Decoupling Simulation Accuracy from Mesh Quality

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Simulating Physics

Modeling [Terzopoulos et al. 1987]

Dedormation [Bargteil et al. 2014]

Fluid dynamics [Pan et al. 2013]

Topology Optimization [Zhu et al. 2017]

Aerodynamics
Partial Differential Equation (PDE)
Partial Differential Equation (PDE)

\[ \Delta u = f, \quad f = 12x^2 \]
PDE Solving

\[ \Delta u = 12x^2 \]

\[ u = x^4 \]
Finite Element Method (FEM)

\[ \Delta u = f \]

\[ u = \frac{x^4}{x} \]
Finite Element Method (FEM)

\[ \Delta u = f \]

\[ u = \frac{4}{x} \quad ? \quad U = \sum_{i=1}^{n} u_i \phi_i \]
Finite Element Method (FEM)

\[ \Delta u = f \]

\[ u = x^4 \approx U = \sum_{i=1}^{n} u_i \phi_i \]
Three Factors Affect FEM Accuracy

- Mesh resolution
- Basis order
- Element quality
Quality Matters???
Quality Matters??
Quality Matters?
Quality Matters
Quality Matters!
Quality Matters!!
Quality Matters!!!

![Quality Matters!!]
Our Solution

Linear

Quadratic

Cubic

Quartic
Posteriori Refinement

• h-refinement [Wu 01], [Simnett 09], [Wicke 10], [Pfaff 14], ...

• p-refinement [Babuška 94], [Kaufmann 13], [Bargteil 14], [Edwards 14], ...
Priori Refinement

We increase order only based on the input
Overview

\[ k = \frac{\ln \left( B \hat{h}^{k+1} \frac{\sigma_E^2}{\hat{\sigma}^2} \right) - \ln h_E}{\ln h_E} \]

1. Use formula
Order of an element

\[ k = \frac{\ln \left( B \hat{h}^{k+1} \frac{\sigma^2_E}{\hat{\sigma}^2} \right) - \ln h_E}{\ln h_E} \]
User parameter, $\gamma = 3$

\[
k = \frac{\ln \left(\frac{B \hat{h}^k + 1}{\hat{\sigma}^2} \frac{\sigma_E^2}{\hat{\sigma}^2} \right) - \ln h_E}{\ln h_E}
\]
Average edge length

\[ k = \frac{\ln \left( B \hat{h}^{k+1} \frac{\sigma_E^2}{\hat{\sigma}^2} \right) - \ln h_E}{\ln h_E} \]
Base order, usually \(1\)

\[
k = \frac{\ln \left( B \hat{h}_k^{\text{\#}+1} \frac{\sigma^2_E}{\hat{\sigma}^2} \right) - \ln h_E}{\ln h_E}
\]
\[ k = \frac{\ln \left( B \hat{h}^{k+1} \frac{\sigma_E^2}{\hat{\sigma}^2} \right) - \ln h_E}{\ln h_E} \]

\[ \hat{\sigma}_{2D} = \sqrt{3}/6 \]

\[ \hat{\sigma}_{3D} = \sqrt{6}/12 \]
\[ k = \ln \left( B \hat{h}^{k+1} \frac{\sigma_E^2}{\hat{\sigma}^2} \right) - \ln \hat{h}_E \]

\[ \sigma_E = \frac{\rho_E}{\hat{h}_E} \]

\[ \hat{\sigma}_{2D} = \sqrt{3}/6 \]

\[ \hat{\sigma}_{3D} = \sqrt{6}/12 \]
Overview

\[ k = \frac{\ln \left( B\hat{h}^{k+1} \frac{\sigma_E^2}{\hat{\sigma}^2} \right) - \ln h_E}{\ln h_E} \]

1. Use formula

2. Propagate degrees
Degree Propagation

- For each element $E$
- Compute $k_E$ using formula
- Increase the order (if necessary) of:
  - The element $E$
  - All edge/face neighbors
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1. Use formula
2. Propagate degrees
3. Construct \( C^0 \) basis
Building Continuous Basis

Linear

Cubic

\[ \varphi_{l1} \]

\[ p_{c3} \]

\[ p_{c4} \]

\[ \varphi_{l2} \]
Building Continuous Basis

- Linear
- Cubic
Overview

\[ k = \frac{\ln \left( B \hat{h}^{k+1} \frac{\sigma_E^2}{\hat{\sigma}^2} \right) - \ln h_E}{\ln h_E} \]

1. Use formula

2. Propagate degrees

3. Construct $C^0$ basis

4. Simulate!
Laplace

Standard

Ours
Laplace

Standard

Ours
Laplace

Standard  Ours
Neo-Hookean Elasticity

Standard

Ours
Neo-Hookean Elasticity

Standard

Ours
Large Dataset

- Thingi10k
  [Zhou 17]
- Tetwild
  [Hu 18]
- ~10k Optimized
- ~10k Not Optimized
How to Measure Errors?

- Standard $L_2$ error estimate for linear elements

$$e_h = \| u - u_h \|_0 \leq C h^2 \| u \|_2$$
How to Measure Errors?

- Standard $L_2$ error estimate for linear elements

\[ e_h = \| u - u_h \|_0 \leq C h^2 \| u \|_2 \]

$L_2$ norm or average error
How to Measure Errors?

• Standard $L_2$ error estimate for linear elements

$$e_h = \|u - u_h\|_0 \leq C h^2 \|u\|_2$$

Exact solution
How to Measure Errors?

• Standard $L_2$ error estimate for linear elements

$$e_h = \| u - u_h \|_0 \leq C h^2 \| u \|_2$$

Approximated solution
How to Measure Errors?

- Standard $L_2$ error estimate for linear elements

$$e_h = \|u - u_h\|_0 \leq C h^2 \|u\|_2$$

- Different $h$ for every model!
How to Measure Errors?

- Standard $L_2$ error estimate for linear elements

$$e_h = \| u - u_h \|_0 \leq C h^2 \| u \|_2$$

- Different $h$ for every model!

- $L_2$ efficiency

$$E_{L_2} = \frac{\| u - u_h \|_0}{h^2}$$
How to Measure Errors?

- Standard $L_2$ error estimate for linear elements

$$e_h = \|u - u_h\|_0 \leq C h^2 \|u\|_2$$

- Different $h$ for every model!

- $L_2$ efficiency

$$E_{L_2} = \frac{\|u - u_h\|_0}{h^2}$$

- Independent from $h$
How to Measure Errors?

- Standard $L_2$ error estimate for linear elements

$$e_h = \| u - u_h \|_0 \leq C h^2 \| u \|_2$$

- Different $h$ for every model!

- $L_2$ efficiency

$$E_{L_2} = \frac{\| u - u_h \|_0}{h^2}$$

  Small values are good!

- Independent from $h$
Degree Distribution

Optimized

Not Optimized
Number of DOF

Increase in DOFs

Optimized

Not Optimized
Overall Time (Meshing + Simulation)

Models vs Time

- Not Optimized
- Optimized
Overall Time (Meshing + Simulation)

- Not Optimized: 1 min ~ 5k models
- Optimized: 1 min ~ 3k models
Summary

\[ k = \frac{\ln \left( B\hat{h}^{k+1} \frac{\sigma_E^2}{\hat{\sigma}^2} \right) - \ln h_E}{\ln h_E} \]

1. Use formula

2. Propagate degrees

3. Construct \( C^0 \) basis

4. Simulate!
Thank you!

Code available
http://www.github.com/polyfem