Decoupling Simulation Accuracy from Mesh Quality

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[1]New York University

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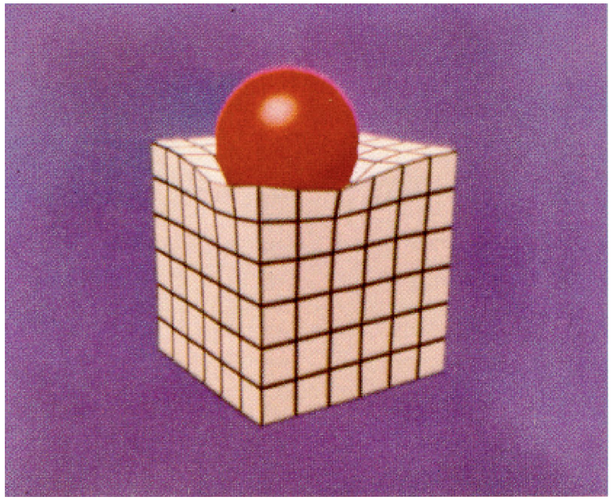
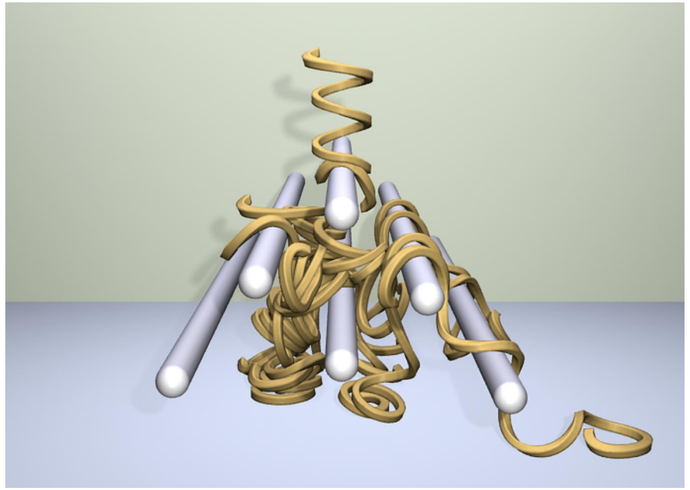
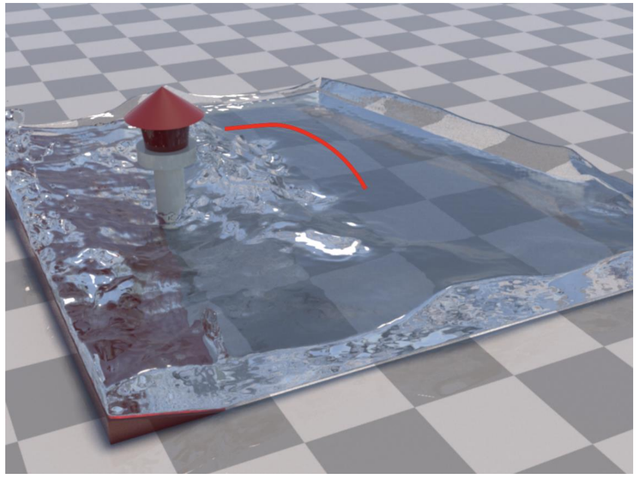
Simulating Physics



Modeling [Terzopoulos et al. 1987]

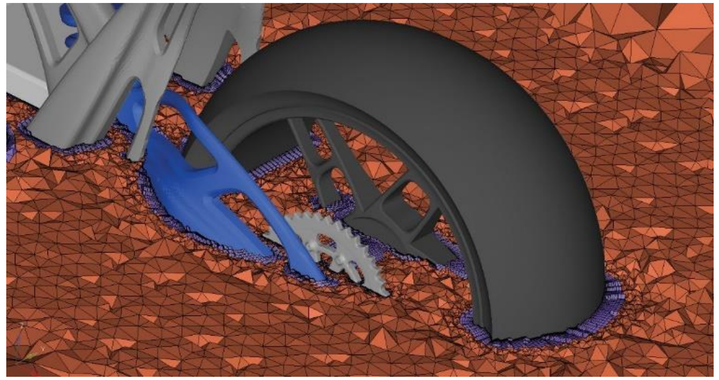


Dedormation [Bargteil et al. 2014]

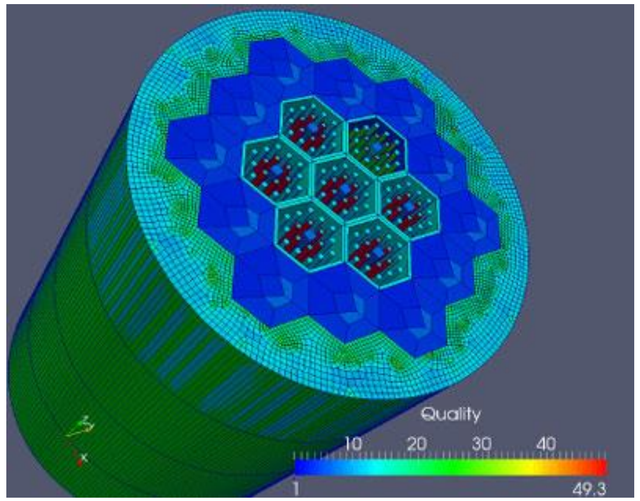


Fluid dynamics [Pan et al. 2013]

Topology Optimization [Zhu et al. 2017]



Aerodynamics



Engineering

Partial Differential Equation (PDE)

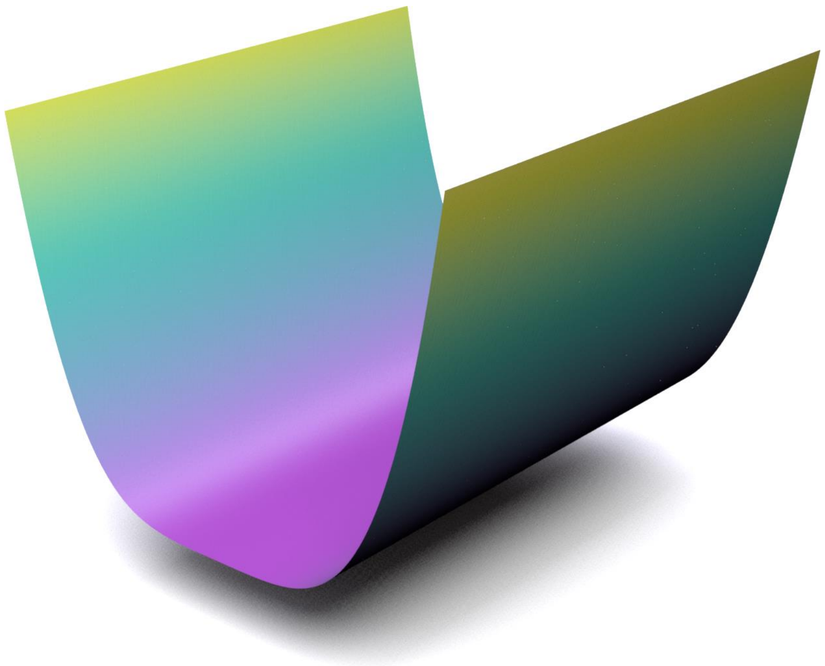
Partial Differential Equation (PDE)

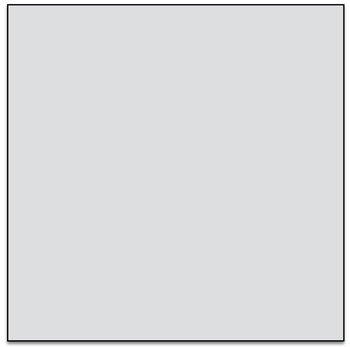
$$\Delta u = f, \quad f = 12x^2$$

PDE Solving

$$\Delta u = 12x^2$$

$$u = x^4$$

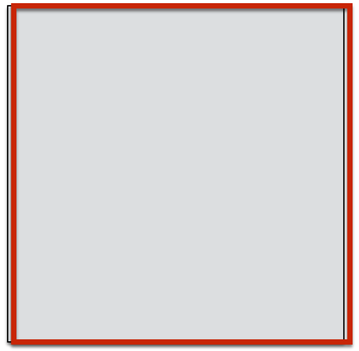




Finite Element Method (FEM)

$$\Delta u = f$$

$$u = \cancel{x^4} \quad ?$$



Finite Element Method (FEM)

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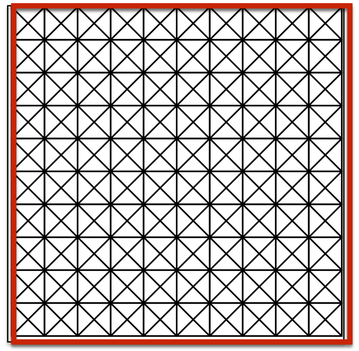
$$U = \sum_{i=1}^{n} u_i \phi_i$$



Finite Element Method (FEM)

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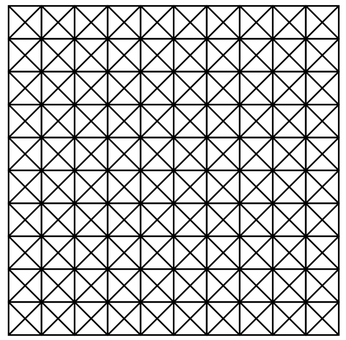
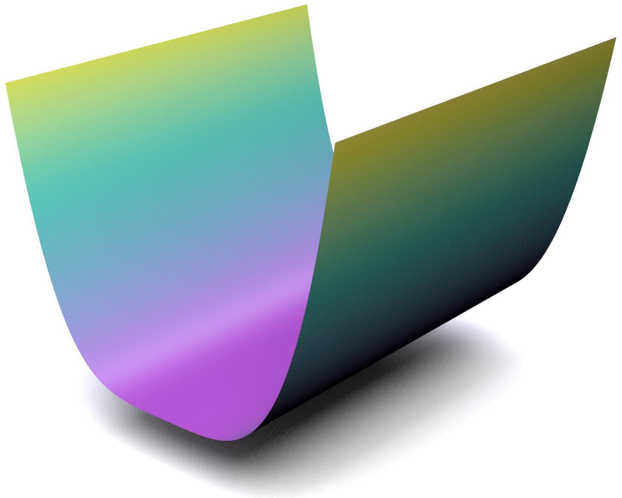
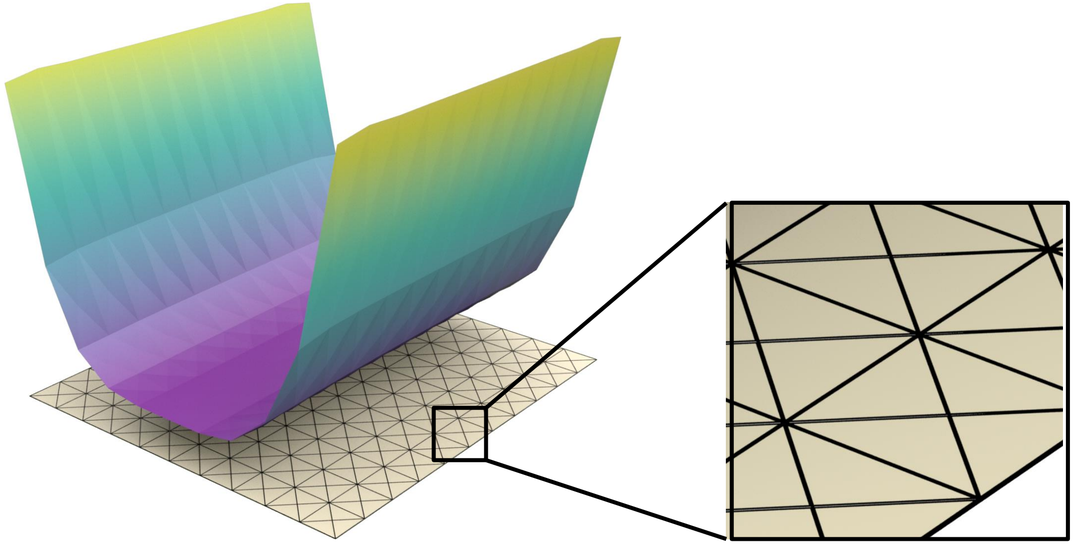
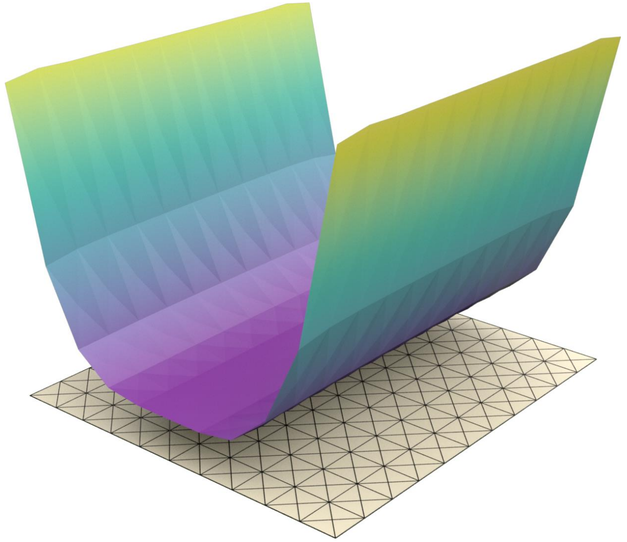




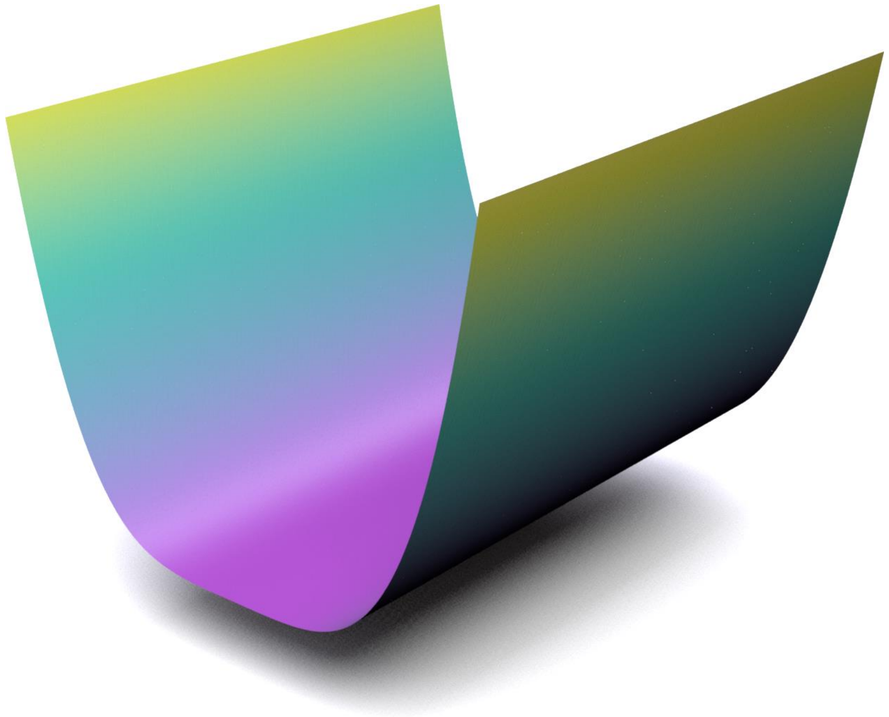


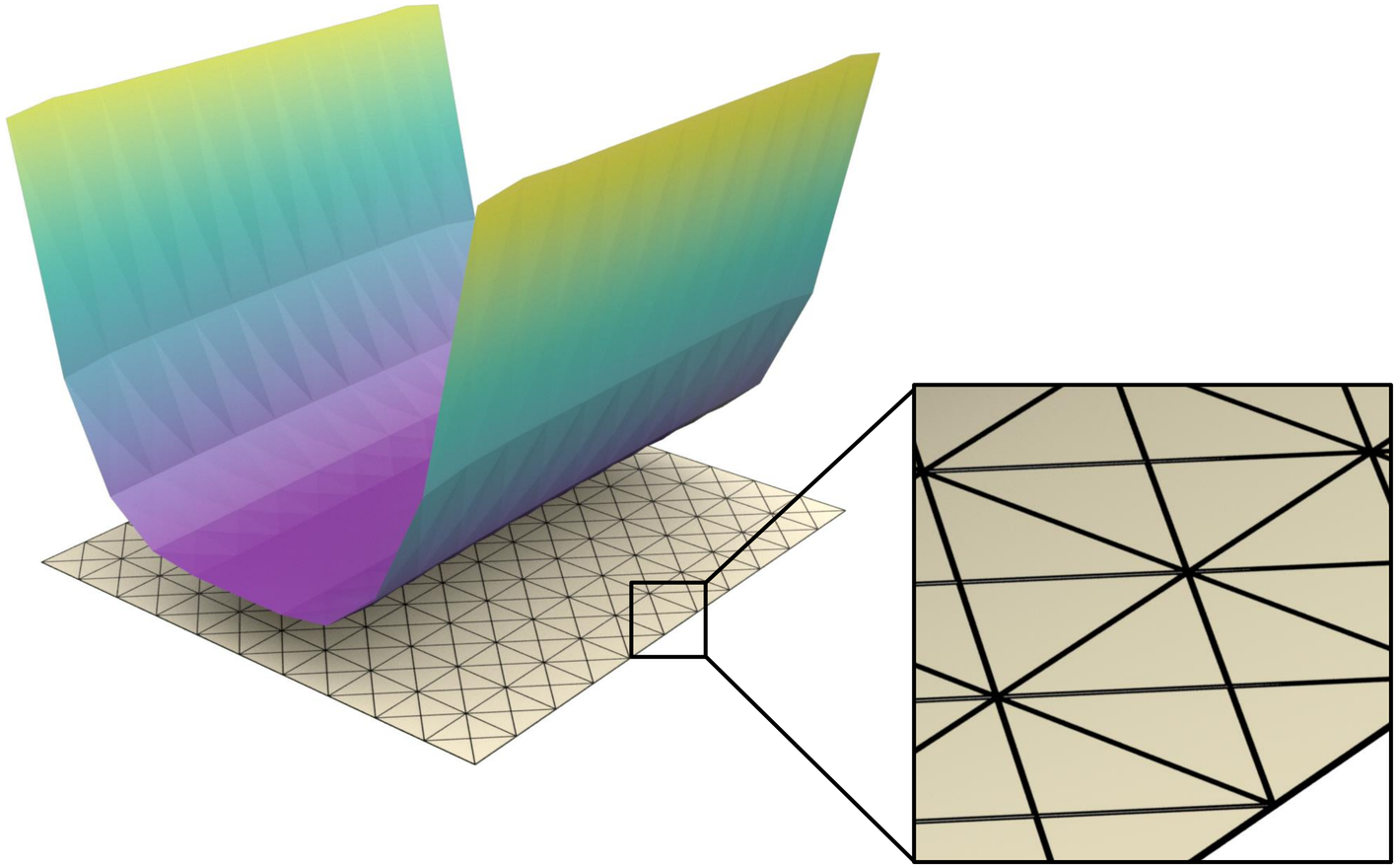
Three Factors Affect FEM Accuracy

- Mesh resolution
- Basis order
- Element quality



Quality Matters???





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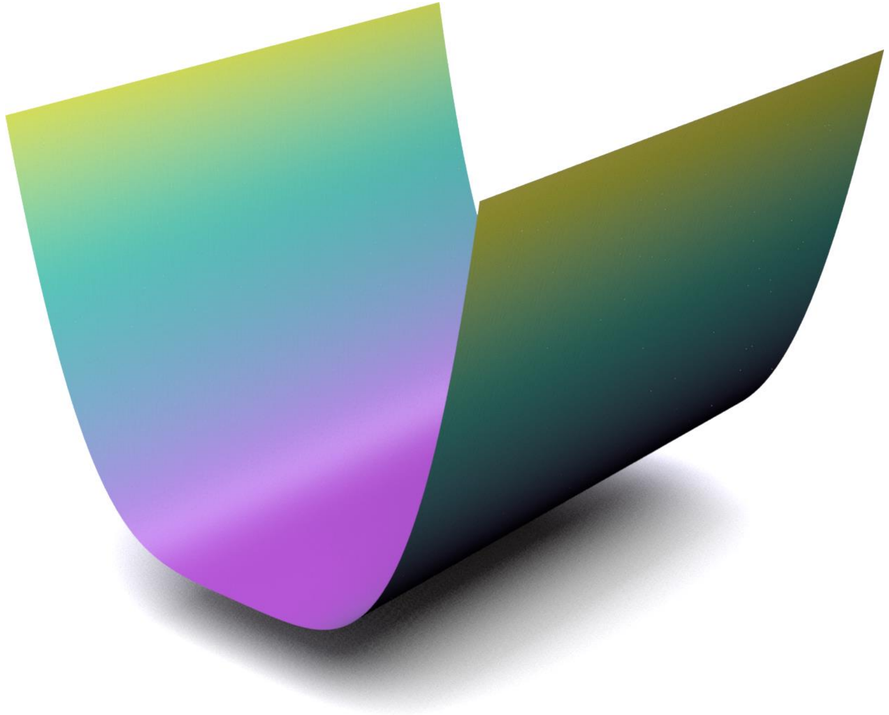

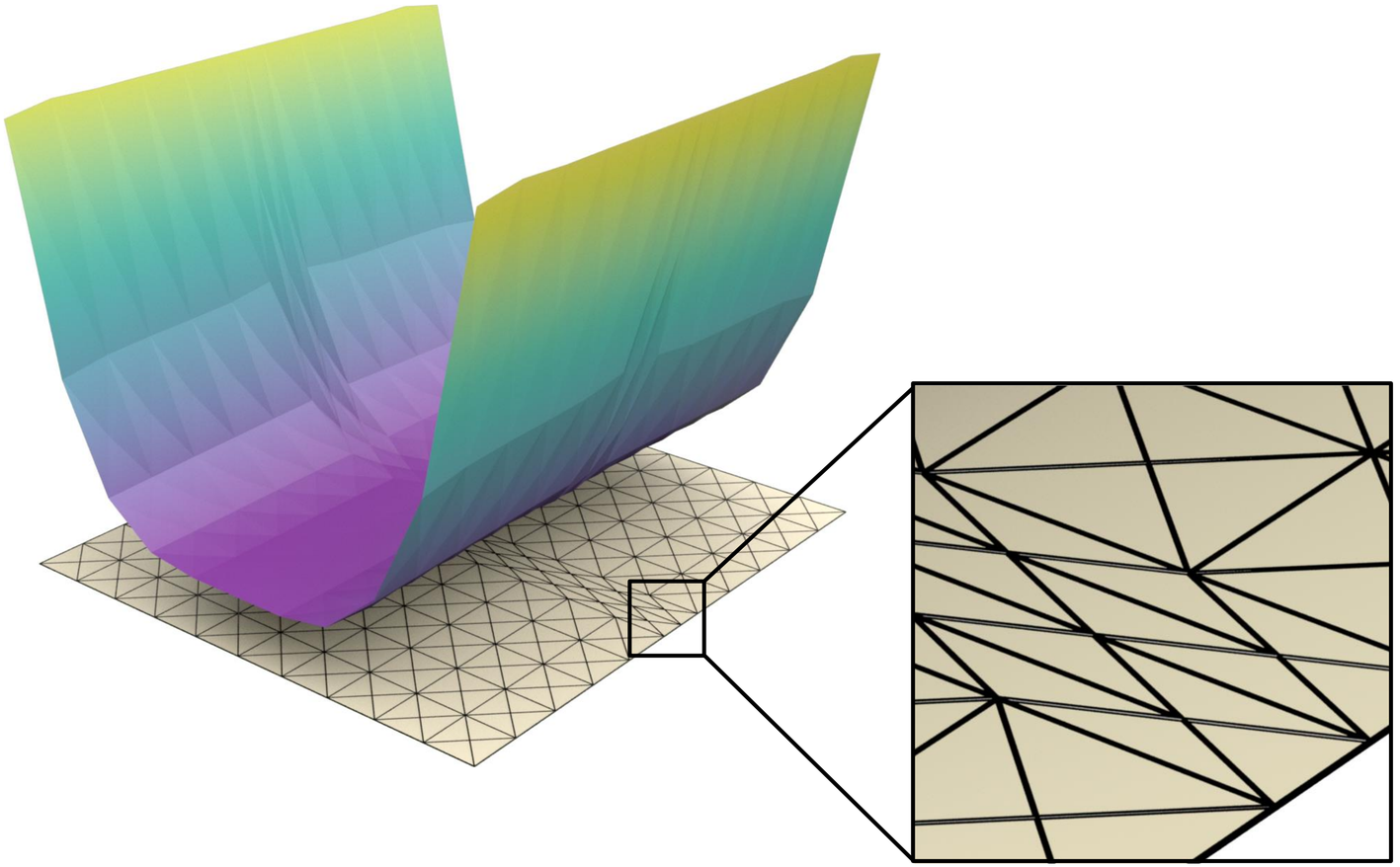

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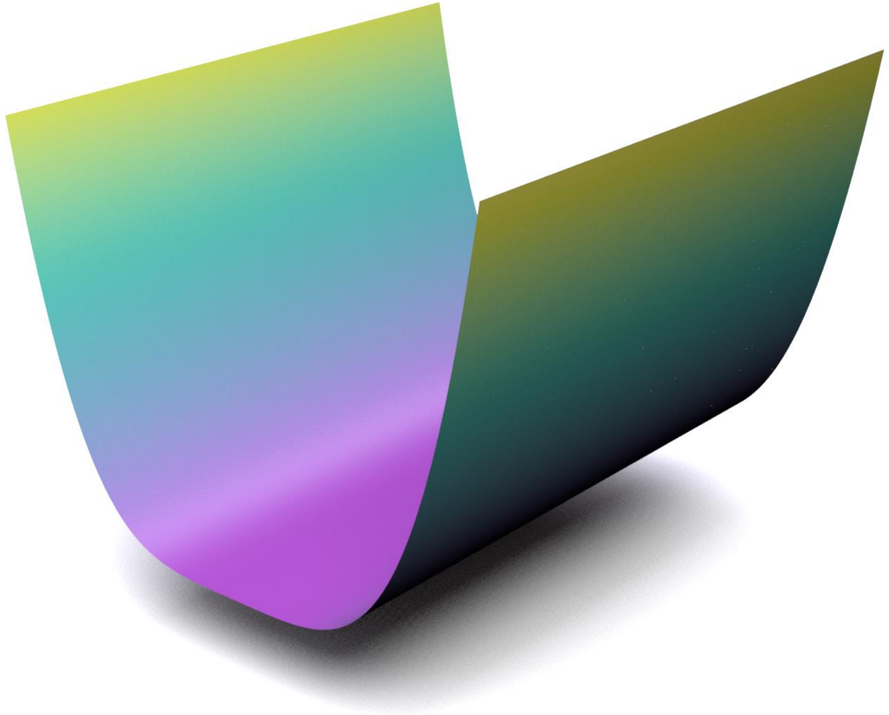

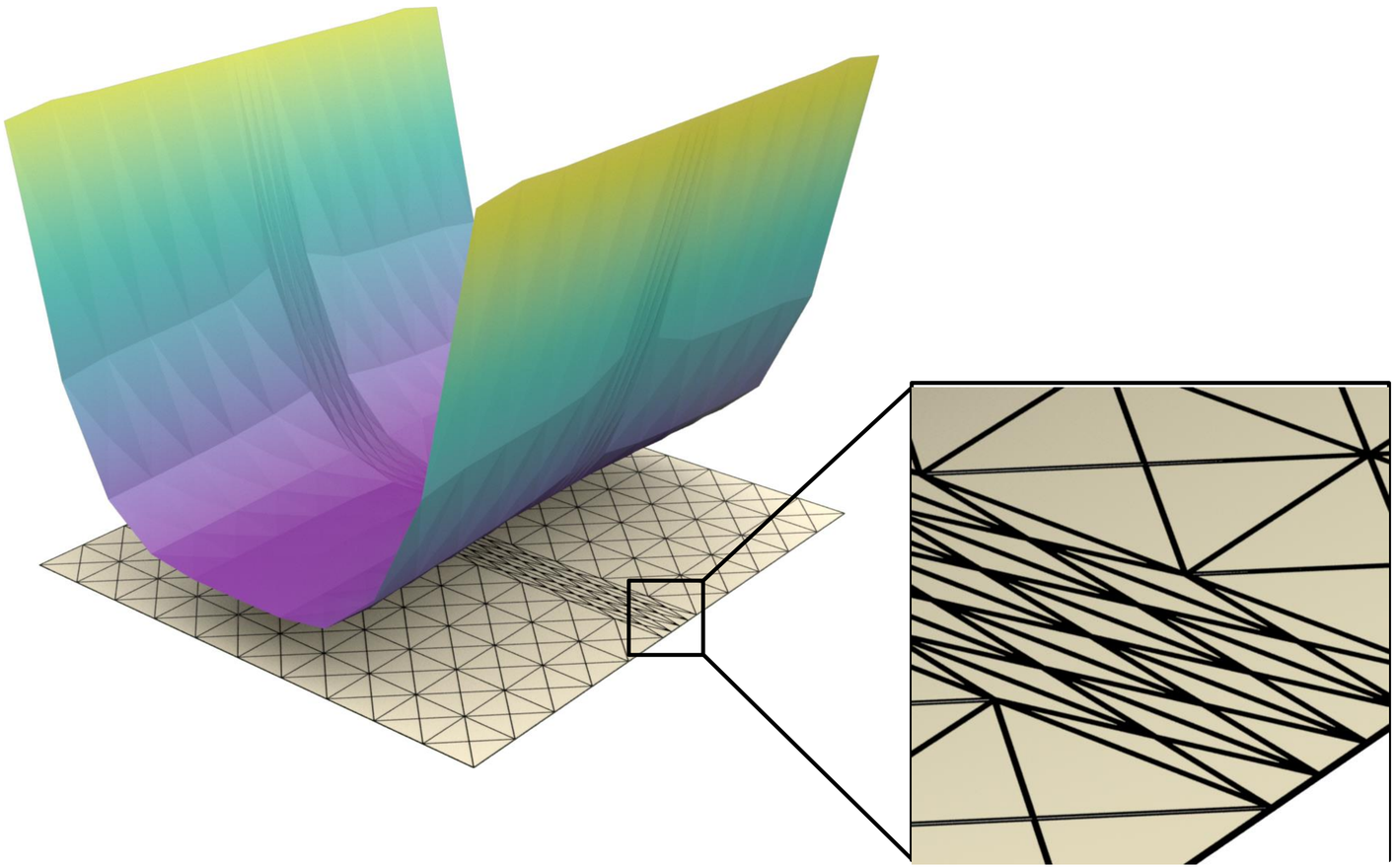

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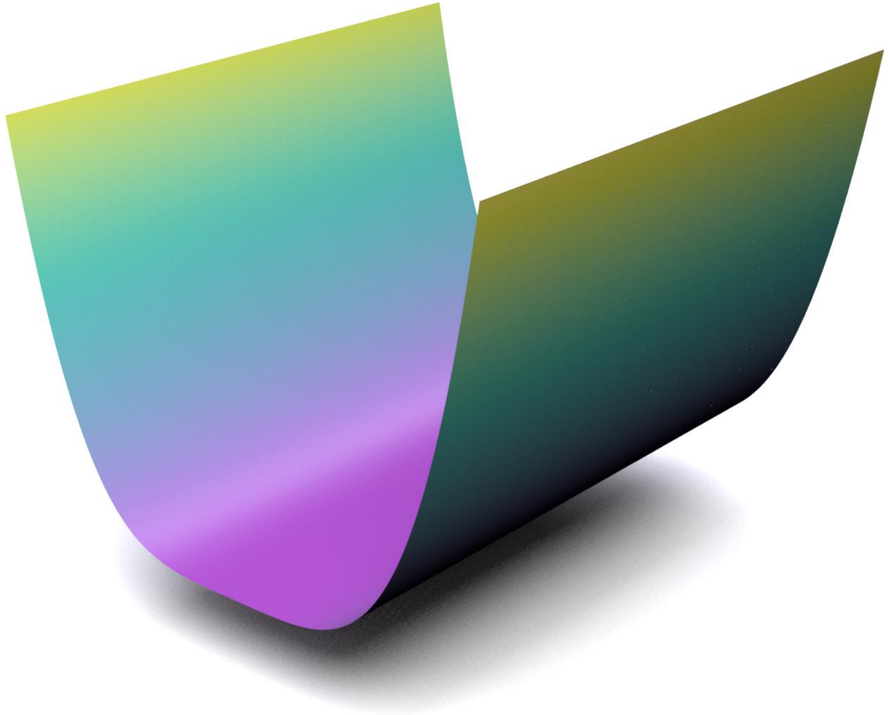

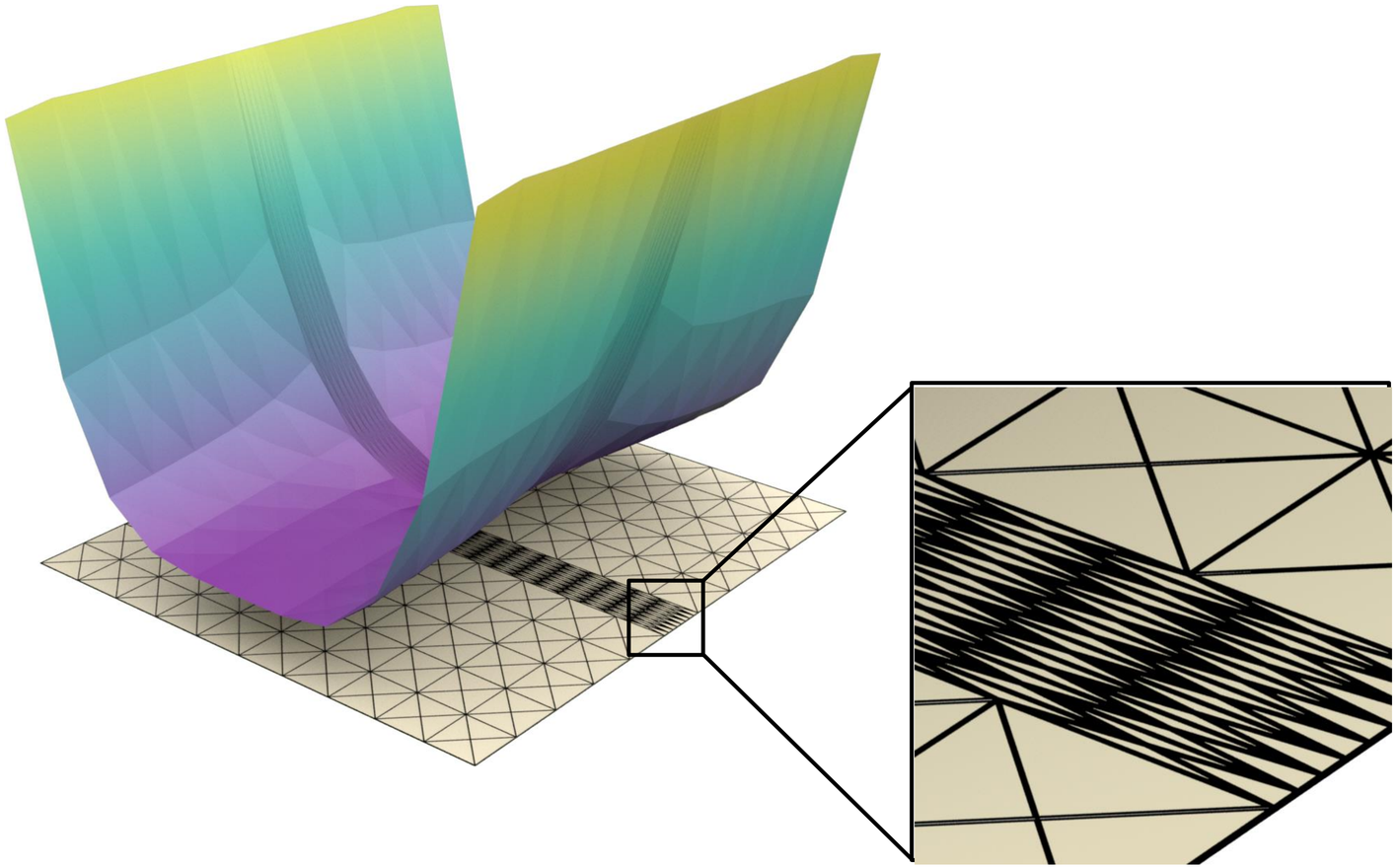

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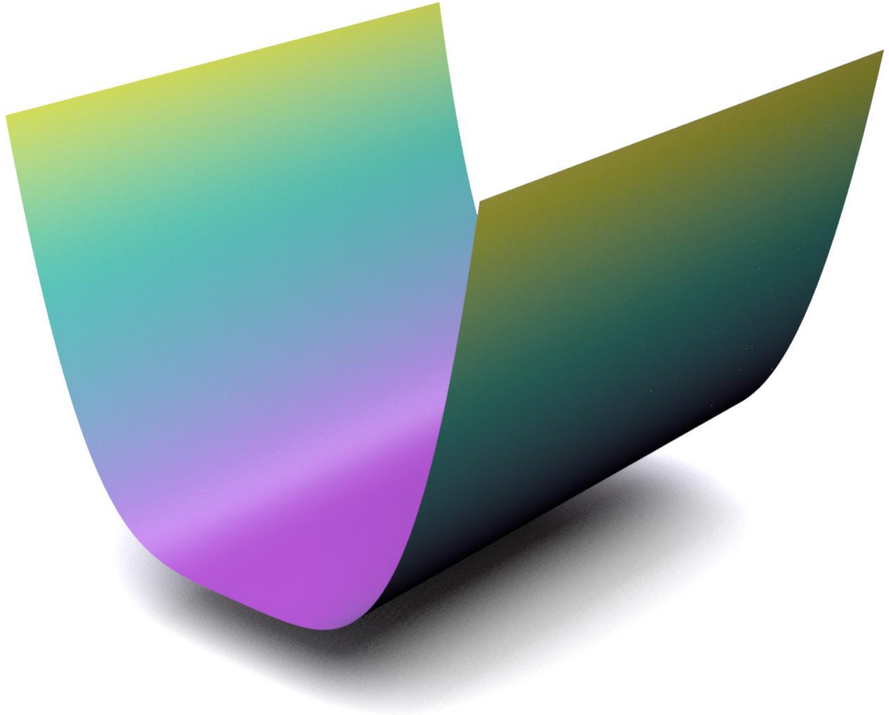

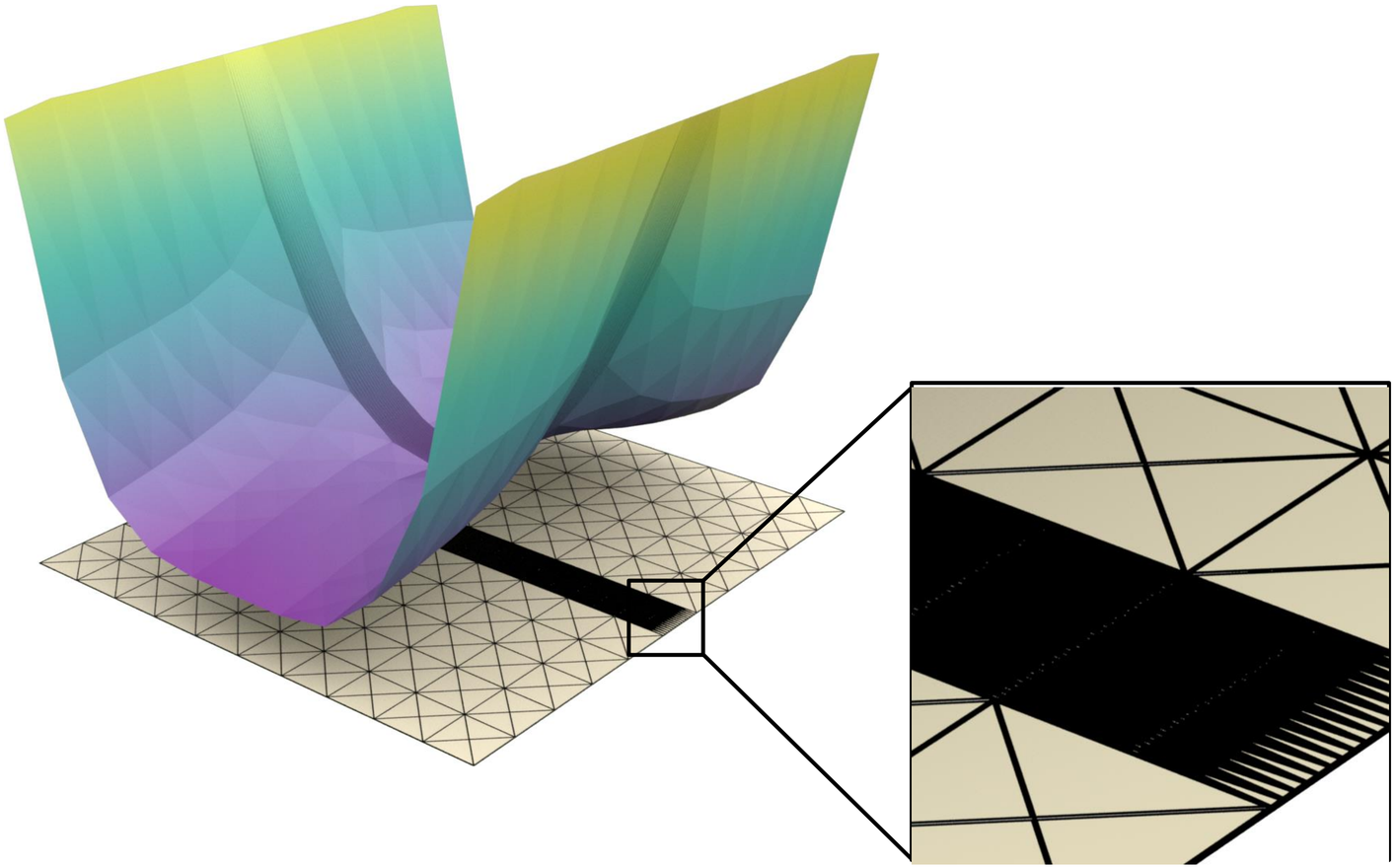

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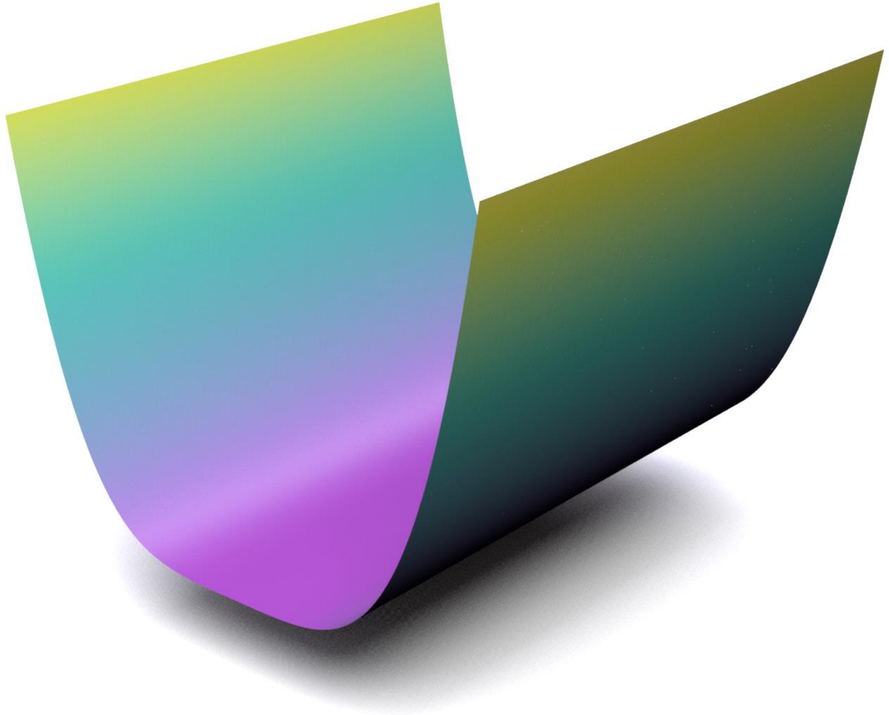

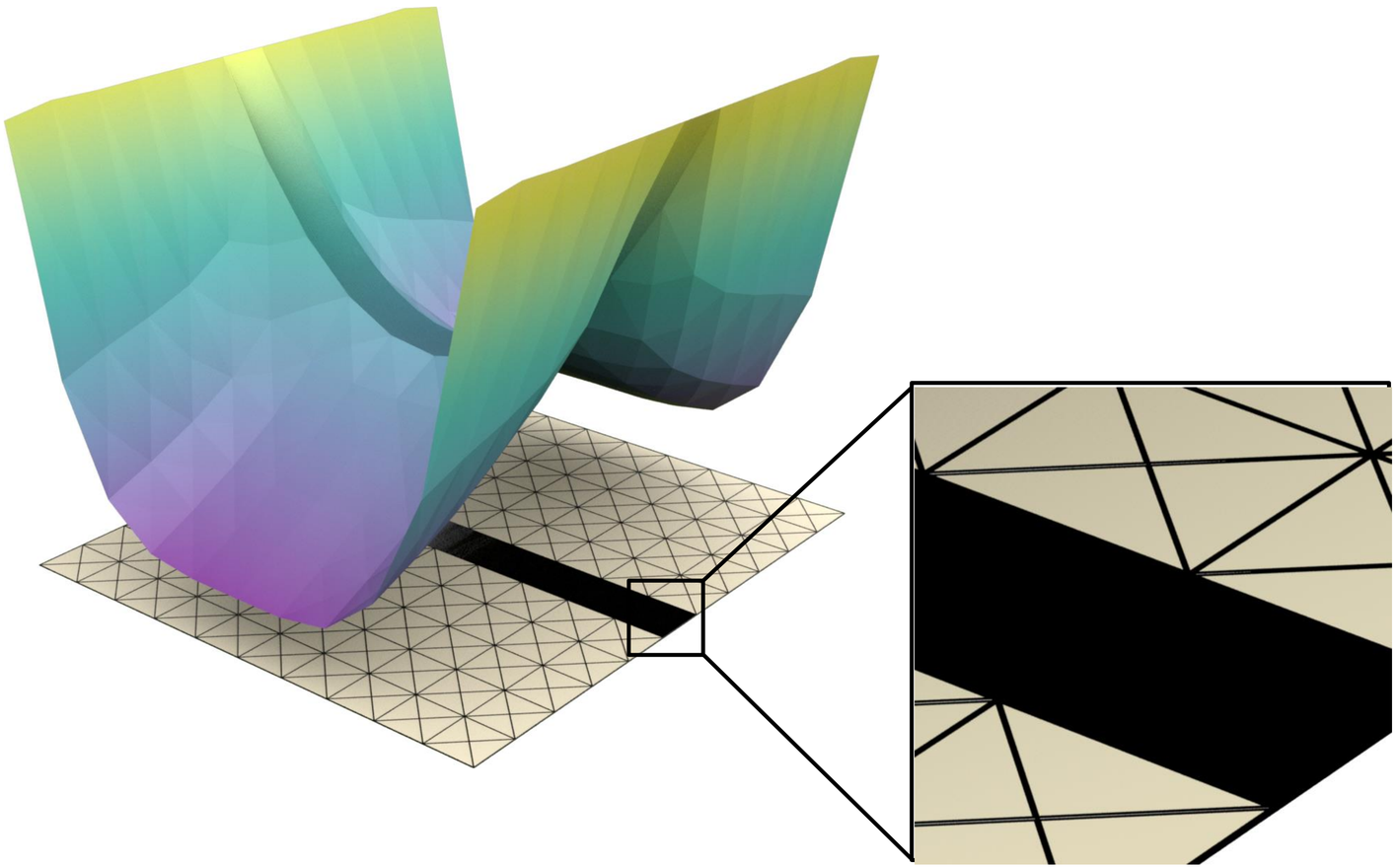

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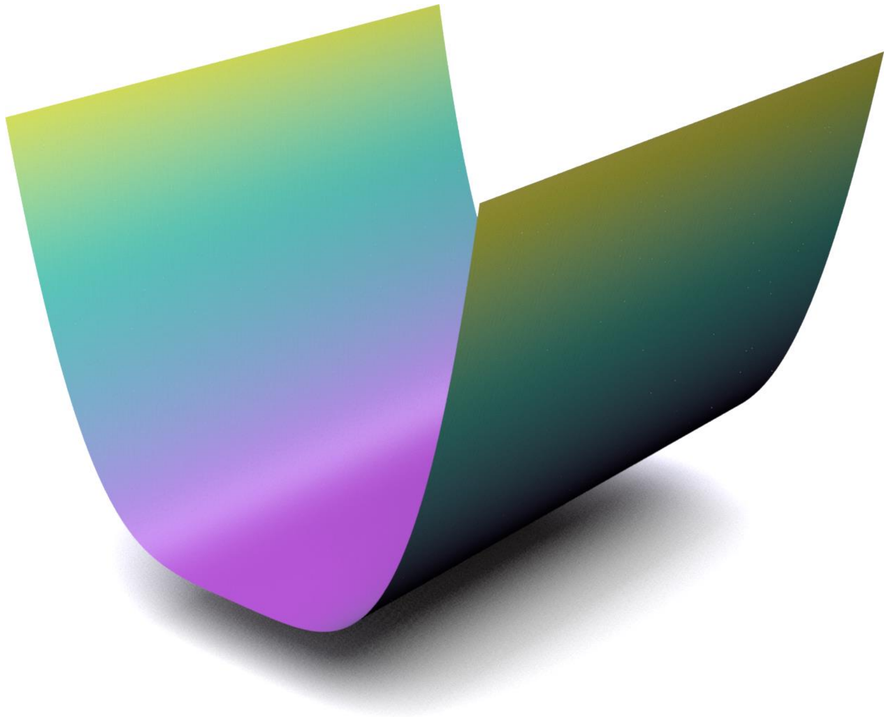
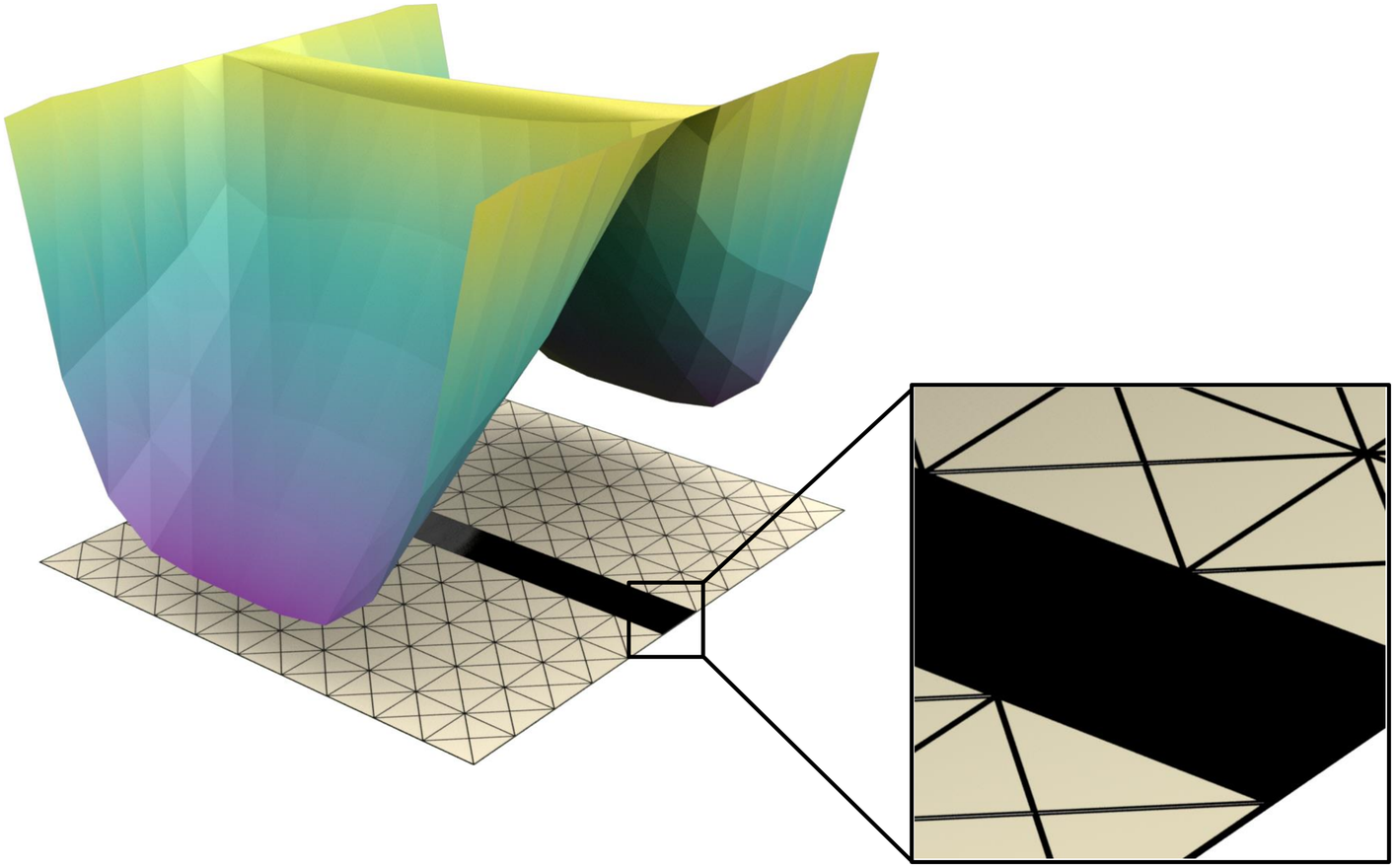
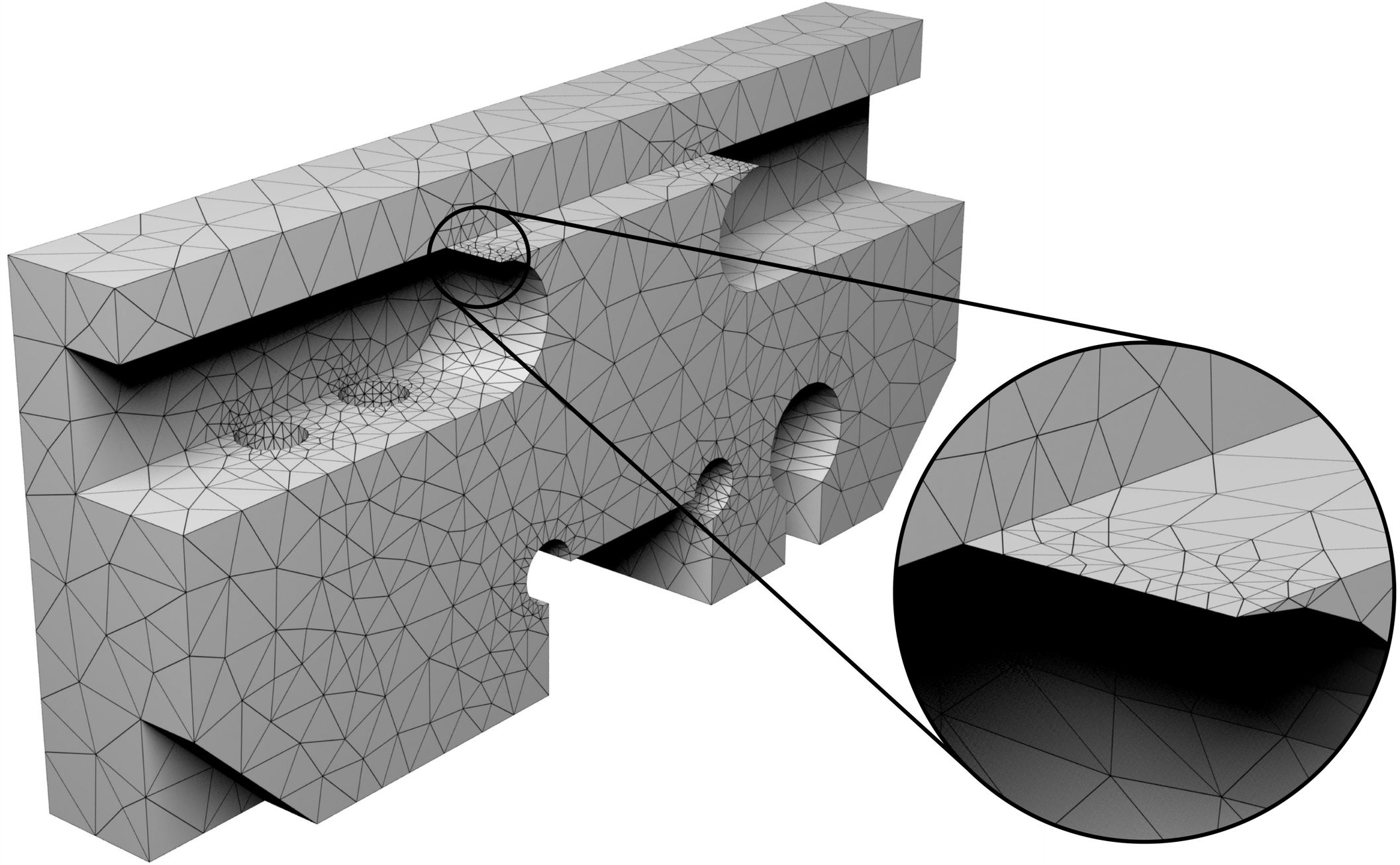
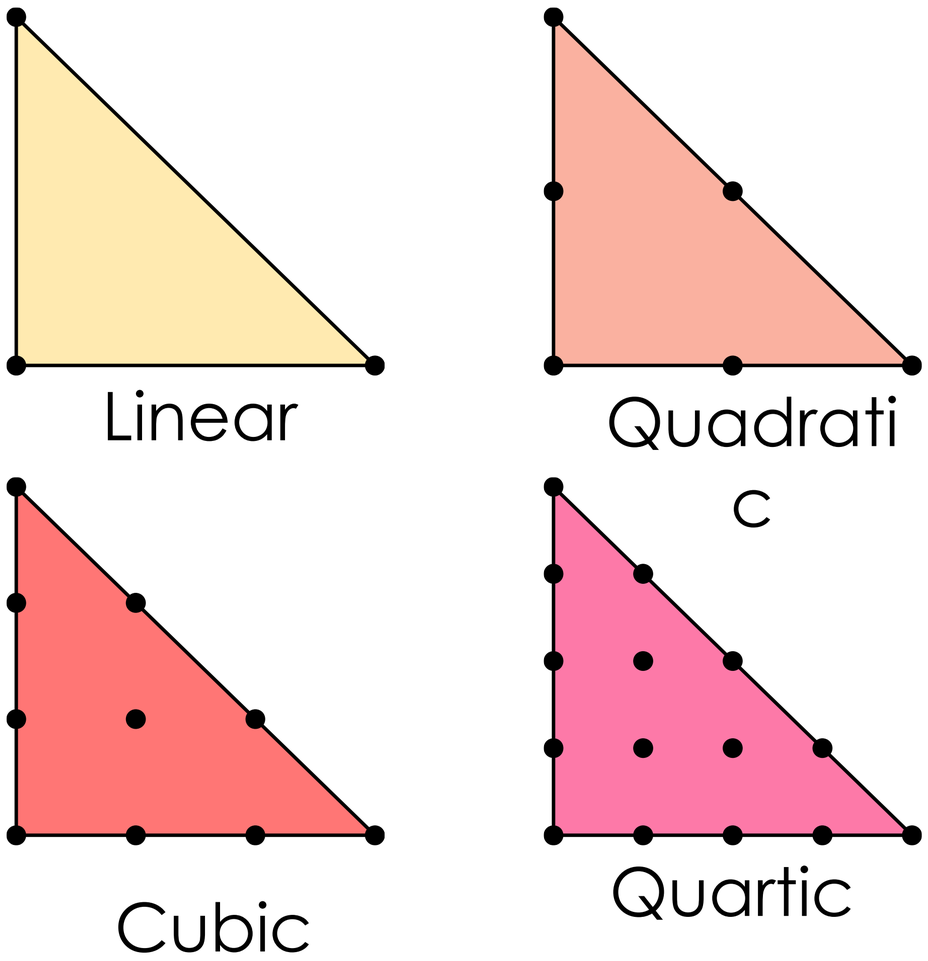
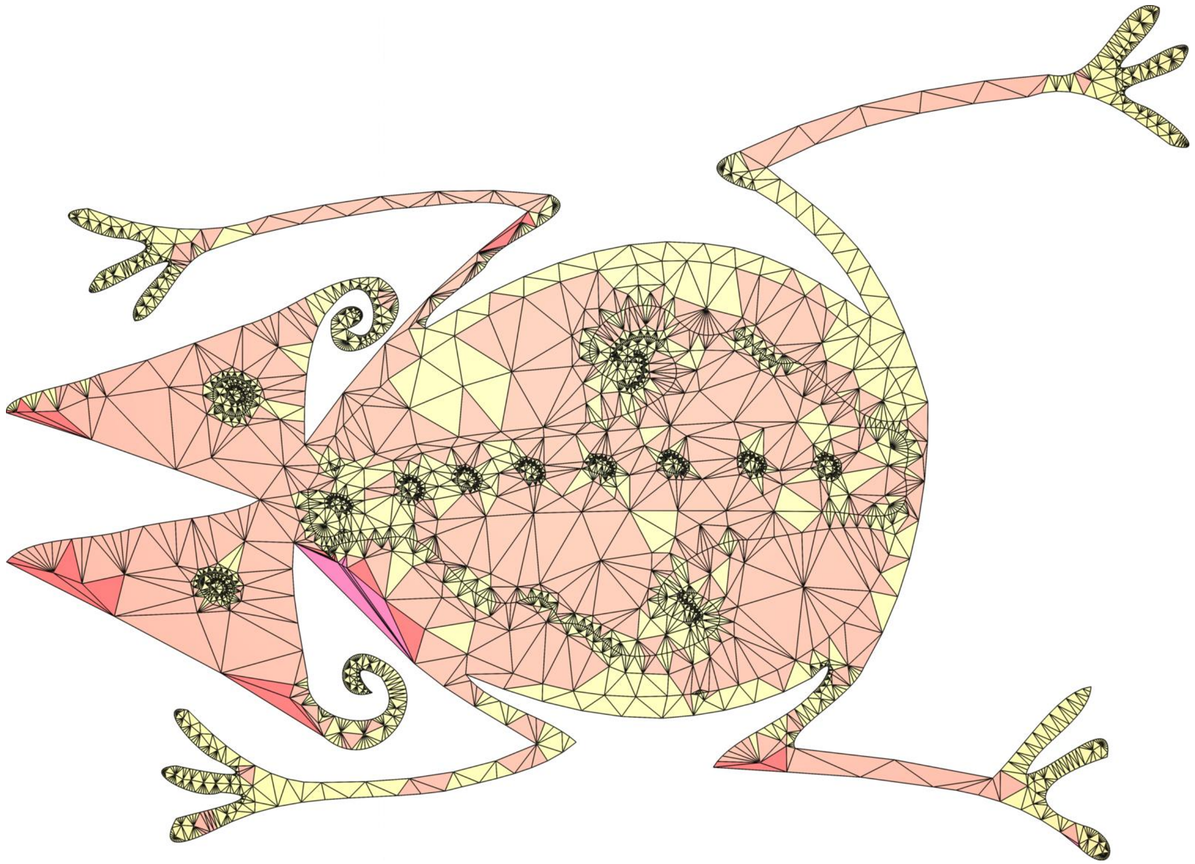






Our Solution





Posteriori Refinement

- h-refinement [Wu 01], [Simnett 09], [Wicke 10], [Pfaff 14], …

- p-refinement [Babuška 94], [Kaufmann 13], [Bargteil 14], [Edwards 14], …

Priori Refinement

We increase order only based on the input

Overview

$$k = \frac{\ln\left(B\hat{h}^{\hat{k}+1}\frac{\sigma_E^2}{\hat{\sigma}^2}\right) - \ln h_E}{\ln h_E}$$

1. Use formula

Order of an element

$$k = \frac{\ln\left(B\hat{h}^{\hat{k}+1}\frac{\sigma_E^2}{\hat{\sigma}^2}\right) - \ln\, h_E}{\ln\, h_E}$$

User parameter, = 3

$$k = \frac{\ln\left(B\hat{h}^{\hat{k}+1}\frac{\sigma_E^2}{\hat{\sigma}^2}\right) - \ln\, h_E}{\ln\, h_E}$$

Average edge length

$$k = \frac{\ln\left(B\hat{h}^{\hat{k}+1}\frac{\sigma_E^2}{\hat{\sigma}^2}\right) - \ln h_E}{\ln h_E}$$

Base order, usually 1

$$k = \frac{\ln\left(B\hat{h}^{\hat{k}+1}\frac{\sigma_E^2}{\hat{\sigma}^2}\right) - \ln h_E}{\ln h_E}$$

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$$\hat{\sigma}_{2D} = \sqrt{3}/6$$

$$\hat{\sigma}_{3D} = \sqrt{6}/12$$

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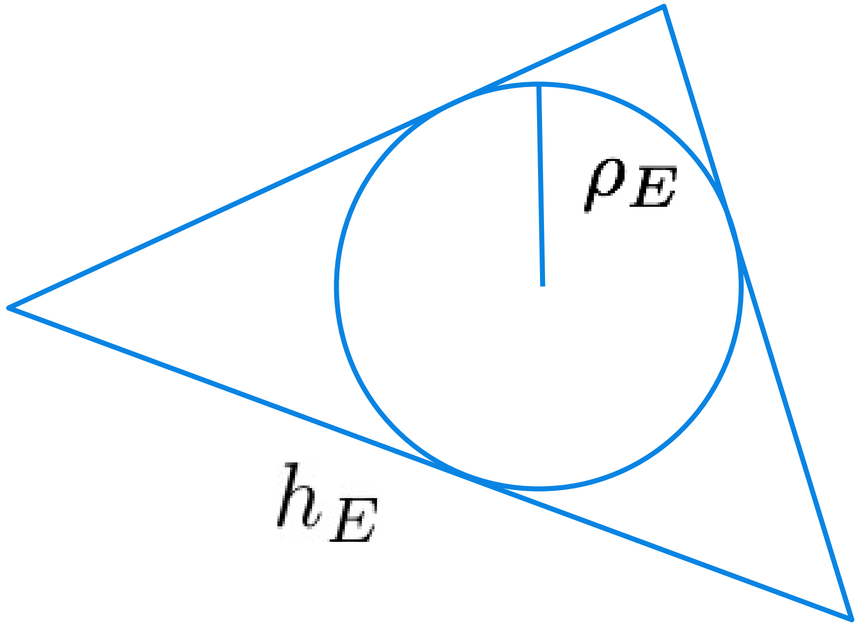
$$k = \frac{\ln\left(B\hat{h}^{\hat{k}+1}\frac{\sigma_E^2}{\hat{\sigma}^2}\right) - \ln h_E}{\ln h_E}$$

$$\sigma_E = \frac{\rho_E}{h_E}$$



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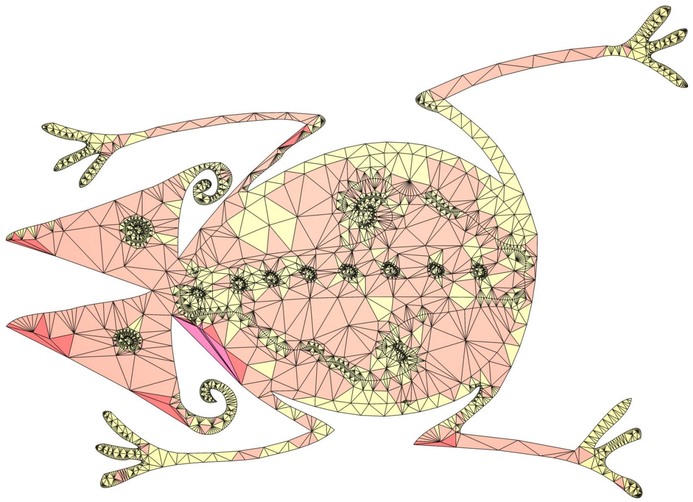
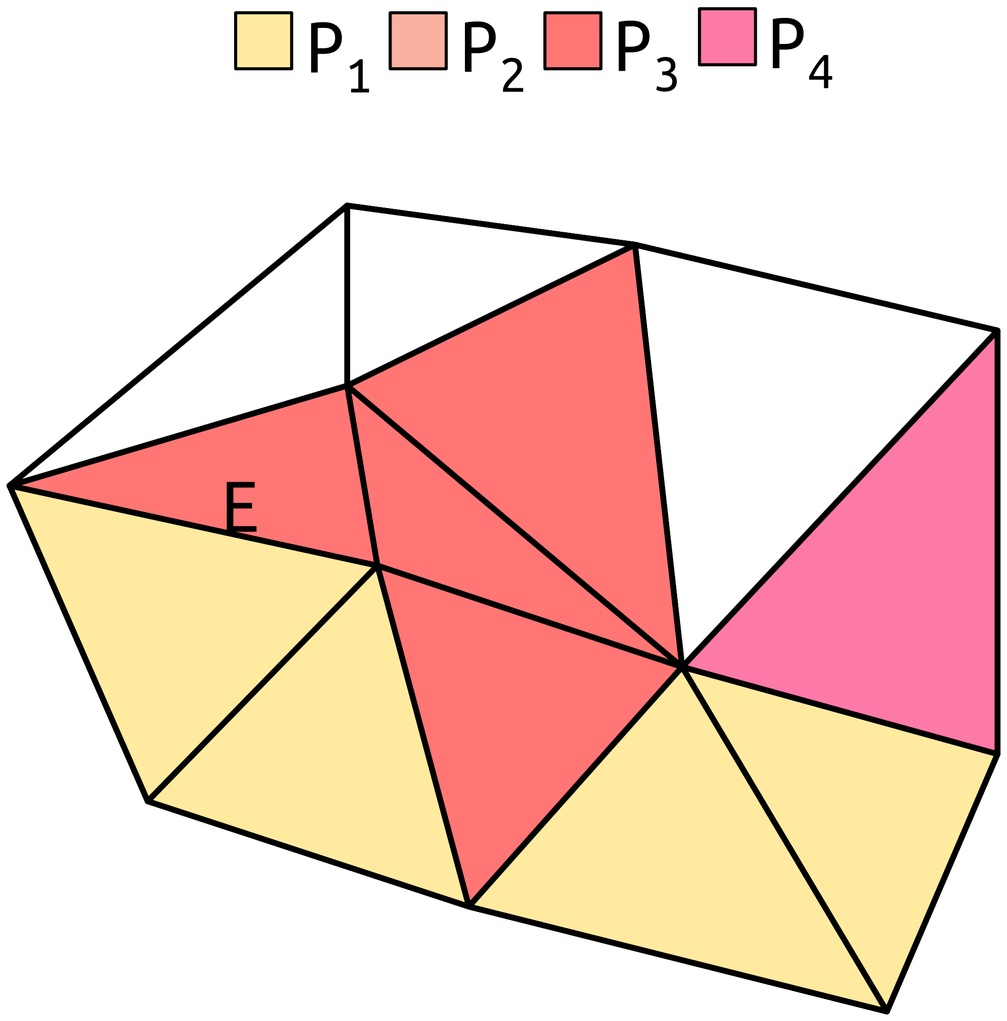


1. Use formula

2. Propagate degrees

Degree Propagation

- For each element E
- Compute k_E using formula
- Increase the order (if necessary) of:
  - The element E
  - All edge/face neighbors



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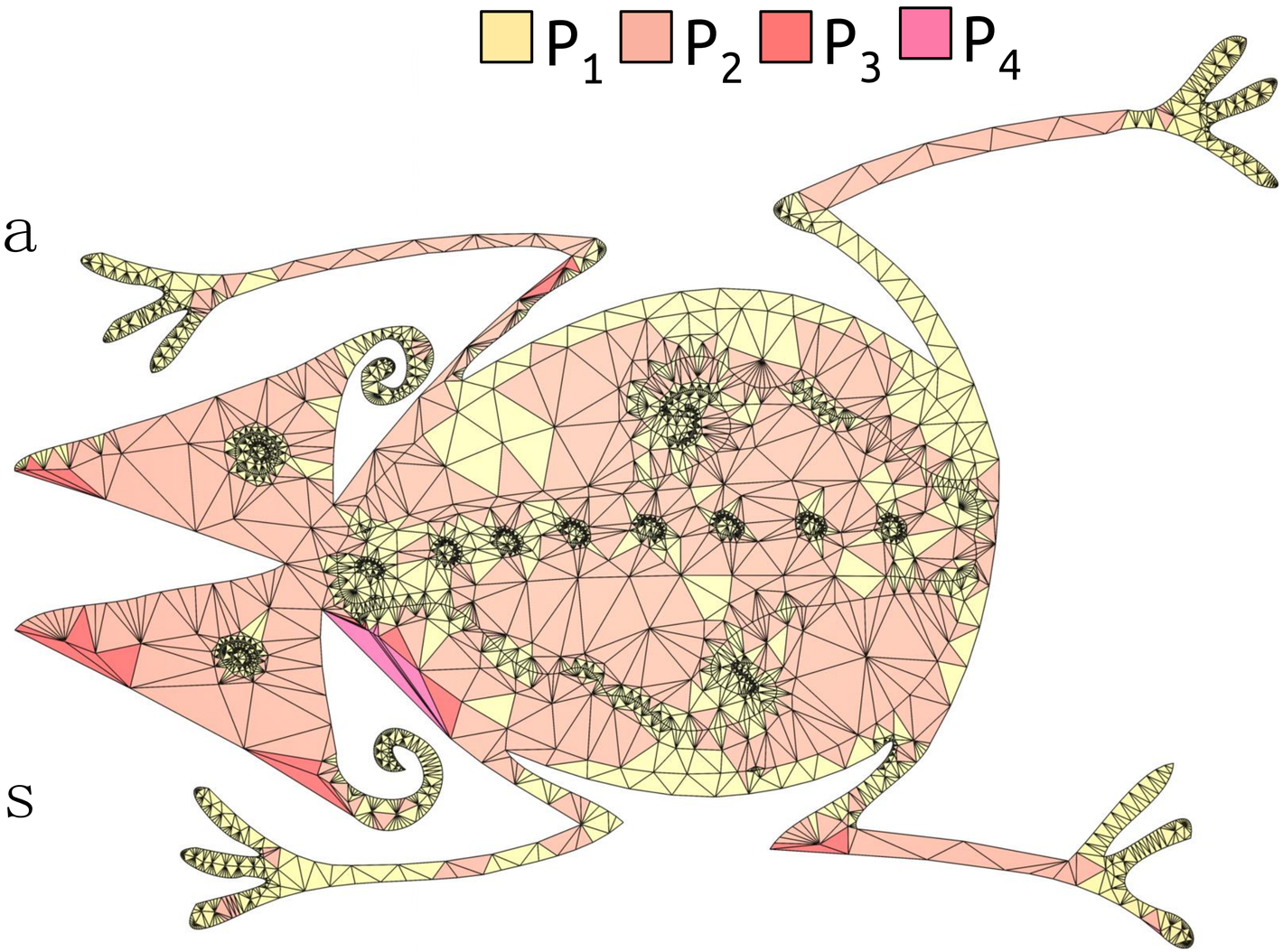

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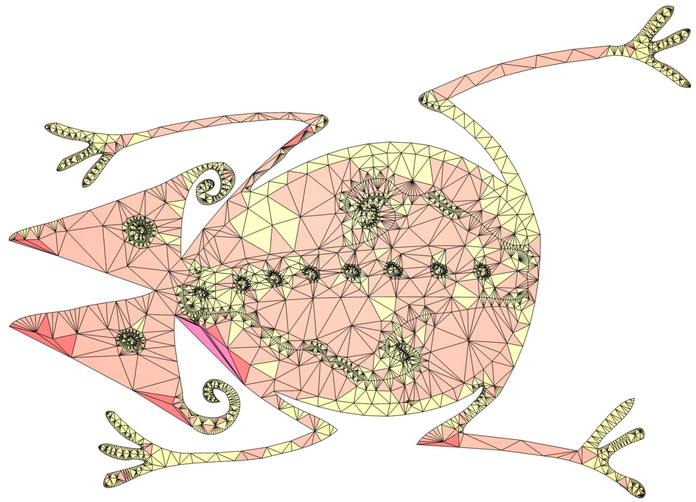
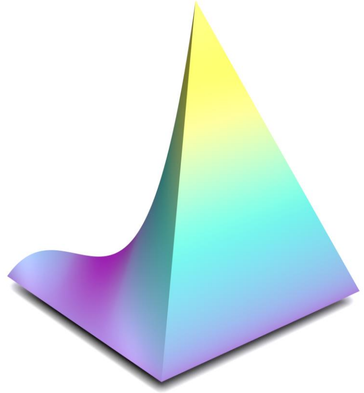
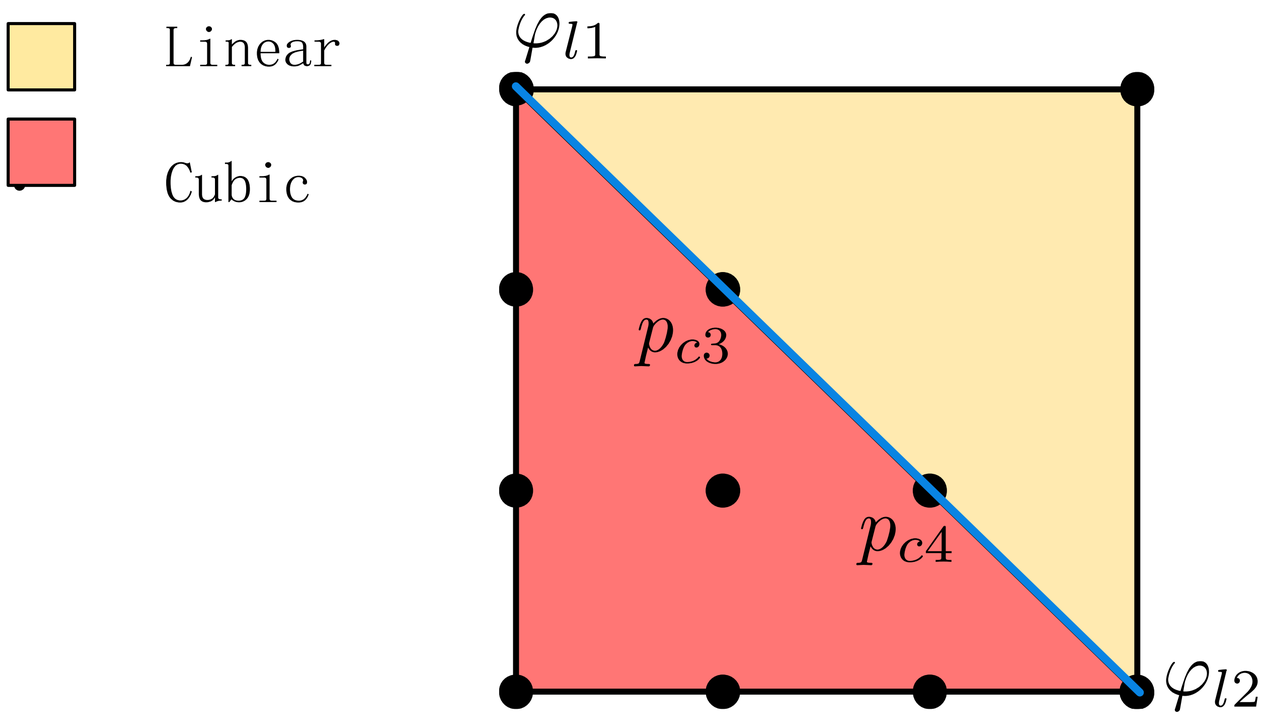


2. Propagate degrees



3. Construct C^0 basis

Building Continuous Basis



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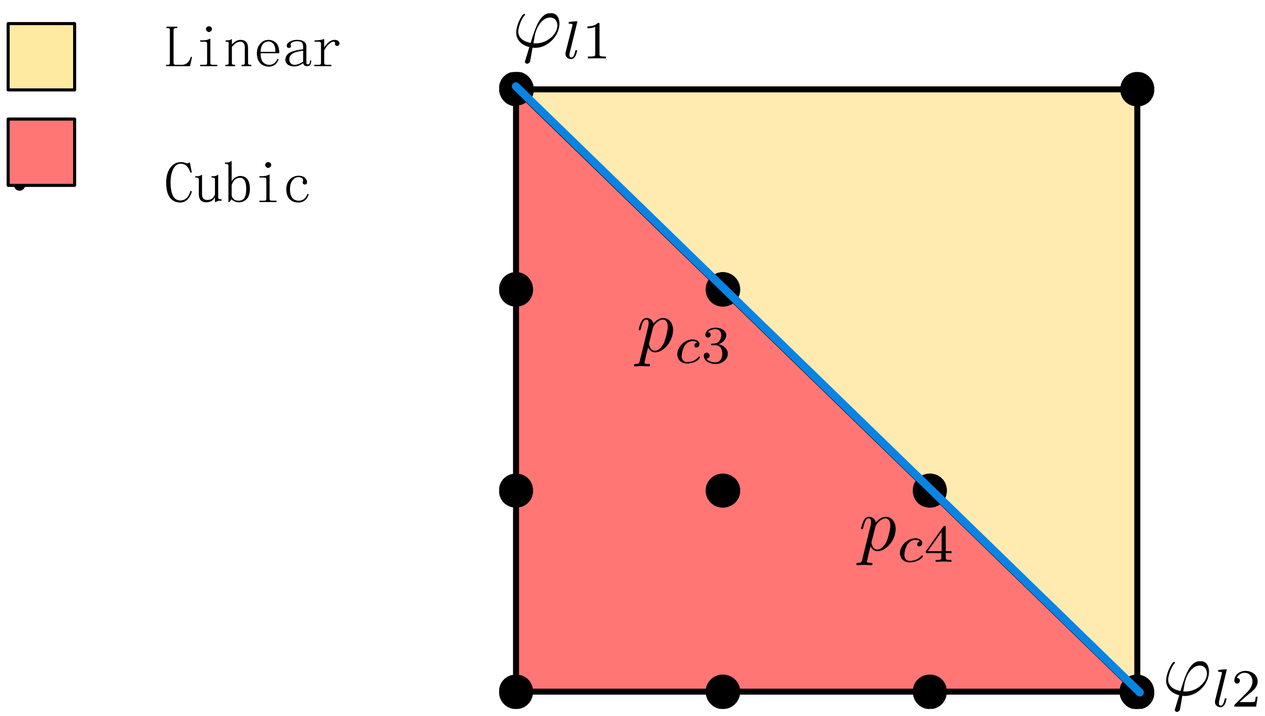

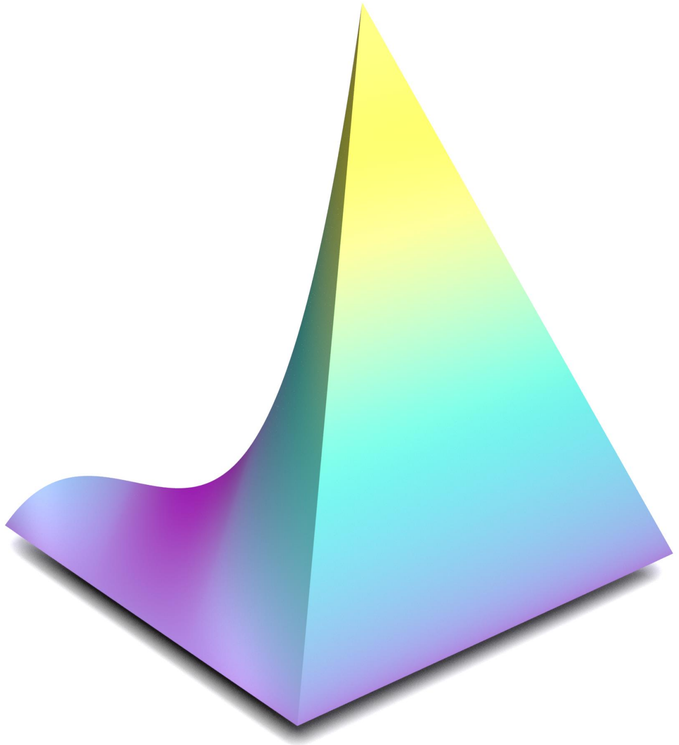

Overview

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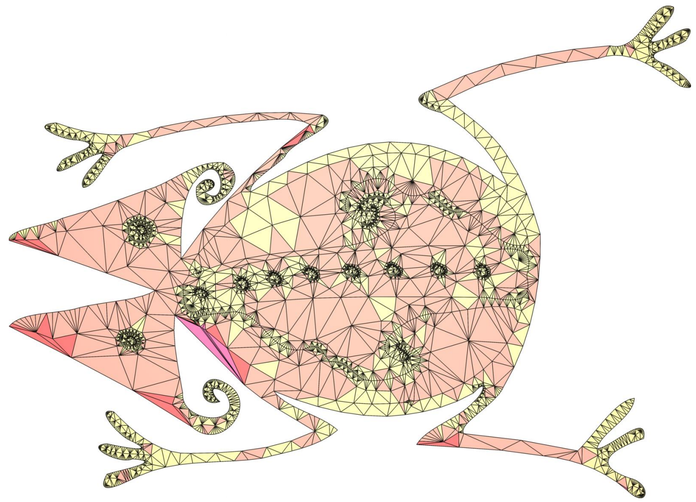
1. Use formula



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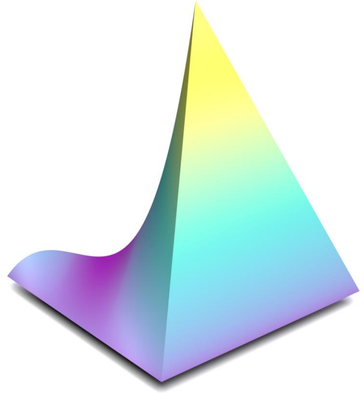

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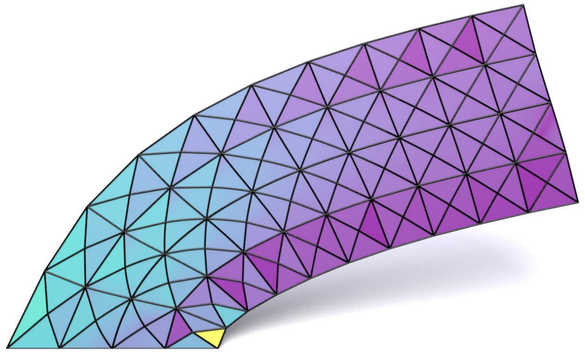
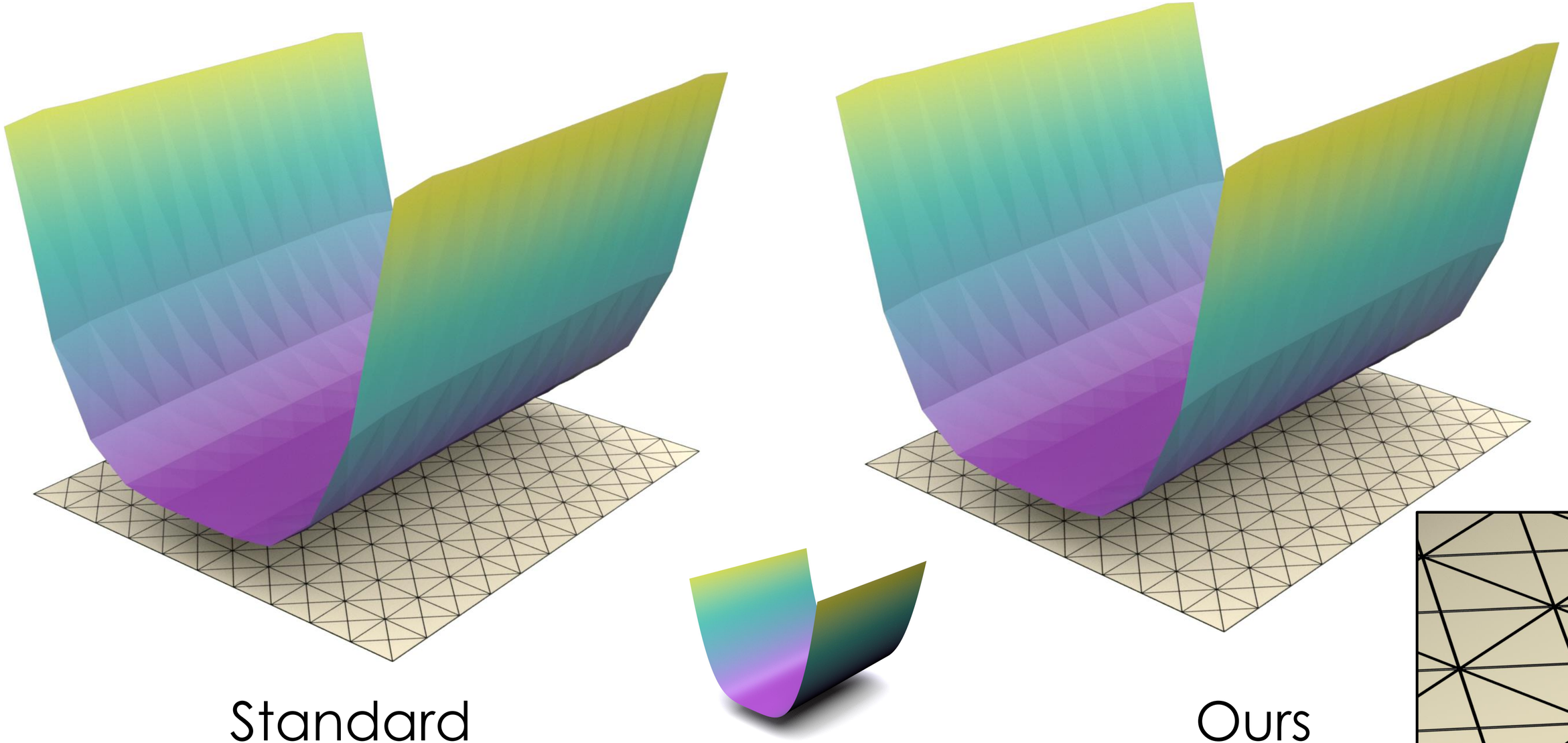
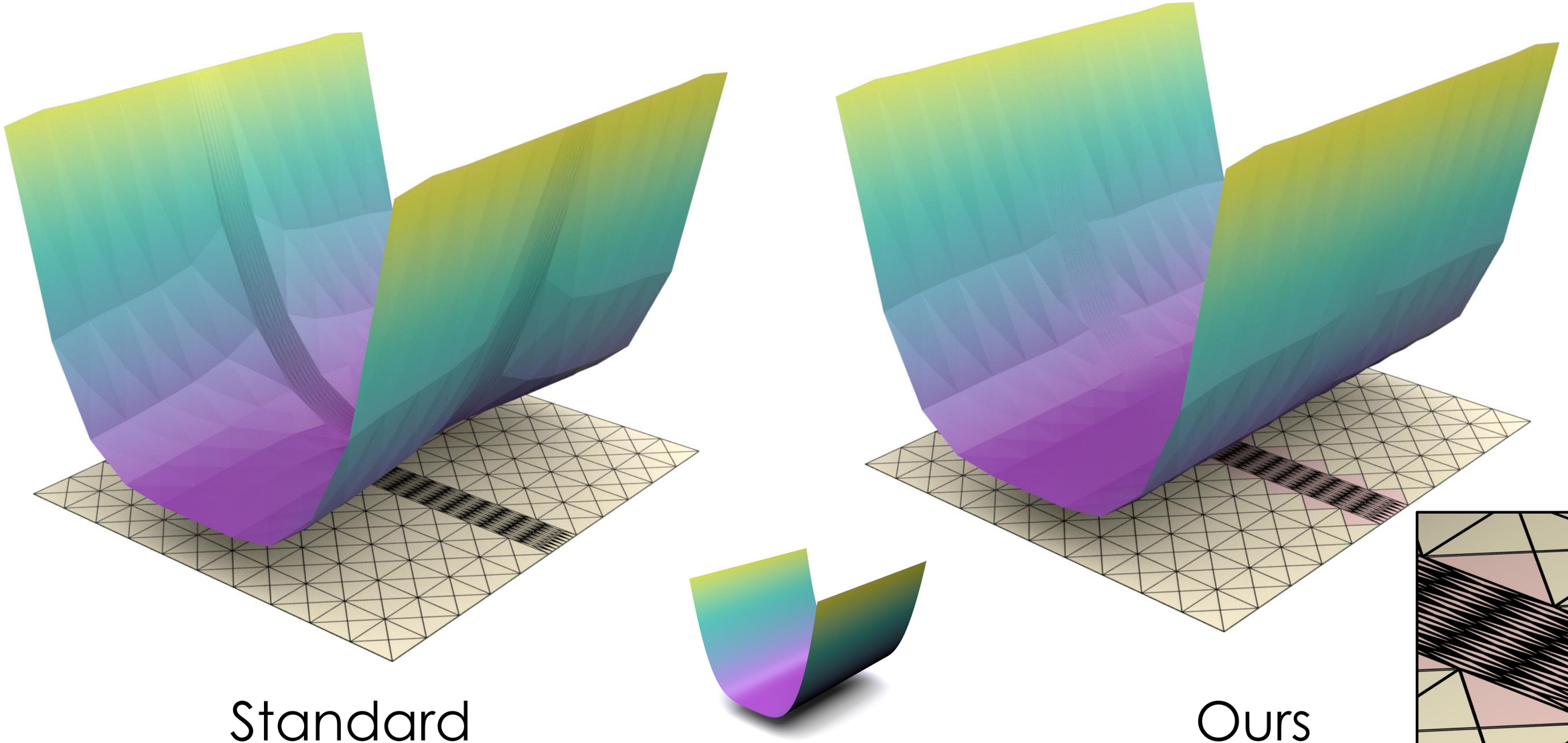


4. Simulate!

Laplace



Laplace



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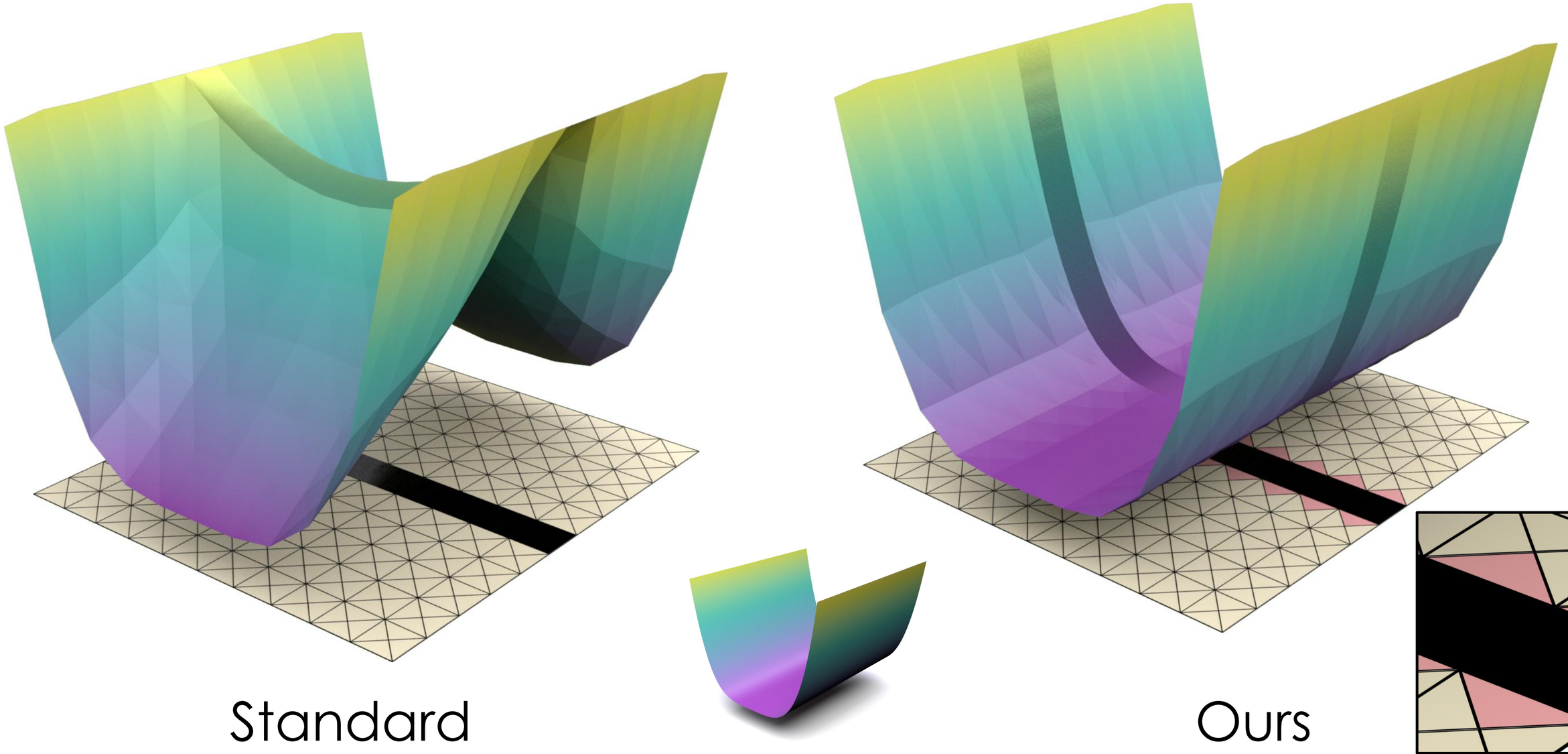
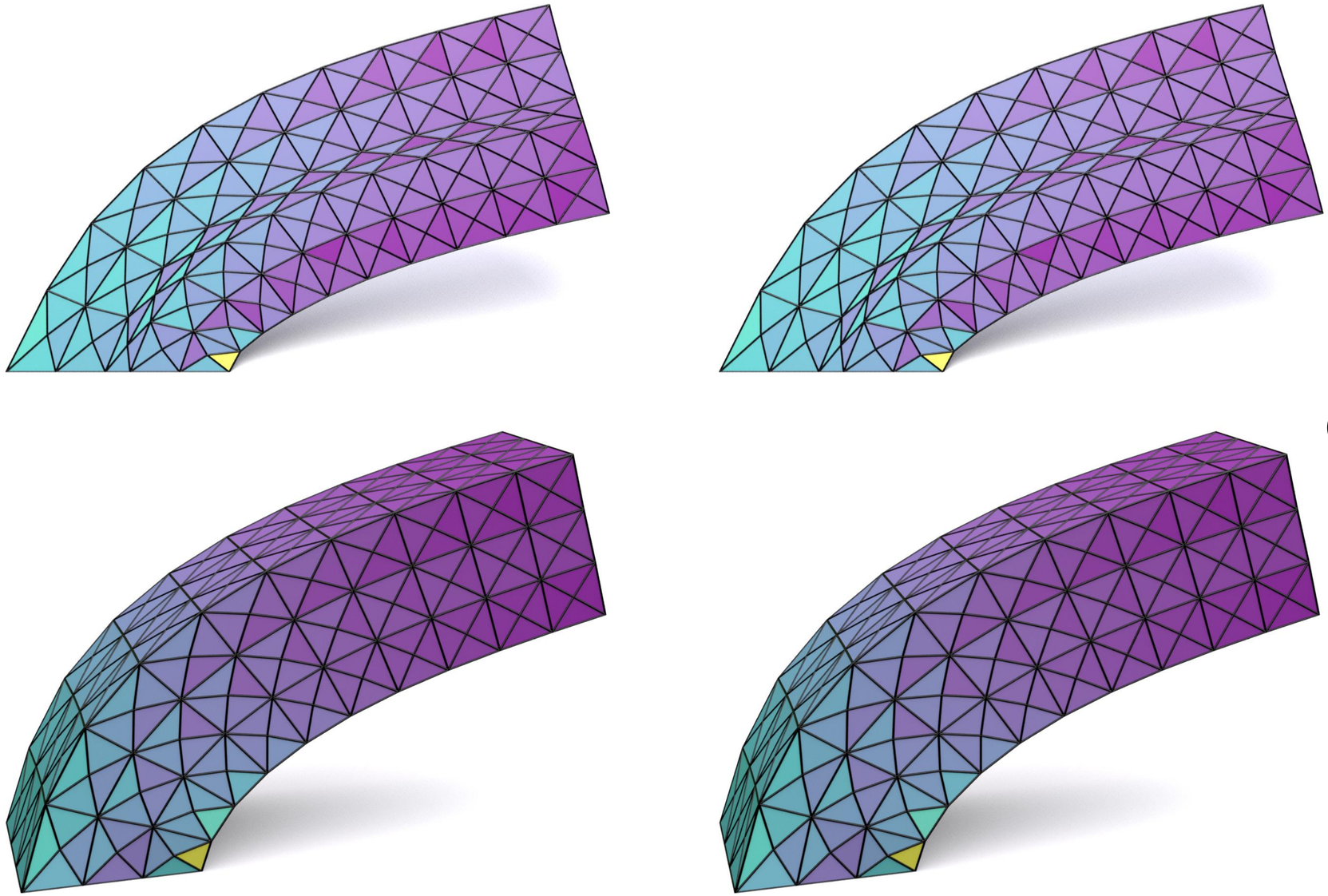
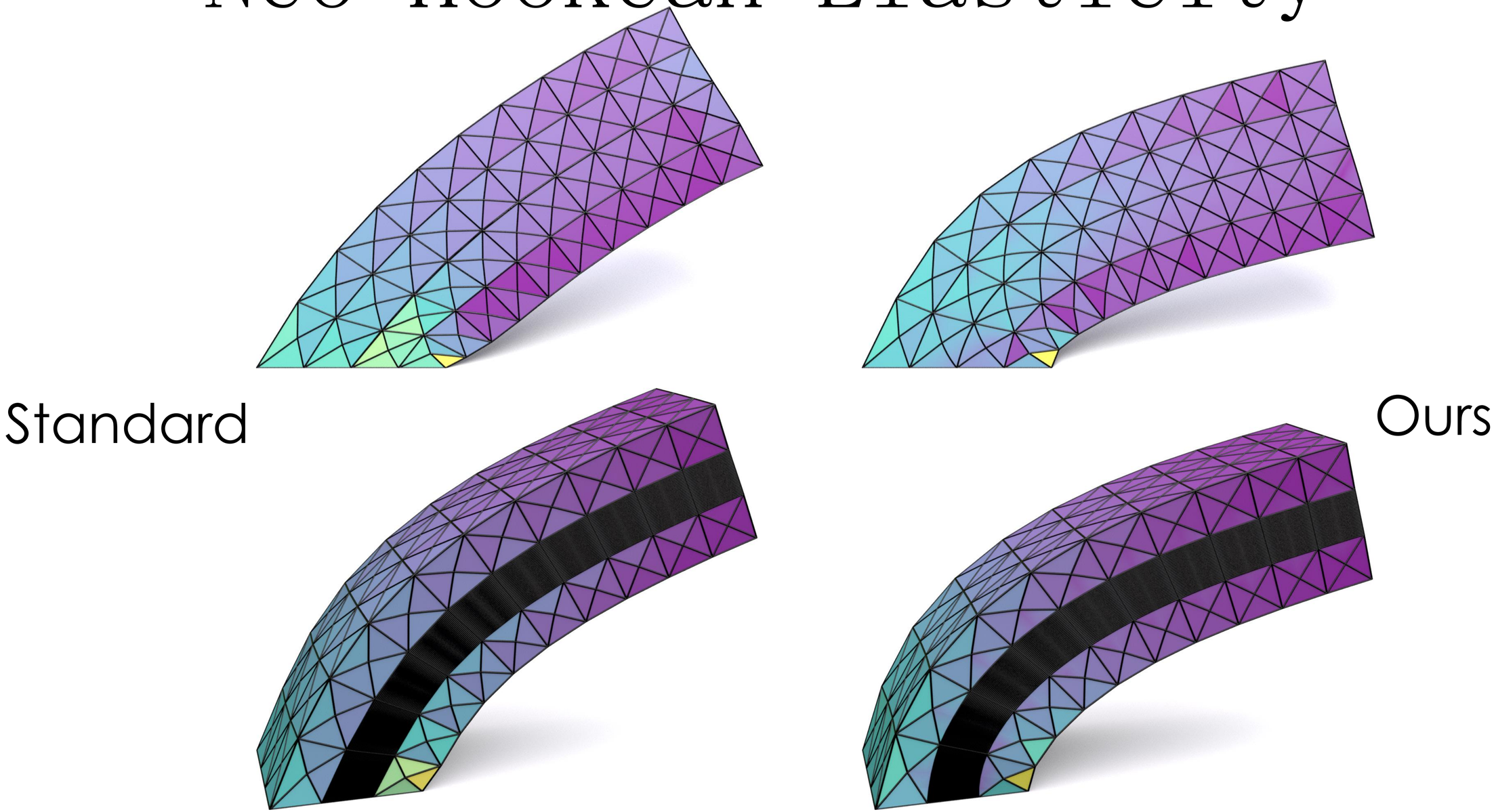


Neo-Hookean Elasticity

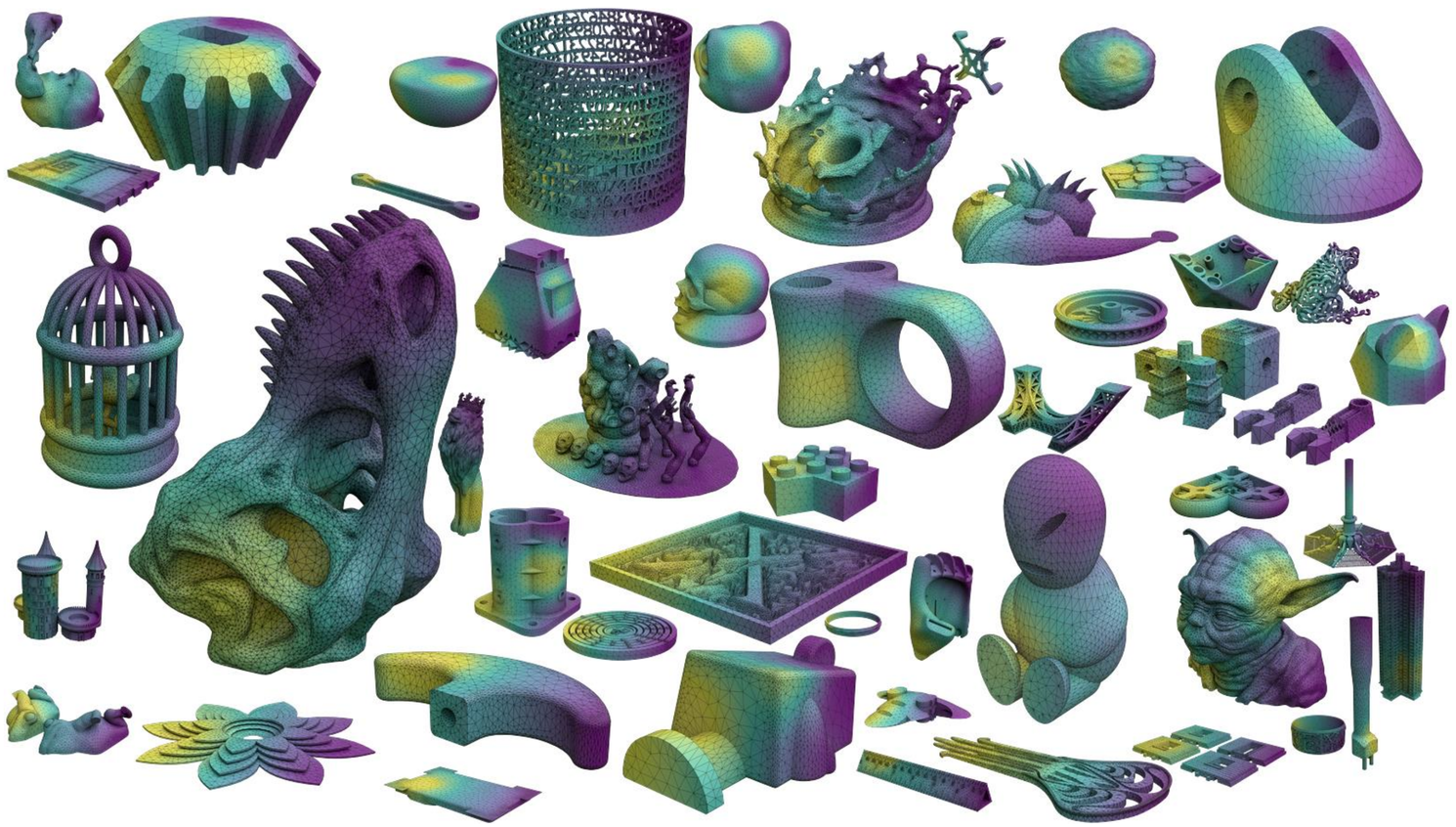


Neo-Hookean Elasticity



Large Dataset

- Thingi10k
  [Zhou 17]
- Tetwild
  [Hu 18]
- ~10k Optimized
- ~10k Not Optimized



How to Measure Errors?

- Standard L_2 error estimate for linear elements

$$e_h = \|u - u_h\|_0 \leq Ch^2 \|u\|_2$$

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L_2 norm or average error

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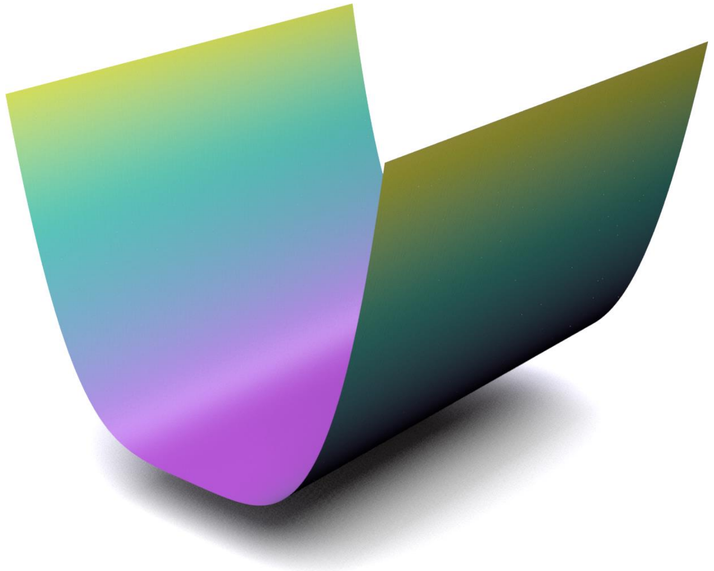
$$e_h = \|u - u_h\|_0 \leq Ch^2\|u\|_2$$

Exact solution



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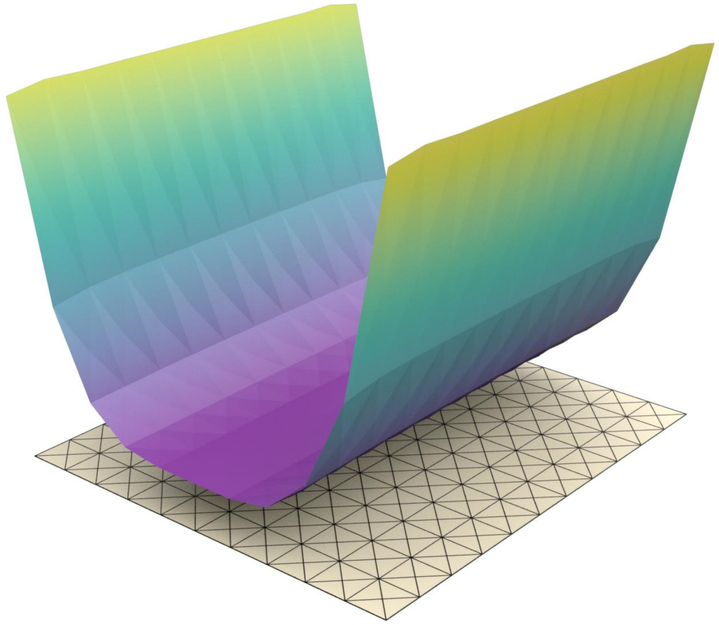
Approximated solution



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  - Different h for every model!

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  - Different **h** for every model!

- L_2 efficiency

$$E_{L_2} = \frac{\|u - u_h\|_0}{h^2}$$

How to Measure Errors?

- Standard L_2 error estimate for linear elements

$$e_h = \|u - u_h\|_0 \leq Ch^2 \|u\|_2$$

  - Different h for every model!

- L_2 efficiency

$$E_{L_2} = \frac{\|u - u_h\|_0}{h^2}$$

  - Independent from h

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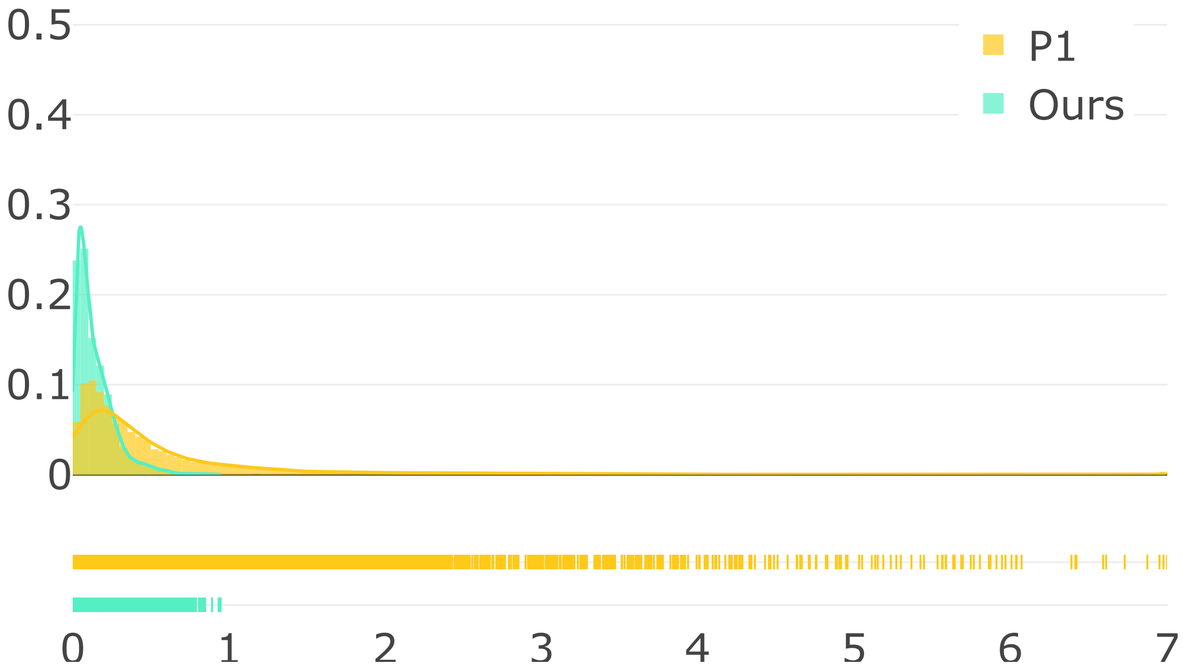
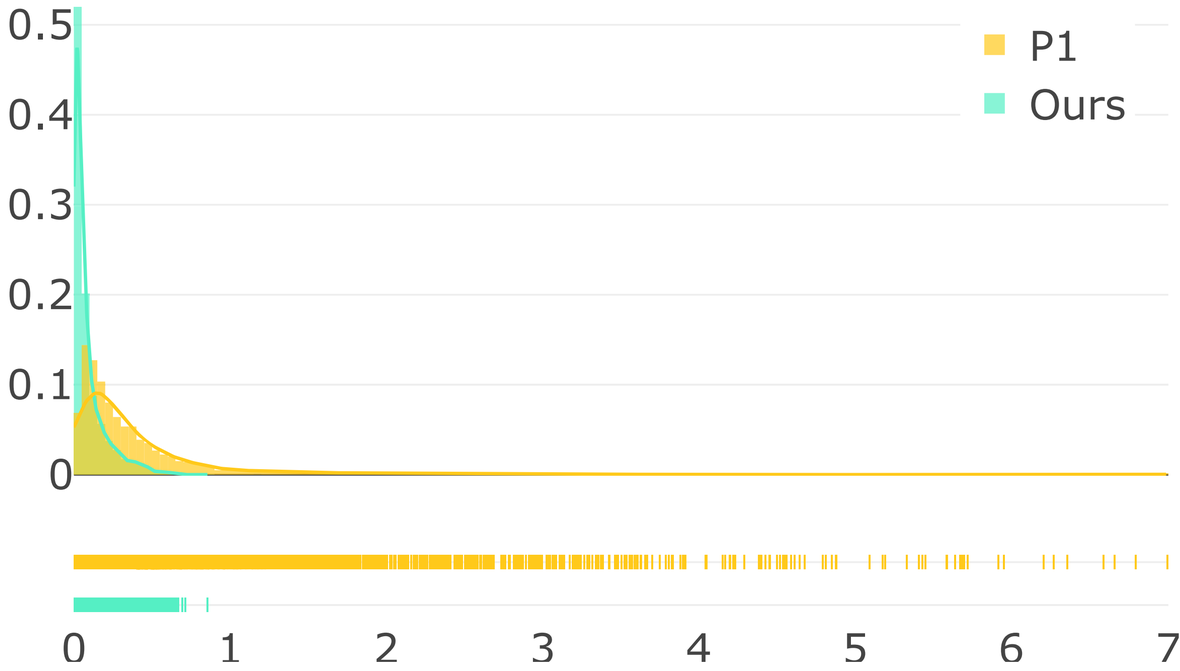
Small values are good!

Efficiency



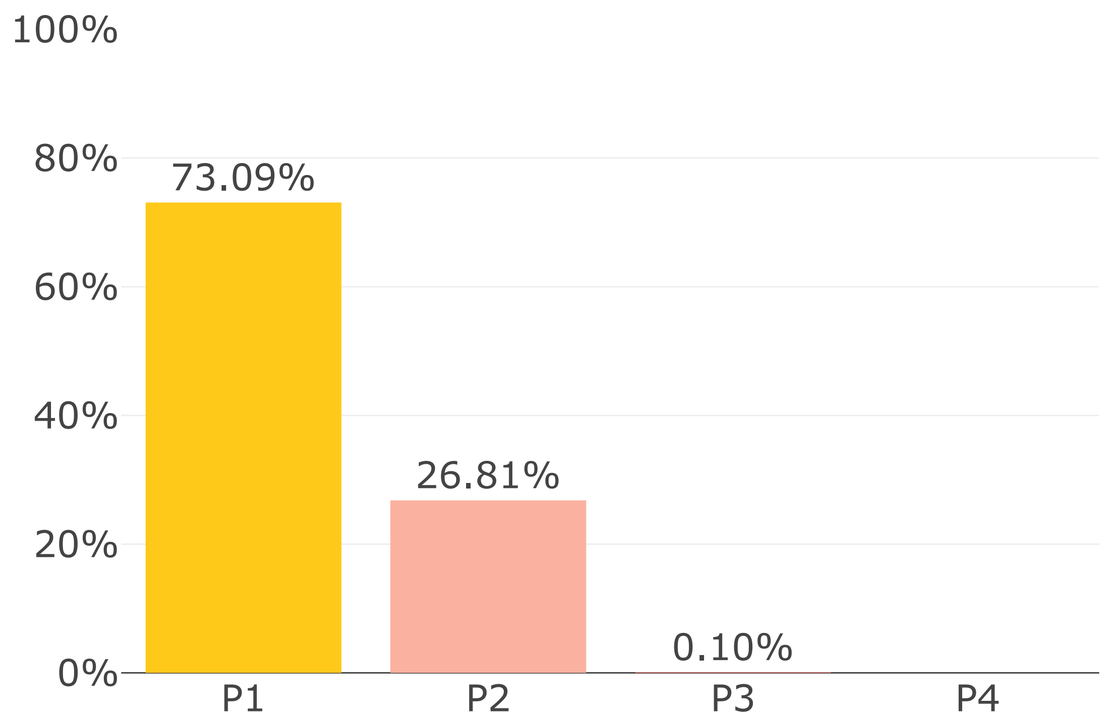


L_2 efficiency

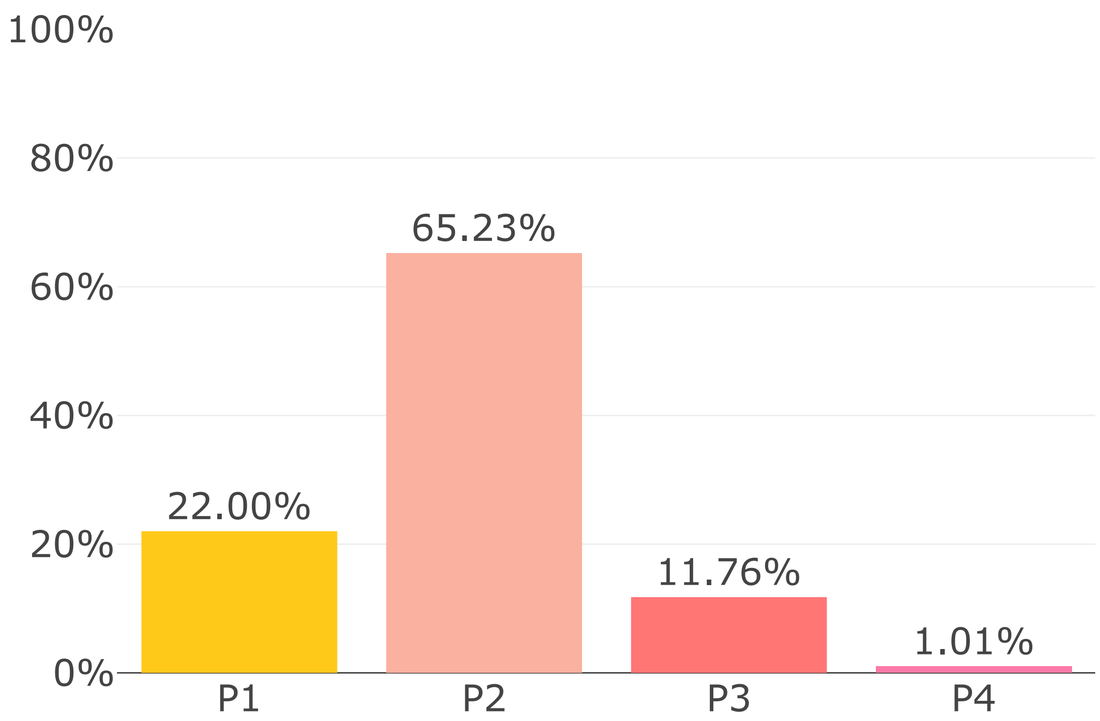
Optimized

Not Optimized

Degree Distribution

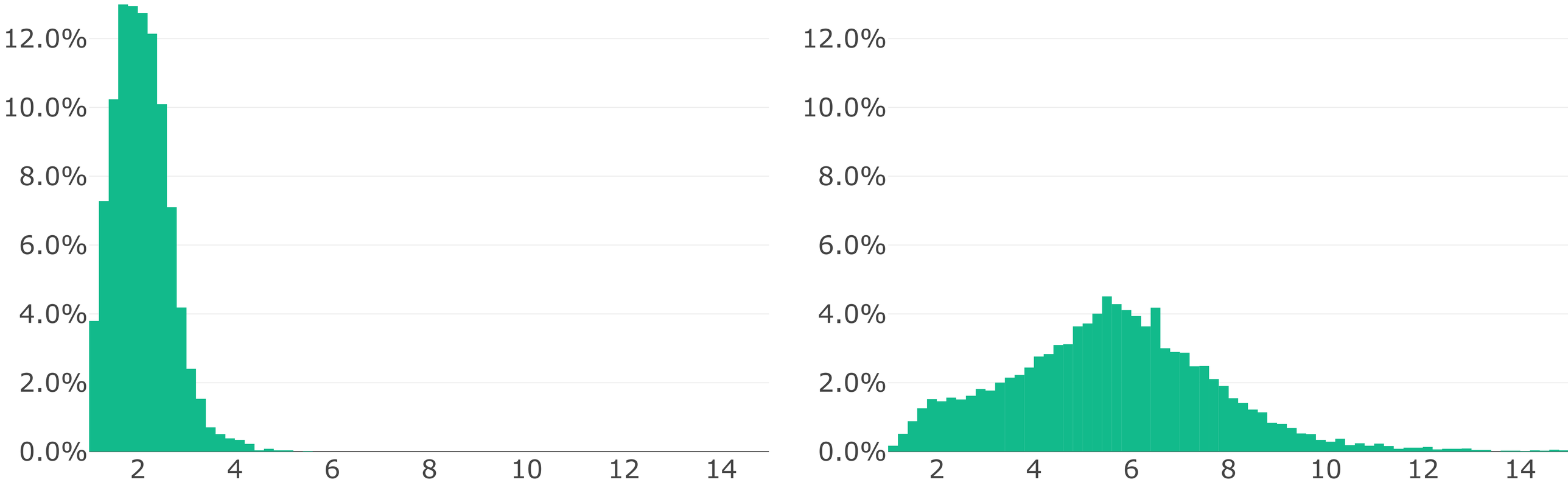


Optimized



Not Optimized

Number of DOF

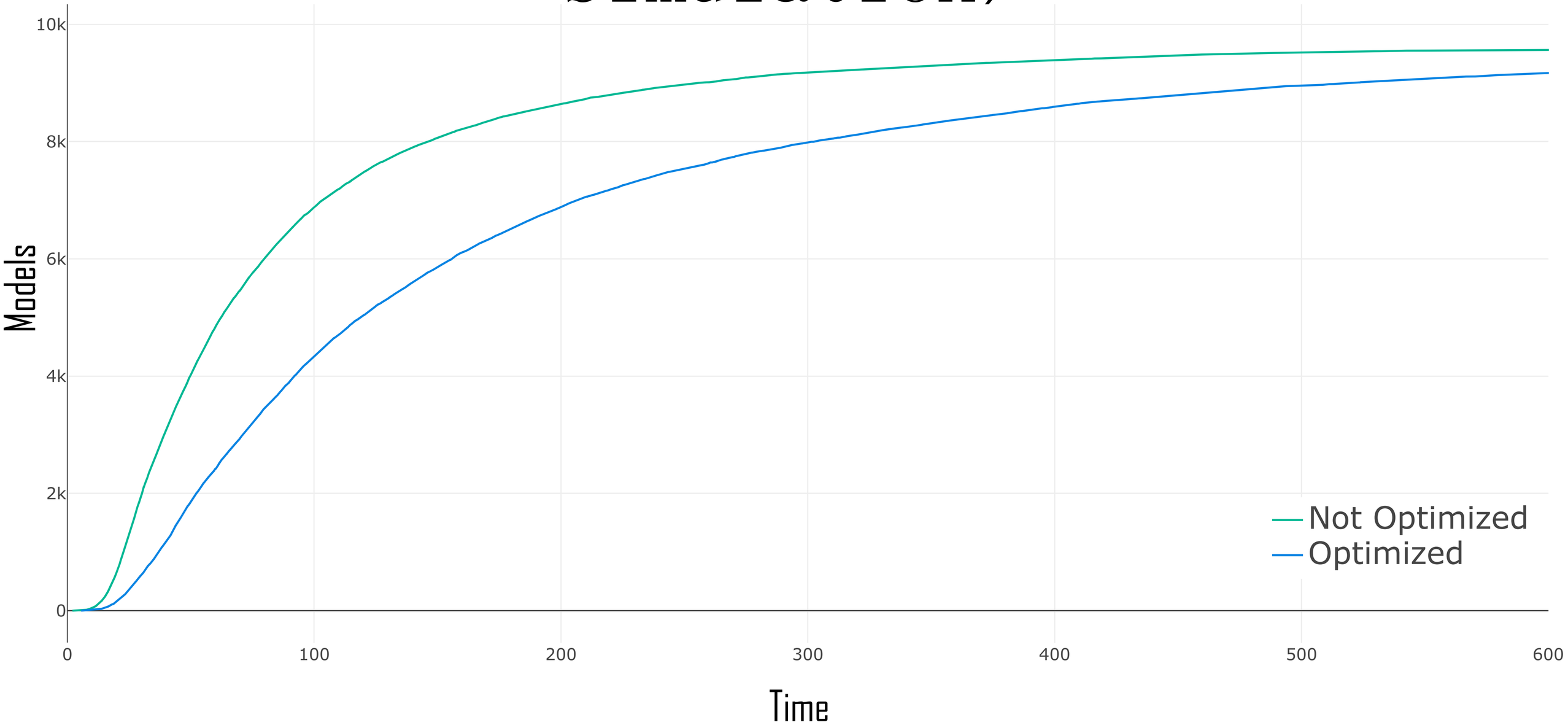


Increase in DOFs

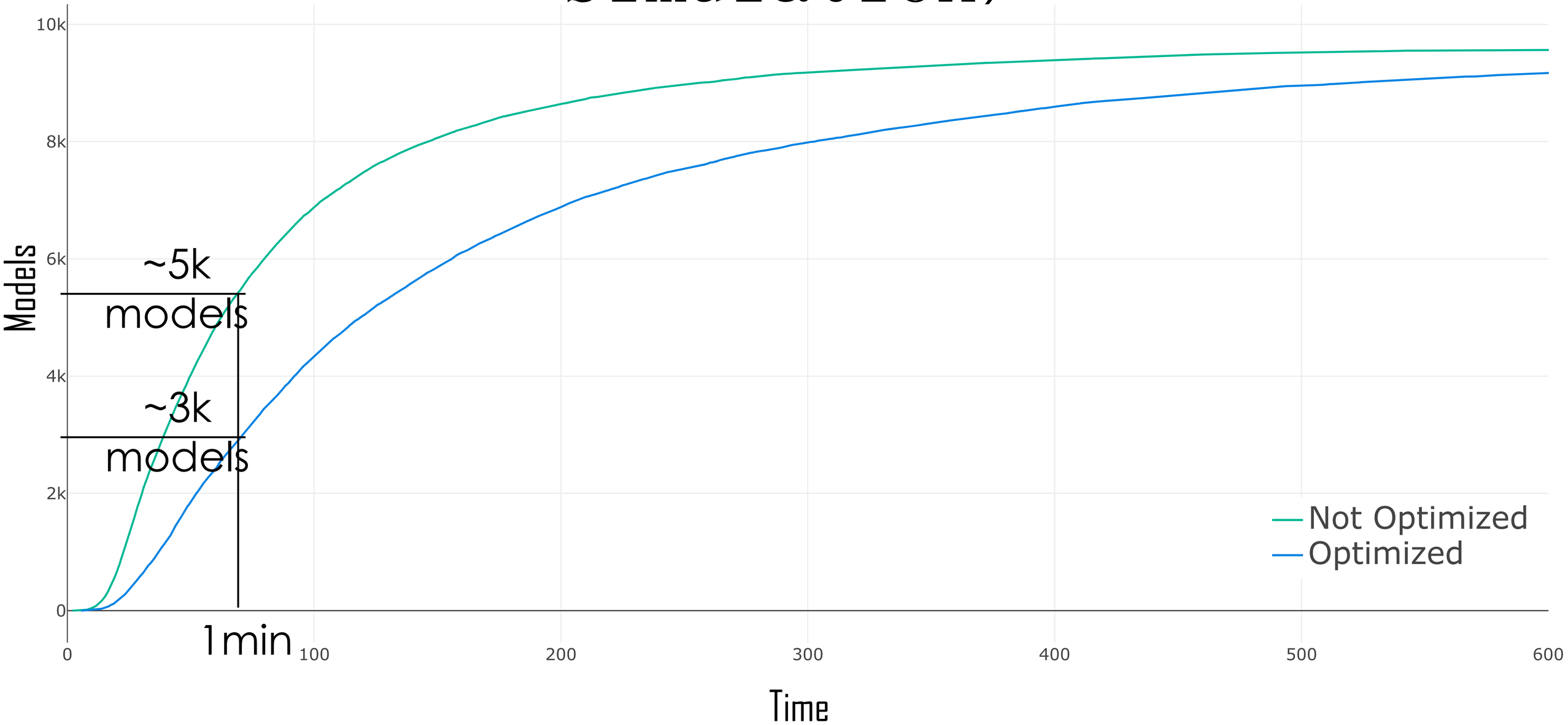
Optimized

Not Optimized

Overall Time (Meshing + Simulation)



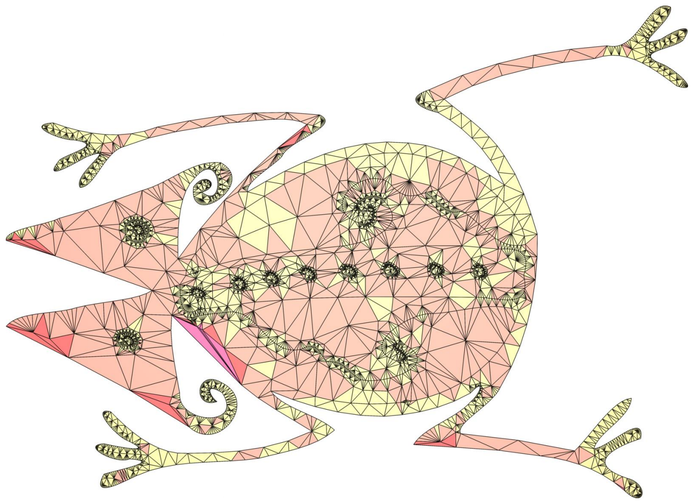
Overall Time (Meshing + Simulation)



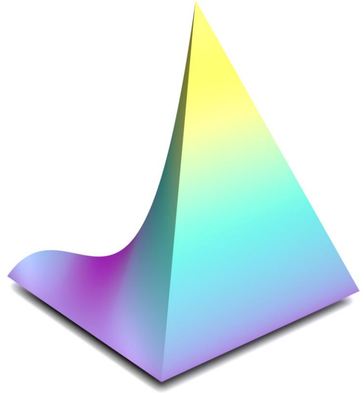
Summary

$$k = \frac{\ln\left(B\hat{h}^{\hat{k}+1}\frac{\sigma_E^2}{\hat{\sigma}^2}\right) - \ln h_E}{\ln h_E}$$

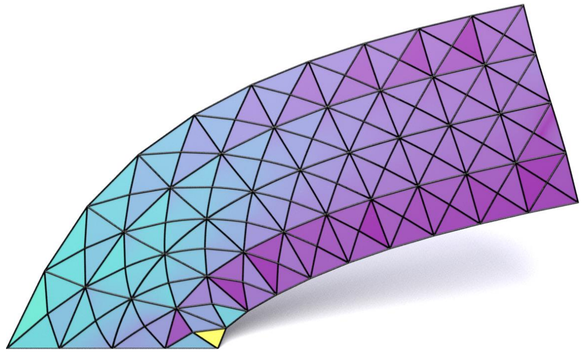
1. Use formula



2. Propagate degrees



3. Construct C^0 basis



4. Simulate!

Thank you!

Code available
http://www.github.com/polyfem