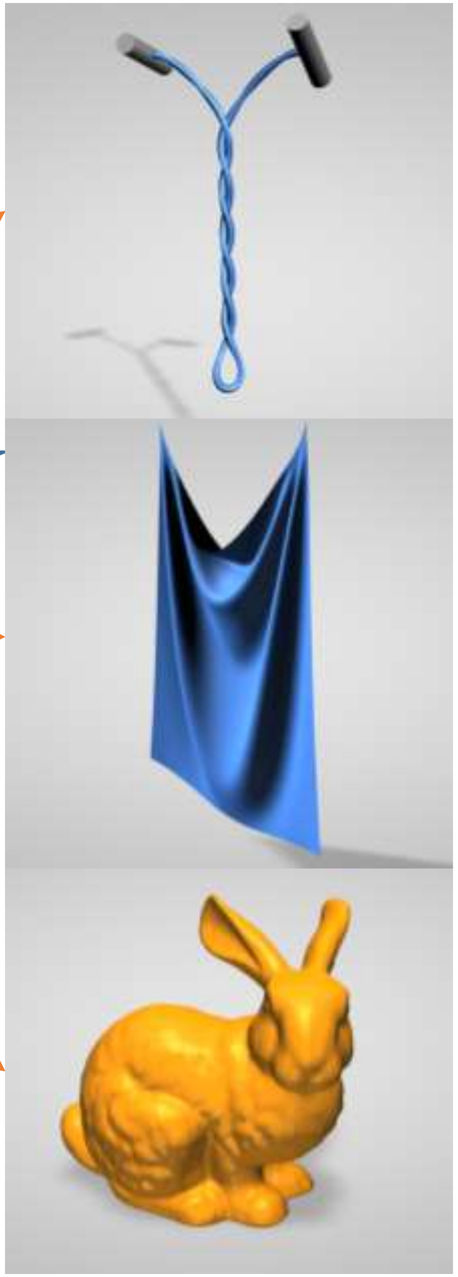
Peridynamics

Previous Elastic Model

$$\Psi = \mu \|\mathbf{F}_E - \mathbf{R}_E\|_F^2 + \frac{\lambda}{2}(J_E - 1)^2$$

$$\mathbf{F} = \begin{bmatrix} 1+\frac{\partial u_1}{\partial X} & \frac{\partial u_1}{\partial Y} & \frac{\partial u_1}{\partial Z} \\ \frac{\partial u_2}{\partial X} & 1+\frac{\partial u_2}{\partial Y} & \frac{\partial u_2}{\partial Z} \\ \frac{\partial u_3}{\partial X} & \frac{\partial u_3}{\partial Y} & 1+\frac{\partial u_3}{\partial Z} \end{bmatrix}$$

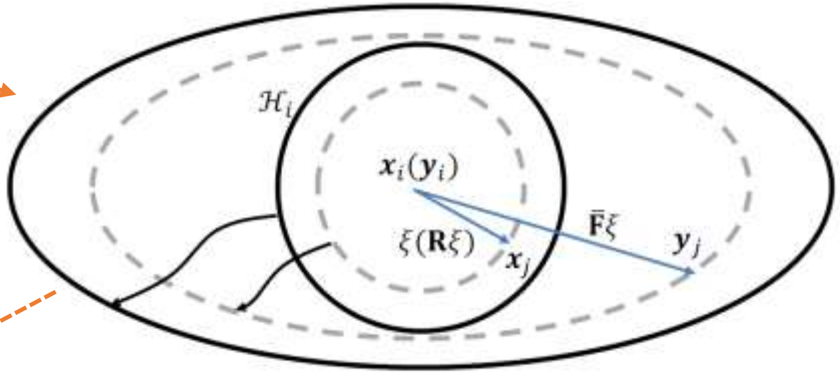
Singular



A. Stomakhin, C. Schroeder, L. Chai, J. Teran, and A. Selle, "A material point method for snow simulation," ACM Trans. Graph. (SIGGRAPH), vol. 32, no. 4, p. 102, 2013.

Our Elastic Model

$$\Psi = \mu \left\| \mathbf{F}_E - \mathbf{R}_E \right\|_F^2 + \frac{\lambda}{2}(J_E - 1)^2$$

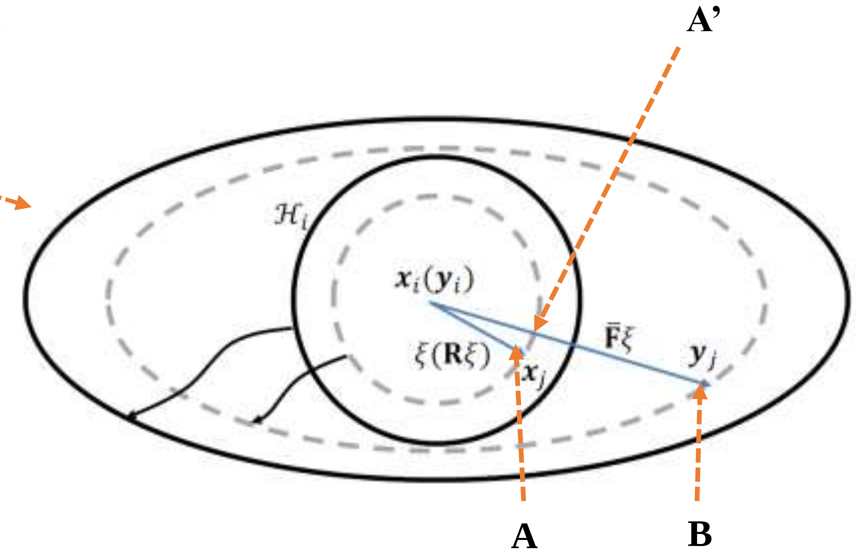


$$\Psi^{dev} = \int_{\mathcal{H}} \underline{w} \langle \xi \rangle \left(\mu \left\| \frac{\bar{\mathbf{F}}\xi - \mathbf{R}\xi}{|\underline{\mathbf{X}}|} \right\|^2 \right) d\xi$$

$$\int_{\mathcal{H}} \underline{w} \langle \xi \rangle \, d\xi = 1$$

Our Elastic Model

$$\Psi = \mu \left\| \mathbf{F}_E - \mathbf{R}_E \right\|_F^2 + \frac{\lambda}{2}(J_E - 1)^2$$



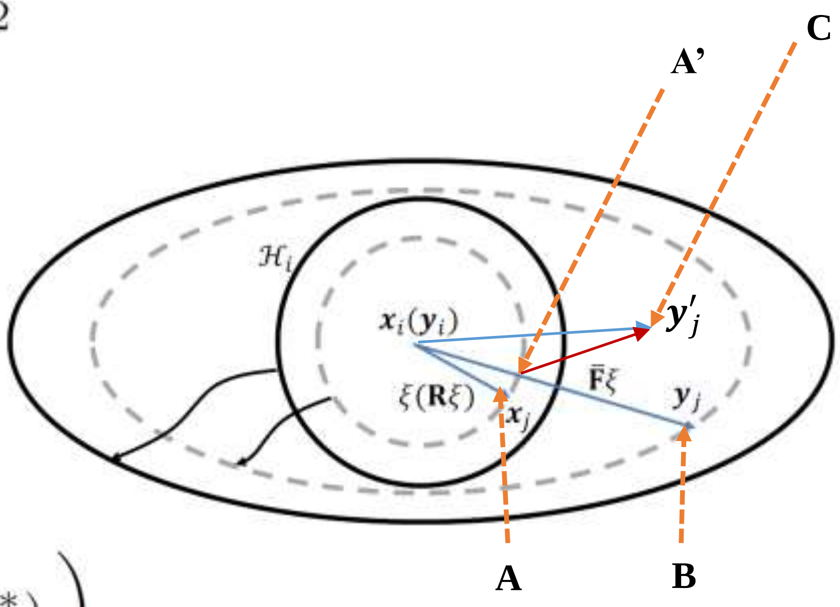
$$\Psi^{dev} = \int_{\mathcal{H}} \underline{w} \langle \xi \rangle \left(\mu \left\| \frac{\bar{\mathbf{F}}\xi - \mathbf{R}\xi}{|\underline{\mathbf{X}}|} \right\|^2 \right) d\xi$$

$$|AB| \approx |A'B| = \left| \left\| \bar{\mathbf{F}}\xi \right\| - \left\| \mathbf{R}\xi \right\| \right|$$

Our Elastic Model

$$\Psi = \mu \|\mathbf{F}_E - \mathbf{R}_E\|_F^2 + \frac{\lambda}{2}(J_E - 1)^2$$

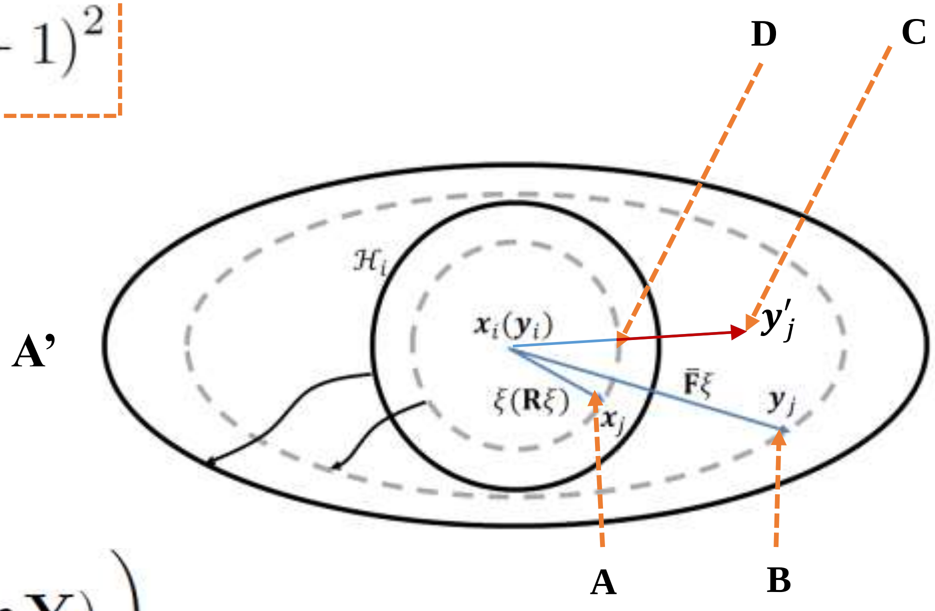
$$\frac{\partial \Psi^{dev}}{\partial \boldsymbol{y}_j} = \underline{\mathbf{T}}^{\mathrm{dev}} \approx \frac{2\mu w}{|\underline{\mathbf{X}}|^2}\left(\underline{\mathbf{Y}} - |\underline{\mathbf{X}}|\,(\mathrm{dir}\,\underline{\mathbf{Y}}^*)\right)$$



Our Elastic Model

$$\Psi = \mu \|\mathbf{F}_E - \mathbf{R}_E\|_F^2 + \frac{\lambda}{2}(J_E - 1)^2$$

$$\frac{\partial \Psi^{iso}}{\partial \boldsymbol{y}_j} = \underline{\mathbf{T}}^{\mathtt{iso}} = \frac{\lambda w}{|\underline{\mathbf{X}}|^2}\left(\underline{\mathbf{Y}} - |\underline{\mathbf{X}}|\,(\mathtt{dir}\,\underline{\mathbf{Y}})\right)$$



Projective Solver

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \nabla \cdot \sigma + g$$

$$\rho_i \ddot{\mathbf{y}}_i = \int_{\mathcal{H}_i} \left\{ \underline{\mathbf{T}}_i \langle \boldsymbol{\xi} \rangle - \underline{\mathbf{T}}_j \langle -\boldsymbol{\xi} \rangle \right\} dV_{\mathbf{x}_j} + \mathbf{b}_i,$$

$$\left\{ \begin{array}{l} \underline{\mathbf{T}}^{\mathrm{dev}} \approx \dfrac{2\mu w}{|\underline{\mathbf{X}}|^2} \left(\underline{\mathbf{Y}} - |\underline{\mathbf{X}}| \left(\mathtt{dir}\, \underline{\mathbf{Y}}^* \right) \right) \\ \underline{\mathbf{T}}^{\mathrm{iso}} = \dfrac{\lambda w}{|\underline{\mathbf{X}}|^2} \left(\underline{\mathbf{Y}} - |\underline{\mathbf{X}}| \left(\mathtt{dir}\, \underline{\mathbf{Y}} \right) \right) \end{array} \right.$$

Nonlinear

Linear

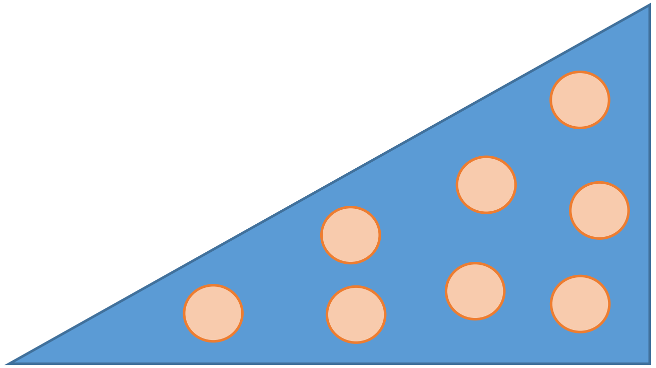
Rod

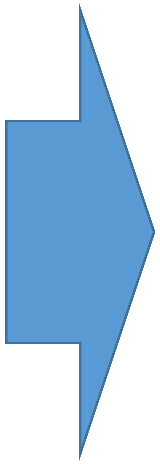
Cloth

Mass-Spring System　　Shape Matching　　Our Method

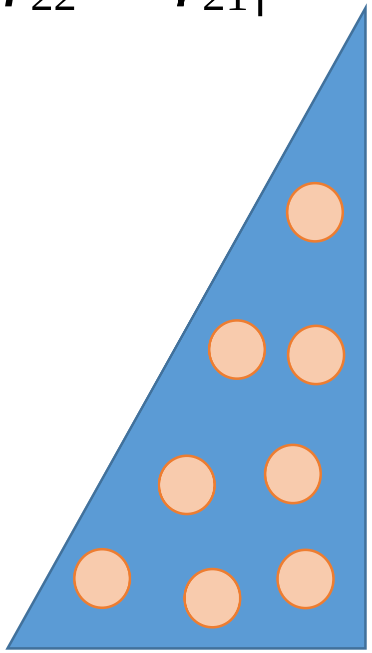
Recovering from Shape Inversion

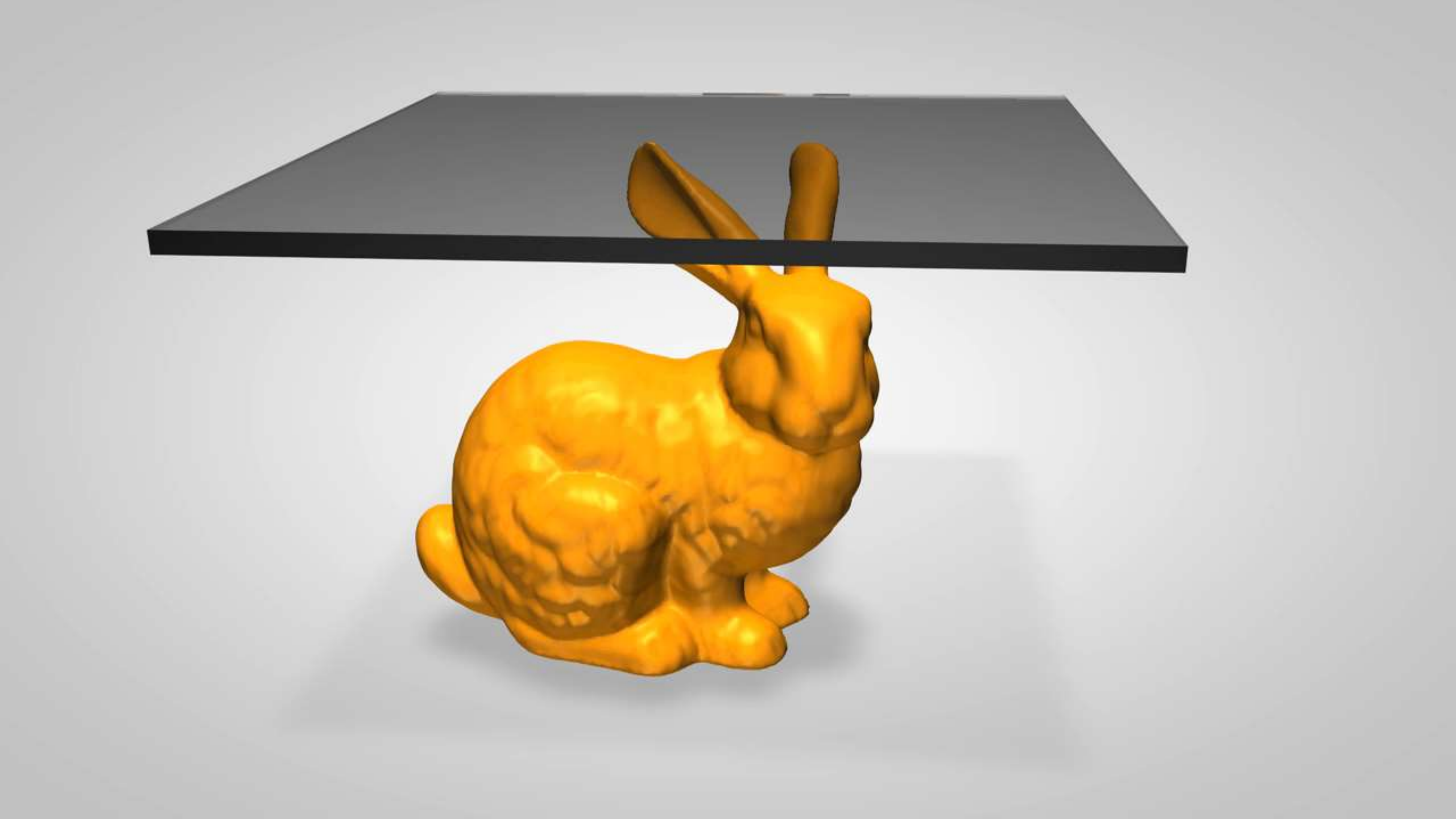
$$\left|\overline{\mathbf{F}}\right| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix} < 0$$





$$\left|\overline{\mathbf{F}}\right| = \begin{vmatrix} f_{12} & f_{11} \\ f_{22} & f_{21} \end{vmatrix} > 0$$





Multiplicative Plastic Model

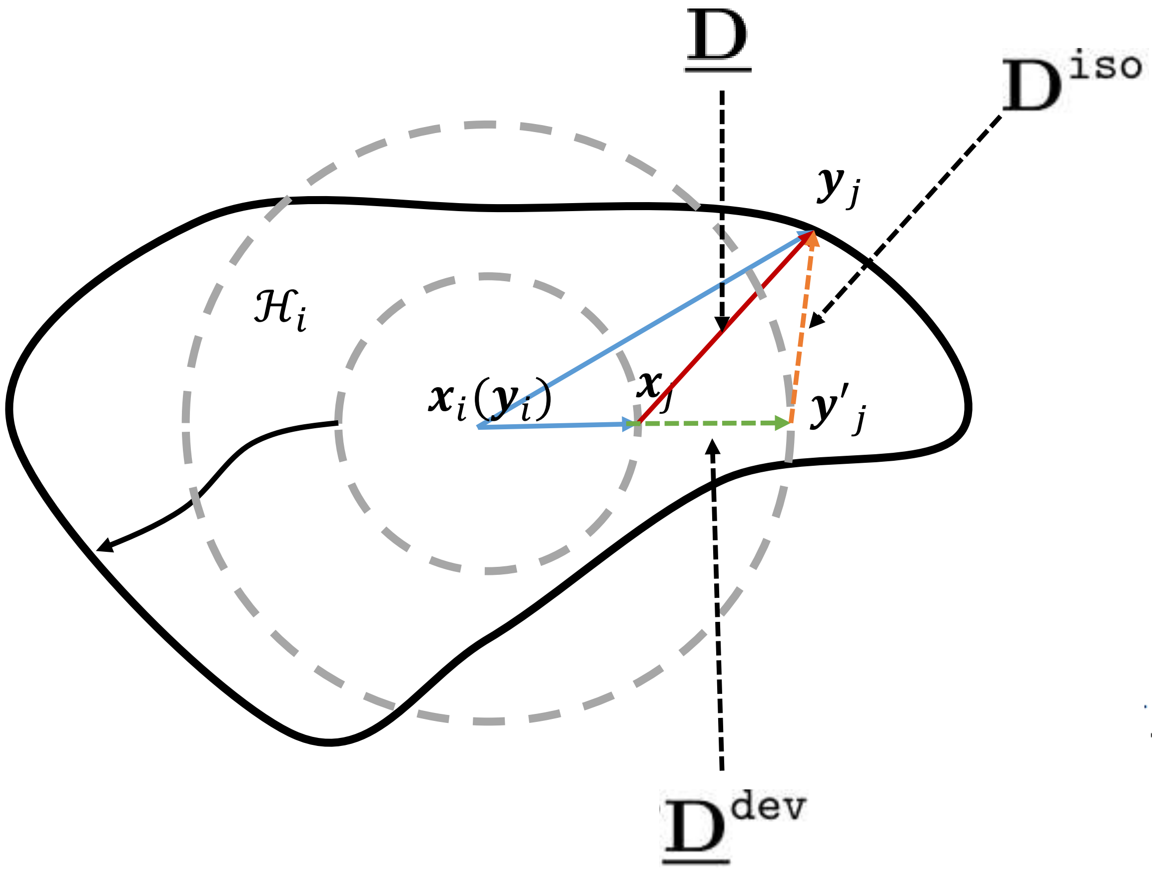
Singular →

$$\boldsymbol{F} = \boldsymbol{F}_e \boldsymbol{F}_p$$

Elastic

Plastic

Our Plastic Model



$$\underline{\mathbf{D}} = \underline{\mathbf{Y}} - \bar{\mathbf{R}}\underline{\mathbf{X}}$$

$$\underline{\mathbf{D}}^{\mathtt{dev}} + \mathbf{D}^{\mathtt{iso}}$$

$$\underline{\mathbf{D}}_{\mathrm{p}} = \alpha \mathbf{D}^{\mathtt{iso}} + \beta \underline{\mathbf{D}}^{\mathtt{dev}}$$

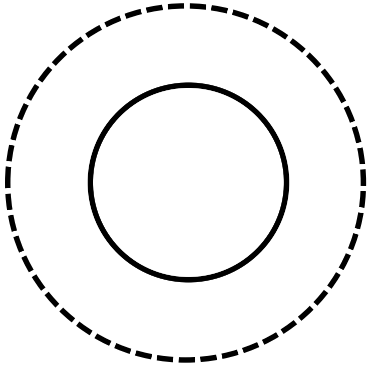
Drucker-Prager Yielding Condition

$$\sqrt{J_2} \leq A + BI_1$$

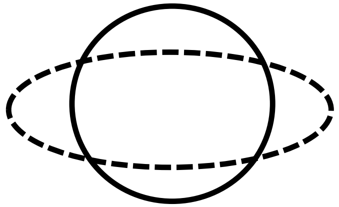
First Invariant: $I_1 = -\lambda e^{\mathtt{iso}}$

Second invariant: $J_2 = \mu \int_{\mathcal{H}} \underline{e}^{\mathtt{dev}} \langle \xi \rangle \, \underline{e}^{\mathtt{dev}} \langle \xi \rangle \, d\xi$

$$\{\alpha, \beta\} = \begin{cases} \left\{0, \frac{\sqrt{J_2} - A - BI_1}{\sqrt{J_2}}\right\}, & \text{if } A + BI_1 > 0 \\ \left\{\frac{A+BI_1}{BI_1}, 1\right\}, & \text{otherwise,} \end{cases}$$



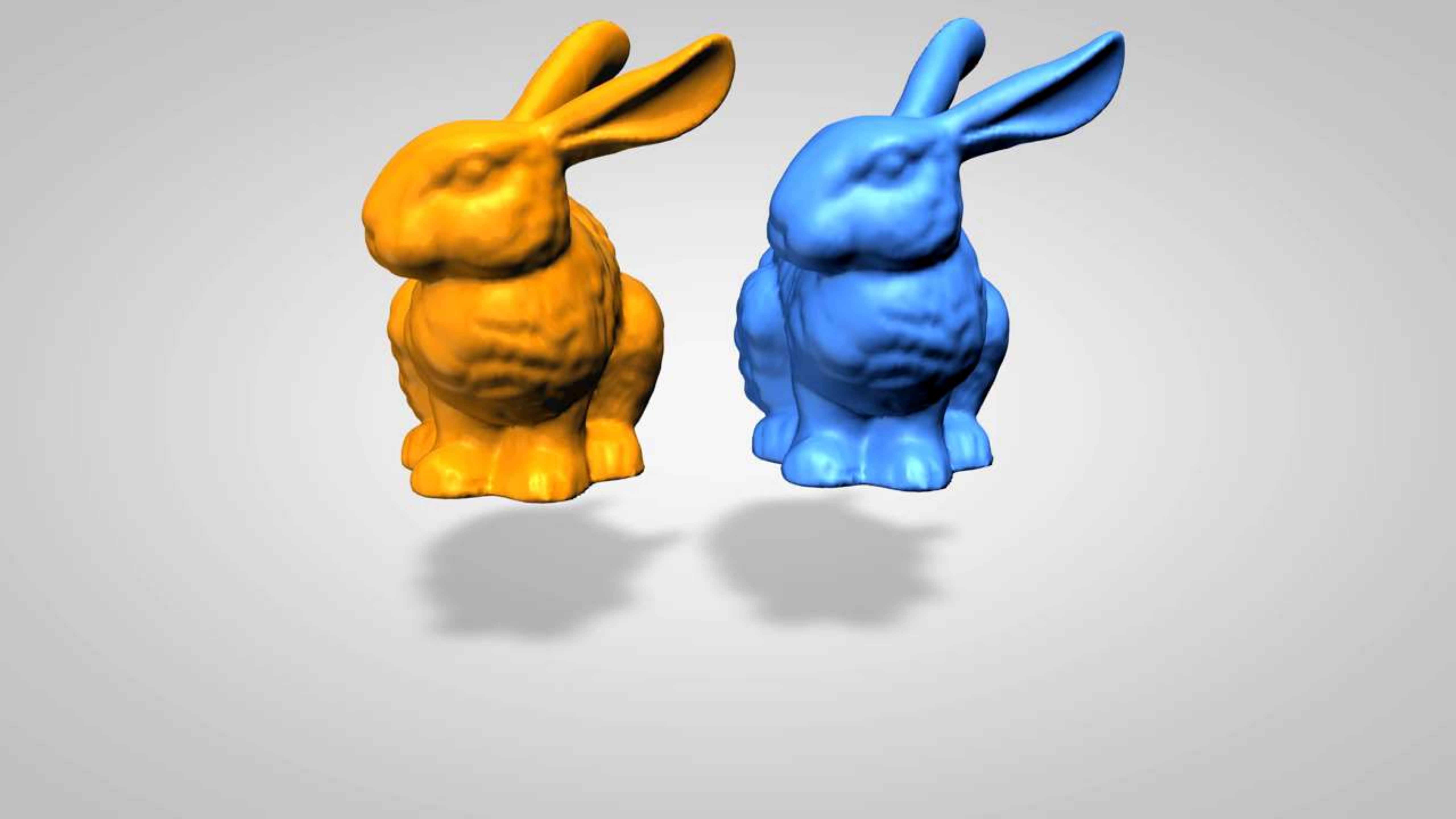
I_1



J_2

Case 1: Elastoplastic Material

$$A > 0 \qquad B > 0$$



Case 2: Homogeneous Viscoelastic Fluids

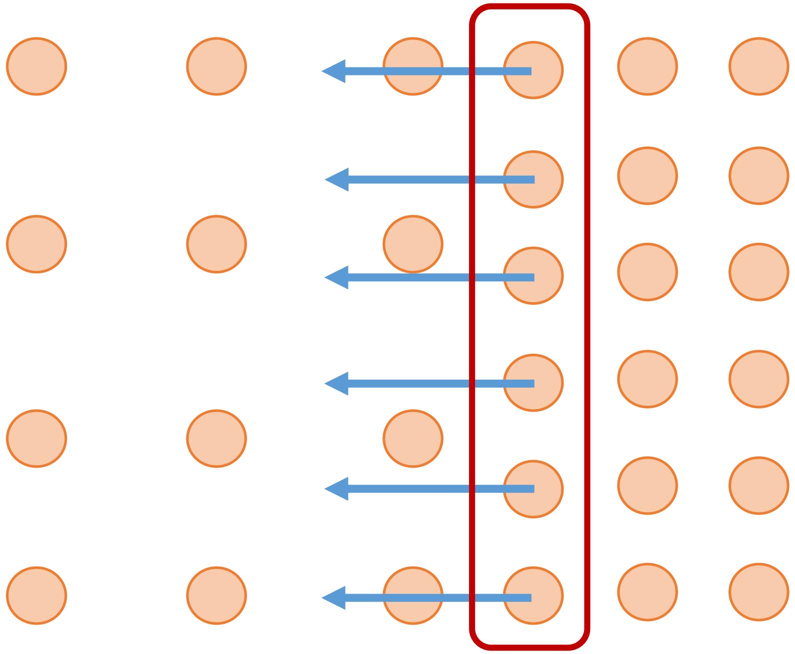
$$A > 0 \quad B = 0$$

Case 2: Viscoelastic Fluids

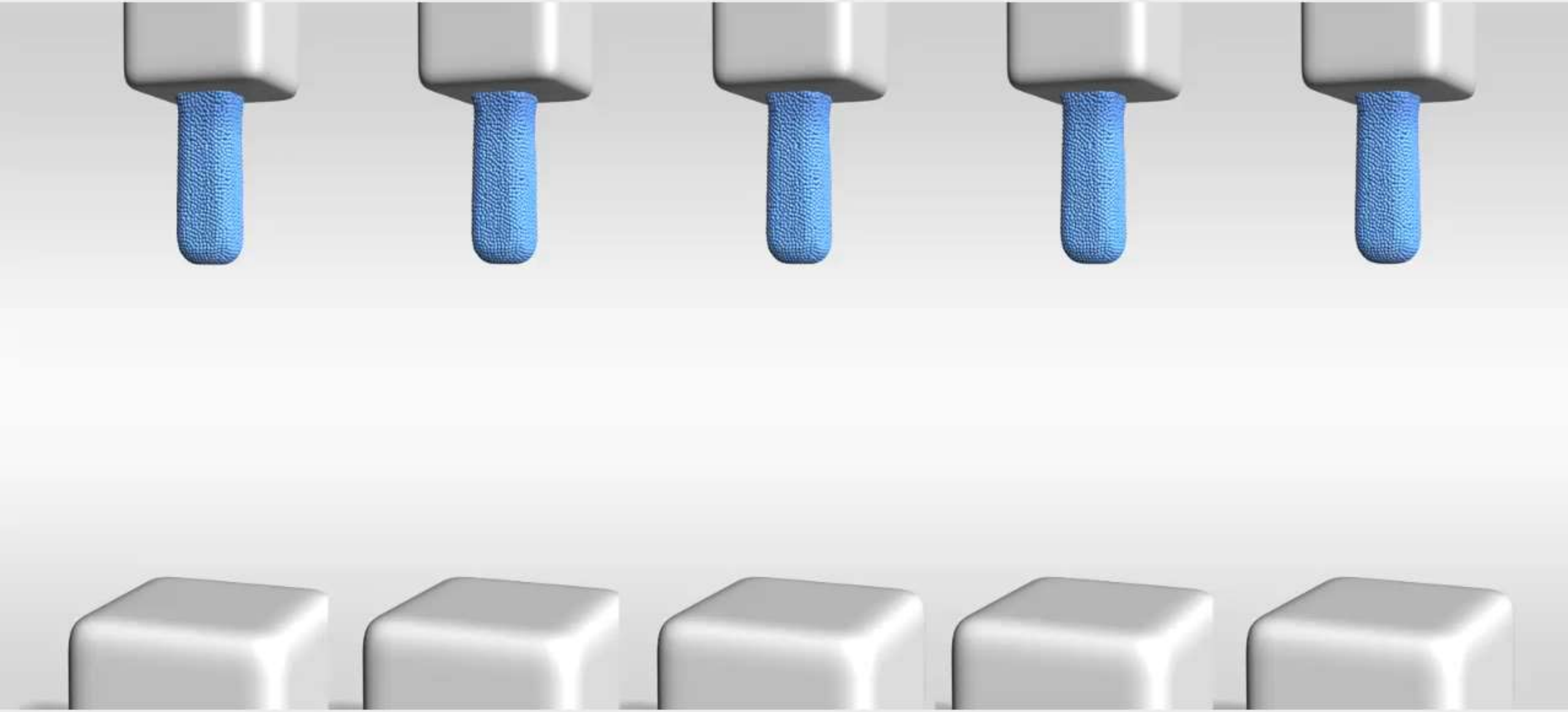
$$C_i = \|\nabla_i \rho\|^2 = 0$$

$$\tau_i = \frac{C_i}{\sum_l \|\nabla_l C_i\|^2 + \varepsilon},$$

$$\mathbf{y}_i \leftarrow \mathbf{y}_i + \delta \sum_j \frac{\tau_i + \tau_j}{2} \nabla_i W_{ij},$$



The Effect of Particle Redistribution Strength



0 0.25 0.5 0.75 1.0

Case 3: Granular Materials

Wet: $A > 0 \quad B > 0$

Dry: $A = 0 \quad B > 0$

Case 3: Granular Materials

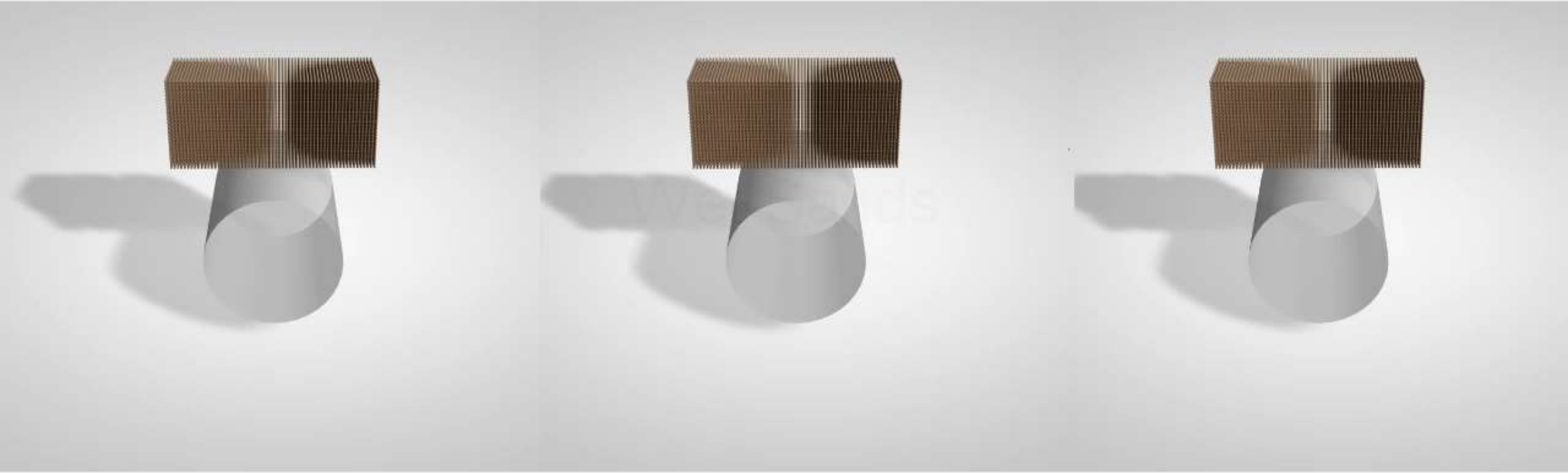
$$\underline{\mathbf{T}}^{\mathtt{dev}} \approx \frac{2\mu w}{|\underline{\mathbf{X}}|^2}\left(\underline{\mathbf{Y}} - |\underline{\mathbf{X}}|\left(\mathtt{dir}\,\underline{\mathbf{Y}}^*\right)\right)$$

$$\underline{\mathbf{T}}^{\mathtt{iso}} = \frac{\lambda w}{|\underline{\mathbf{X}}|^2}\left(\underline{\mathbf{Y}} - |\underline{\mathbf{X}}|\left(\mathtt{dir}\,\underline{\mathbf{Y}}\right)\right)$$

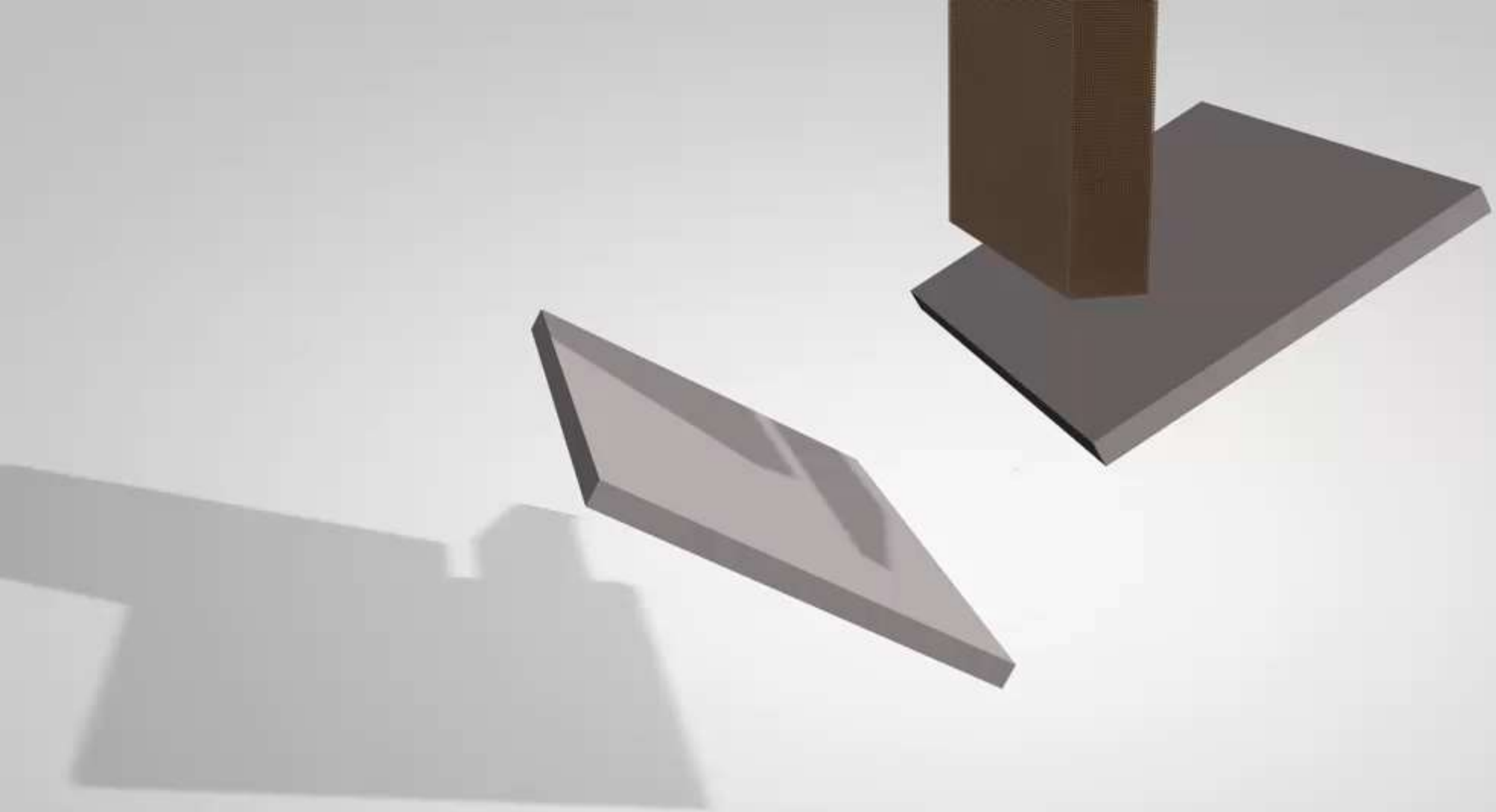
$$k_{\mathtt{f}}(\rho) = \begin{cases} K_{\mathtt{f}} e^{H_{\mathtt{f}}(\rho/\rho_0 - 1)}, & \text{if } \rho \in [\rho_0, +\infty), \\ 0, & \text{otherwise,} \end{cases}$$

$$k_{\mathtt{c}}(\rho) = \begin{cases} K_{\mathtt{c}}, & \text{if } \rho \in [\rho_0, +\infty), \\ K_{\mathtt{c}} e^{H_{\mathtt{c}}(1 - \rho/\rho_0)}, & \text{if } \rho \in [\rho_1, \rho_0), \\ 0, & \text{otherwise.} \end{cases}$$

The Effect of Water Saturation



0.01 | 0.05 | 0.1

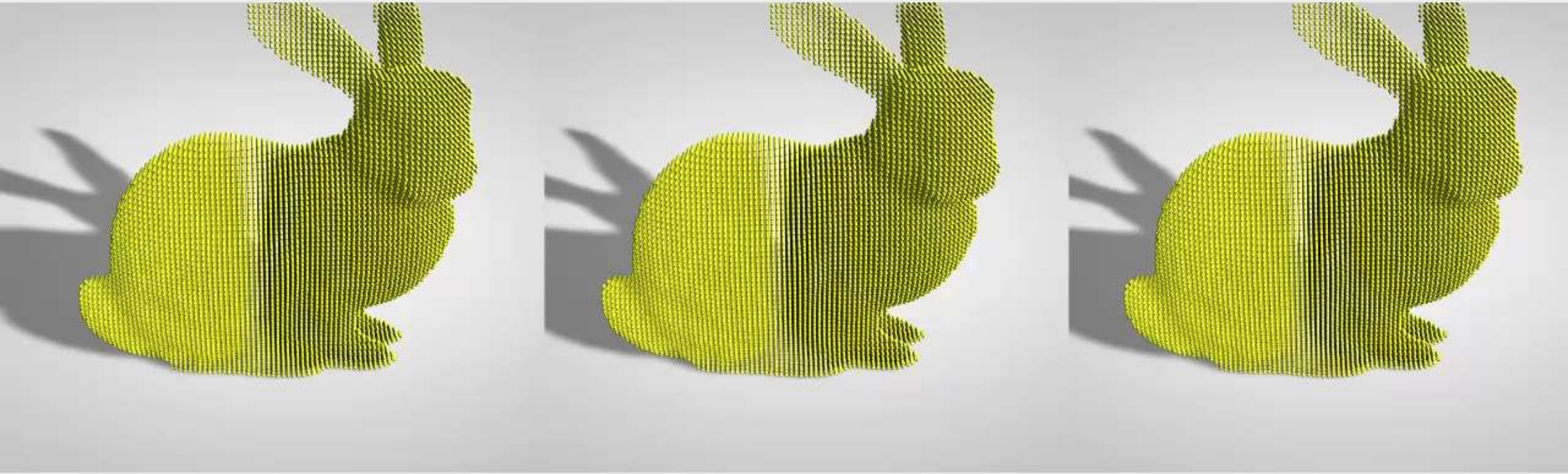


A Large Scale Simulation of Wet Sands

Water Saturation = 0.01

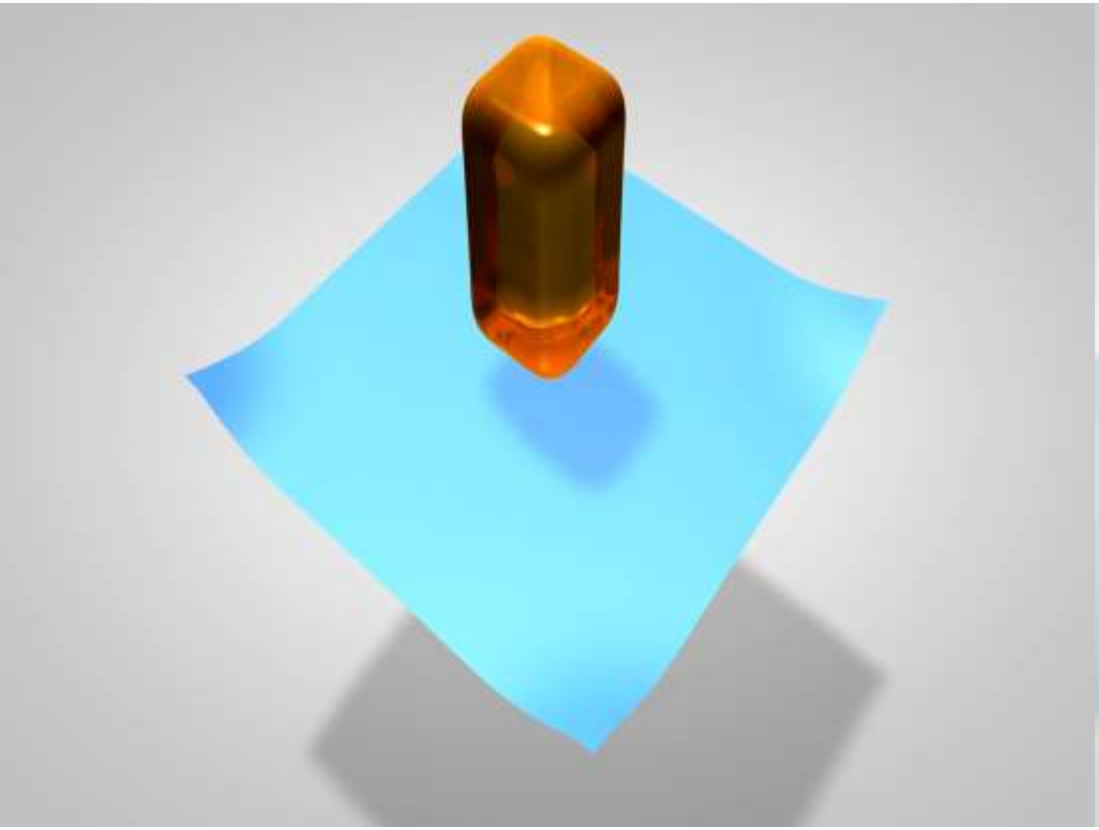
Friction Angle = 15

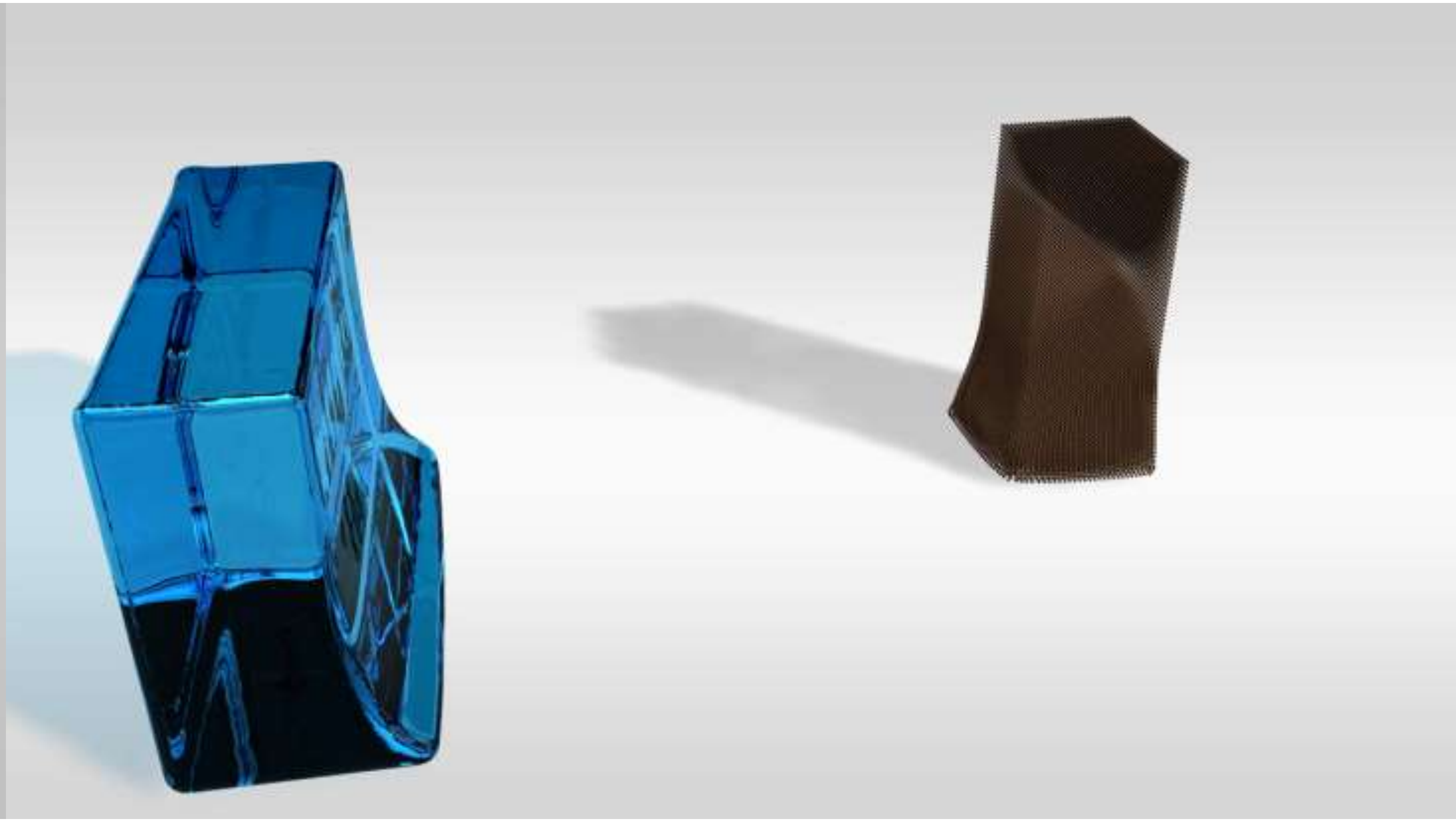
The Effect of Friction Angle



15 30 45

More Examples

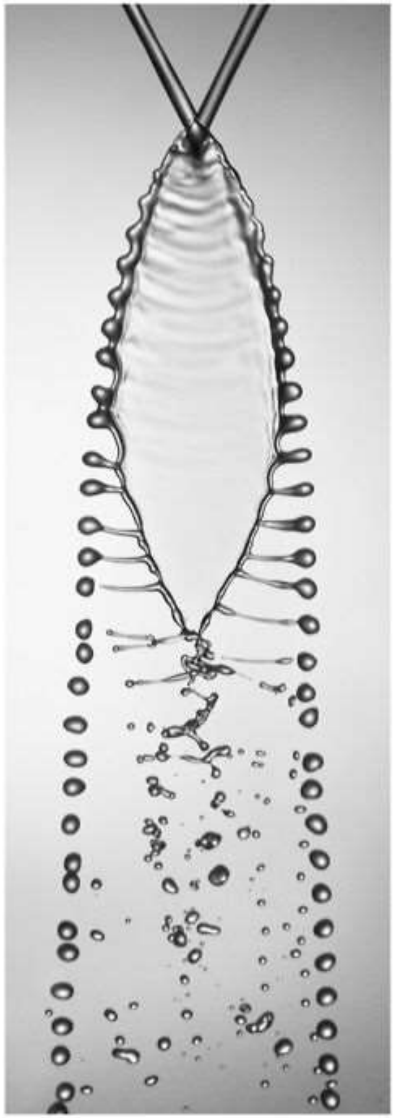




Challenges

- The particle deficiency problem
- Tensile instability
- Accuracy

Challenges



Thank you for your attention!

http://peridynamics.com/

https://github.com/PhysikaTeam/Physika