Rotation-strain Coordinate for Large Elastic Deformation

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Background

Elastic Simulation

- Traditional Industry: car, ship, plane etc.
- Film, Game, VR: Realistic digital objects.
- Modern Fabrication: 3D printing.
- Medicine, etc.



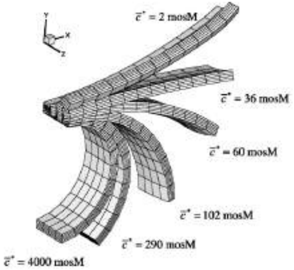
Foundation and Challenge

- Action

$$S[x, \dot{x}] = \int_0^T T - V\,dt$$

- The Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial V}{\partial x} = 0$$



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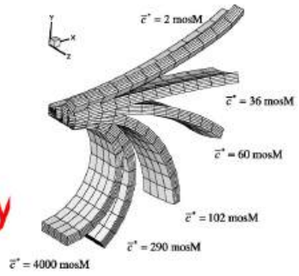
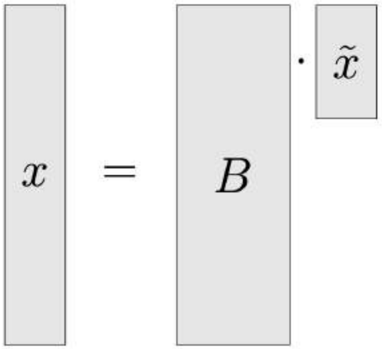
- Key challenges: Dimension and Non-linearity
  - Large number of nodes: high dimensional $x \in \mathbb{R}^N$
  - Non-linear elastic force: $f(x)$ is not linear to x



Linear Subspace

- In much smaller space $B = [B_1, B_2, \cdots, B_r]$
  - Linear combination of a few possible deformations B_i
  - A few number of variables $\tilde{x} \in \mathbb{R}^r, r \ll N$
  - Variable substitution $x = B\tilde{x}$



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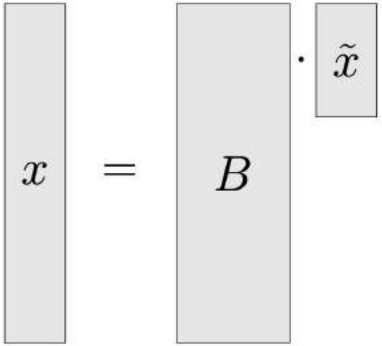
- Many methods to construct B
  - Modal Analysis
  - Modal Derivatives
  - Mass PCA
  - ...



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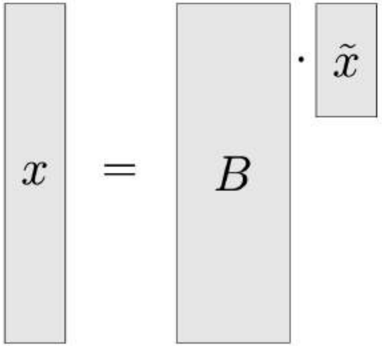
  - ...
- But nothing to do with Non-linearity



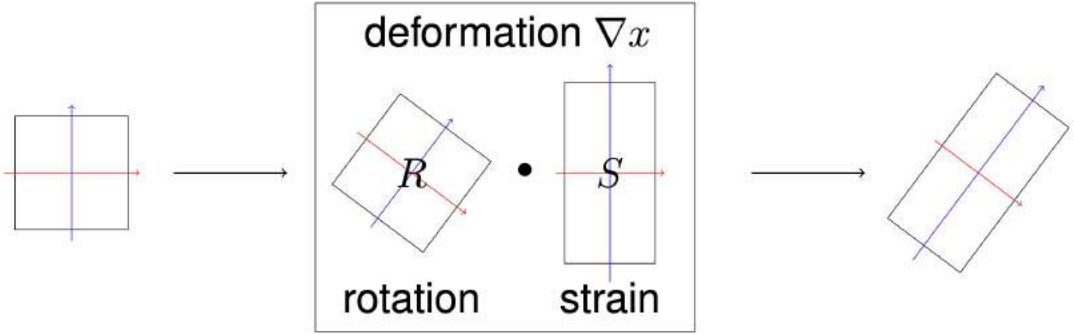
The Source of Non-linearity

- Geometric nonlinearity
  - displacement-strain relationship (Green strain, etc.)
- Material nonlinearty
  - strain-stress relationship (StVK, Neohookean, etc.)

We only consider geometric nonlinearity, with linear strain-stress relationship $\sigma = C : \epsilon$

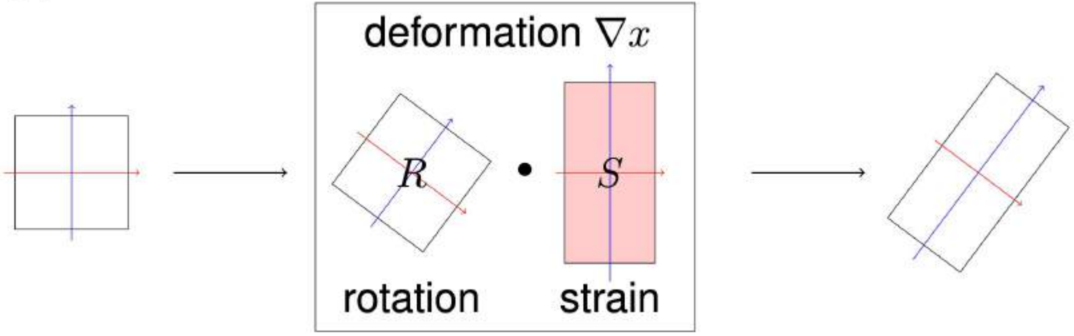
The Source of Non-linearity

- In the deformation gradient $\nabla x = RS$
  - Rigid rotation R, non-rigid strain S



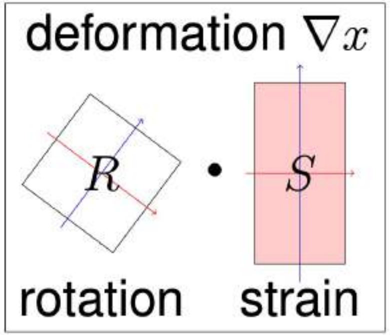
The Source of Non-linearity

- In the deformation gradient $\nabla x = RS$
  - Rigid rotation R, non-rigid strain S
- The potential energy just measures the strain S
  - $V(x)$ should not depend on rotation R



The Source of Non-linearity

- Typical schemes to remove R from V
  - Co-rotational: $R^T \nabla x = S$
  - Cauchy-Green: $\nabla x^T \nabla x = S^2$
  - Invariants: e.g. $\det(\nabla x)$
- Rotation and strain are mixed in ∇x
  - Non-linear operation needed to decouple them



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- Rotation and strain are mixed in ∇x
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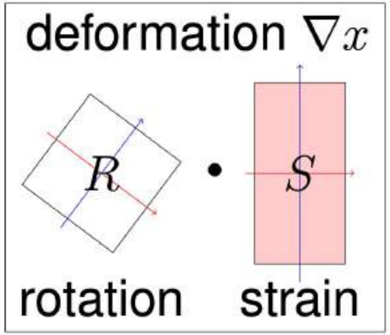
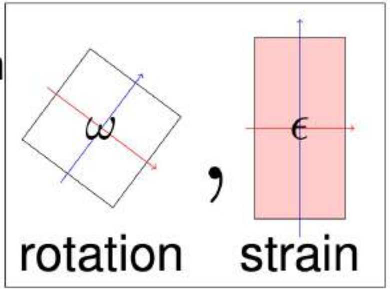
- Rotation-Strain Coordinates $y = (\omega, \epsilon)$
  - Separately encode the rotation and strain

$$\boxed{\omega = \log(R), \epsilon = S - Id}$$





From y back to x

- Solve Poisson problem

$$x = \underset{x^*}{\operatorname{argmin}} \int_\Omega \|\nabla x^* - \underbrace{\exp(\omega)}_{\color{red}{\mathbf{R}}} \underbrace{(\epsilon + Id)}_{\color{blue}{\mathbf{S}}} \|^2, \quad \text{s. t.} \frac{\int_\Omega x dV}{\int_\Omega dV} = c$$

From y back to x

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$$x = \Psi(y)$$

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$$x = \Psi(y)$$

- Part of nonlinearity in $V(x)$ has been shifted into Ψ

Reduced RS coordinate

- For infinitesimal deformation

$$\nabla x = Id + \nabla u = Id + \underbrace{\frac{\nabla u + (\nabla u)^T}{2}}_{\epsilon(u)} + \underbrace{\frac{\nabla u - (\nabla u)^T}{2}}_{\omega(u)}$$

Reduced RS coordinate

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- Modal basis $W \in \mathbb{R}^{3N \times r}$
  - A subset of small deformation modes

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- Modal basis $W \in \mathbb{R}^{3N\times r}$
  - A subset of small deformation modes
- Form modal coordinate to Euclidean coordinate

$$z \xrightarrow{W} \bar{u} \xrightarrow{Q} (\omega, \epsilon) \xrightarrow{\exp} \nabla \bar{x} \xrightarrow{\Psi} x$$

As geometric warping

- Fast integration: Cubature *[Ann et al. 2008]*

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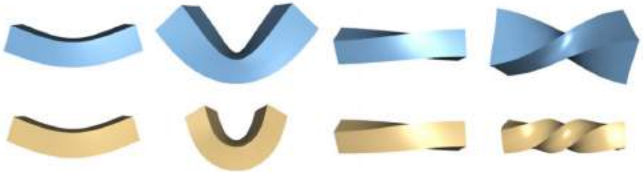
- Warping for subspace deformation



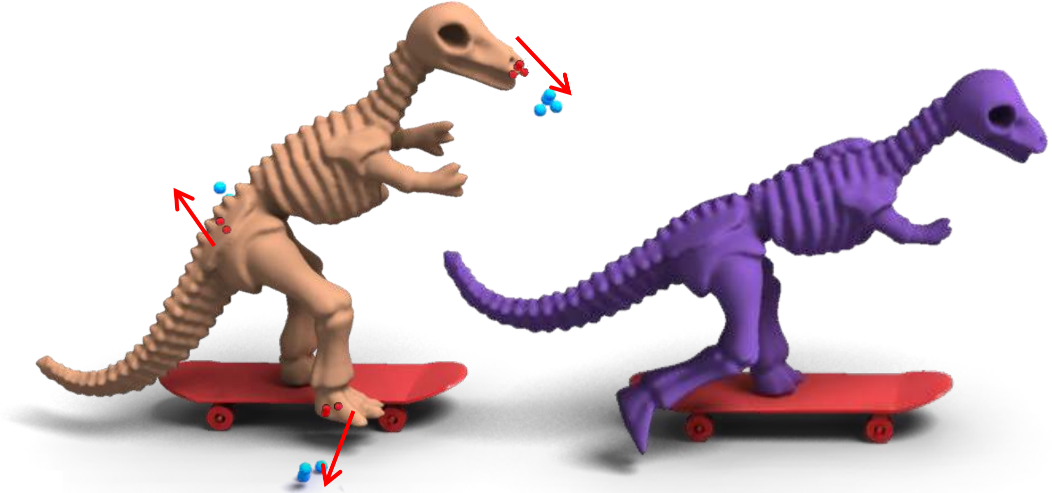
Space-time editing [Li 2014]

Introduction

Space-time editing

- Powerful tool for animation editing
- Seeking minimal control forces
- Matching the constraints in space-time.

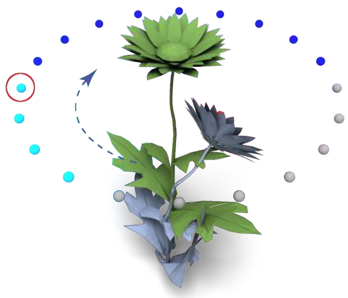
Dynamic or static input animation



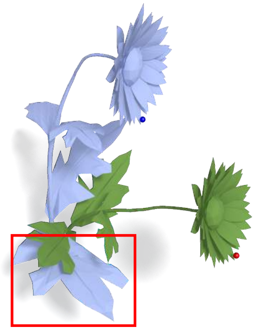
Positional and/or keyframe constraints

Unsolved problems in practice

- Scalability for complex models
- Lack of control due to linearization



[Barbič et al. 2012]



[Li et al. 2013]

- Elastic material significantly affects animation
  - What is the *right* material?

Technical contributions

We propose two new techniques to solve these problems.

- ***Reduced RS (Rotation-Strain) approach***

  Provides tight positional constraints under large deformation.

- ***Material Optimization***

  Provides physically plausible and consistent results.

Space-Time Editing

For efficiency, we formulate the problem in modal coordinates

$$\arg\min_{z} E_f(z) + \gamma E_c(z)$$

Modal coordinates → z

$$\begin{cases} E_f(z) = \dfrac{1}{2} \displaystyle\sum_{i=2}^{Frames-1} \|\ddot{z}_i + D\dot{z}_i + \Lambda z_i\|_2^2 \\ E_c(z) = \dfrac{1}{2} \displaystyle\sum_{(i,j)\in C} \left\| u_i^j(W, z_i) - \hat{u}_j^i \right\|_2^2 \end{cases}$$

Measures control forces

Measures error in constraints

Space-Time Editing

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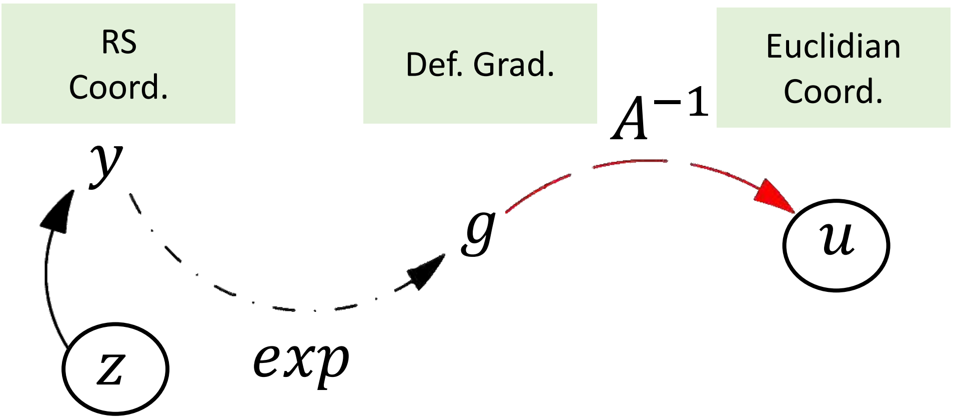
Euclidean coordinates reconstruction

- Must be robust to large deformation.
- Should only require local evaluations for efficiency.

Rotation-Strain



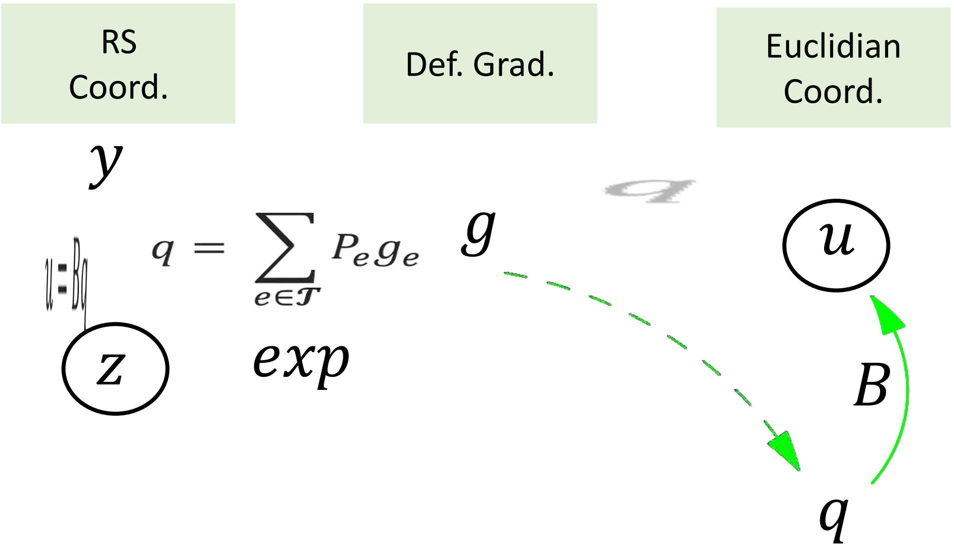
Proposed by [Huang et al. 2011].

- Compute y, g for all elements.
- Solving global linear eq.

$$Au = G^T \bar{V} g(y)$$

Inefficient for local evaluations

Reduced Rotation-Strain



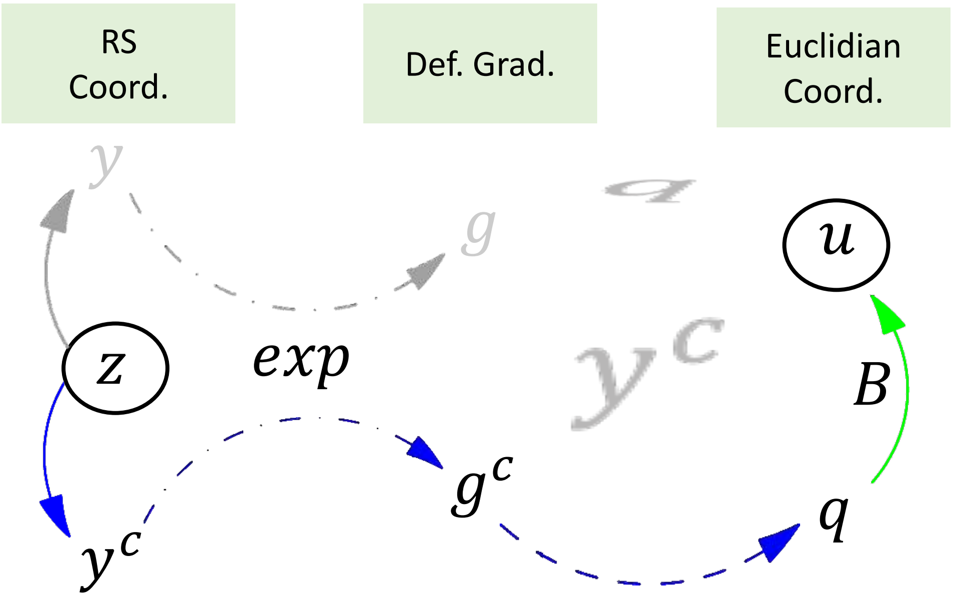
- Geometric reduction.
- Cubature method.

$$u = Bq$$

$$q = \sum_{e \in \mathcal{T}} P_e g_e$$

Avoid global linear solve.

Reduced Rotation-Strain



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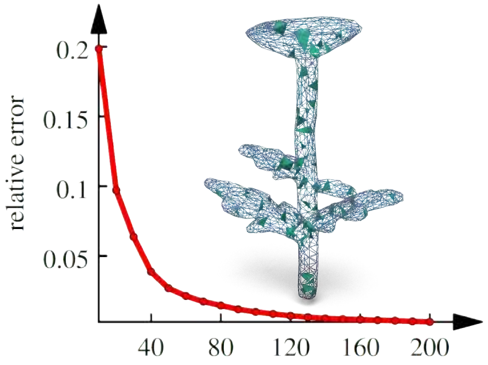
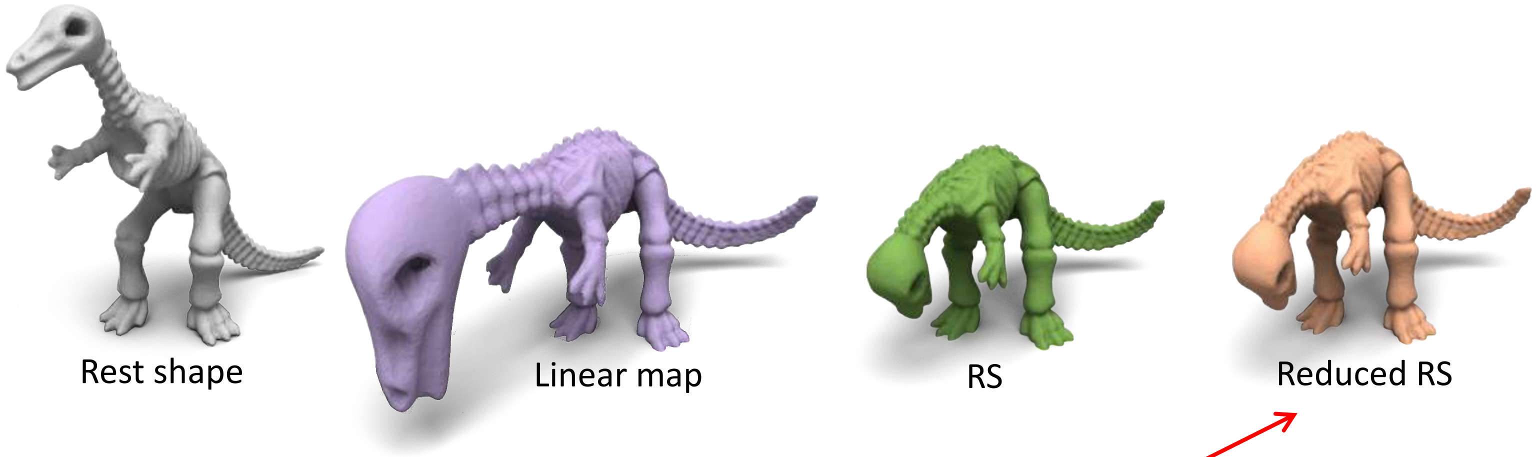


Compute RS for cubature only.
Partial shape reconstruction

Comparison

Different method for the mapping from z to u



Reduced RS method

- Robust to large deformation.
- Two orders of magnitude faster than full RS.
- Allows local evaluations of 3D coordinates.

Material optimization

How to pick a good elastic material?

Introduce material as new DOFs, and Optimize!

$$\arg\min_{z,\Lambda,D,W} E_f(z,\Lambda,D) + \gamma E_c(W,z)$$

Material in modal space: frequency, damping, and modal basis.

Material optimization

How to set the proper elastic material?

Introduce material as new DOFs, and Optimize it!

$$\arg\min_{z,\Lambda,D,W} E_f(z,\Lambda,D) + \gamma E_c(W,z)$$

Material in modal space: frequency, damping and modal basis.

Dimension is too large, so introduce ***basis sampling,***

$$W = \widehat{W} S$$

Optimize smaller sampling basis S instead of W.

Material optimization

Formulation for material optimization

Regularization term.

$$\arg\min_{z,\Lambda,D,S} E_f(z,\Lambda,D) + \gamma E_c(S,z) + \mu E_s(S)$$

$$subject\ to\ \lambda_k, d_k \geq 0\ \forall k \in [1,r]$$

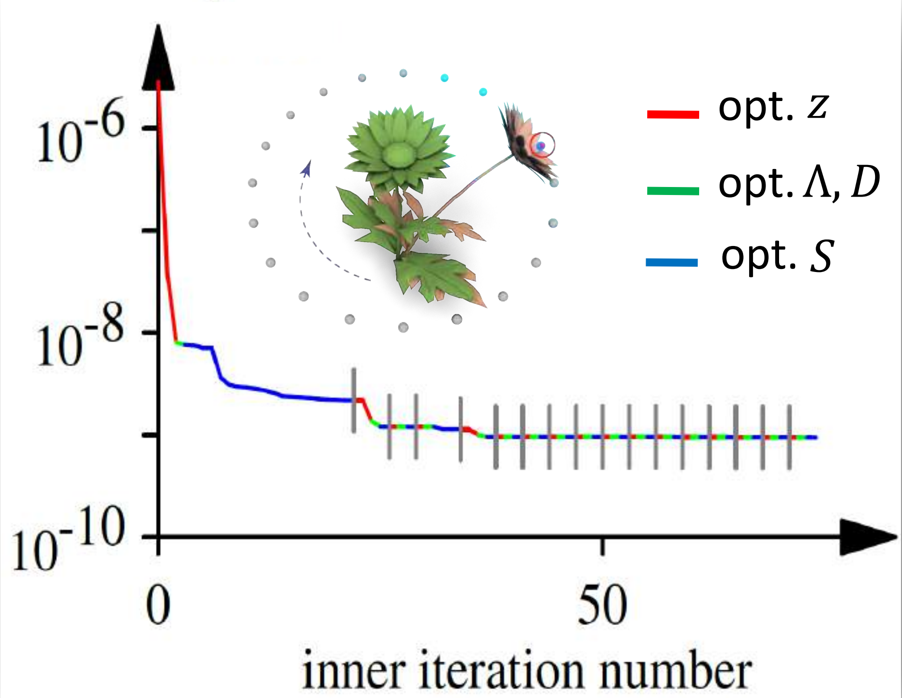
Nonlinear, but all variables are in subspace.

Numerical method

Optimize the variables one by one

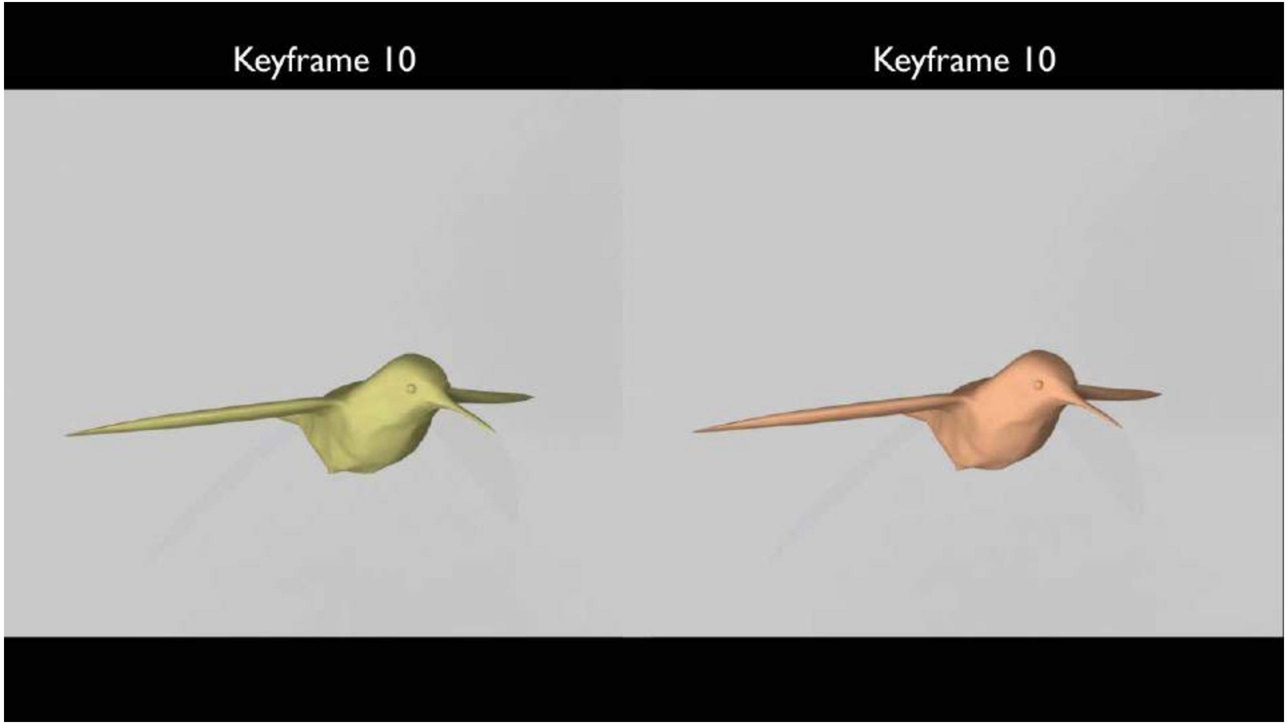
- Fix Λ, D, S, optimize z
- Fix z, S, optimize Λ, D
- Fix z, Λ, D, optimize S

Guarantees monotone decrease!



Animation editing

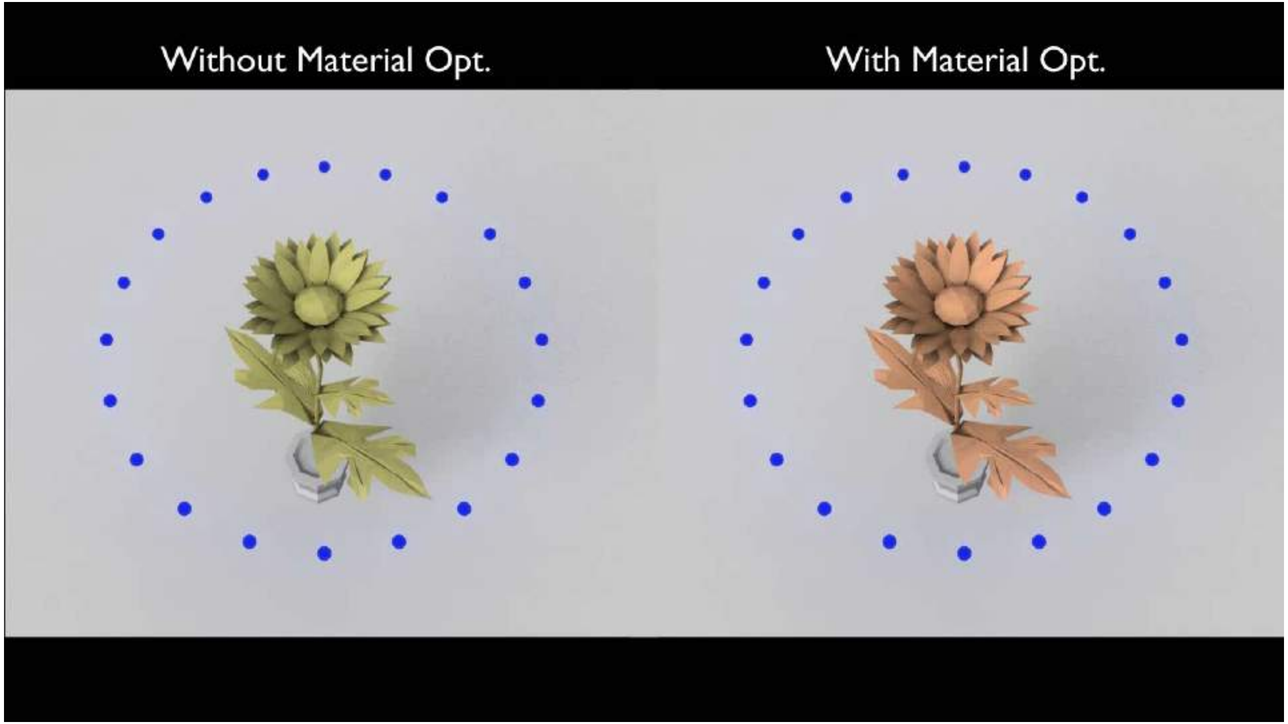
With material optimization, the resulting animation is more consistent!



Animation editing

With material optimization, the resulting animation is more ***consistent***!

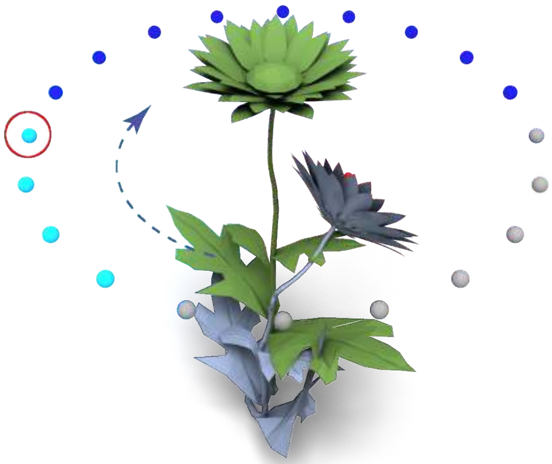
Abrupt suddenly



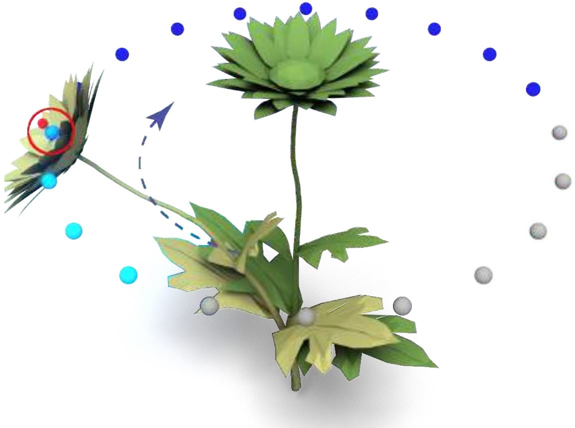
Keep circular motion

Comparison

Our method provides ***tight positional constraints***, even for large edits.



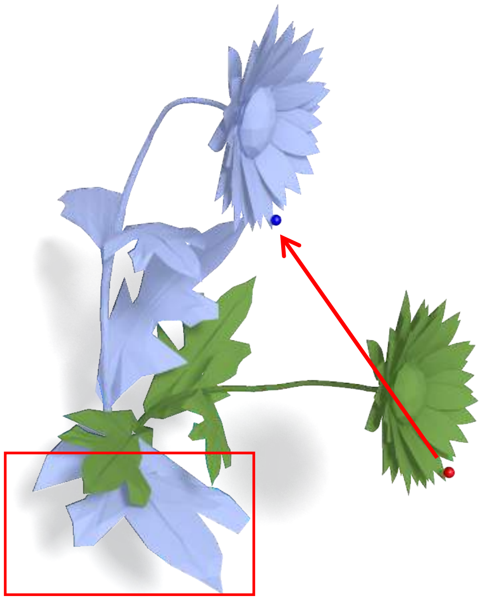
[Barbič et al. 2012]



Our result

Comparison

Our method supports ***large edits*** without visual artifacts.



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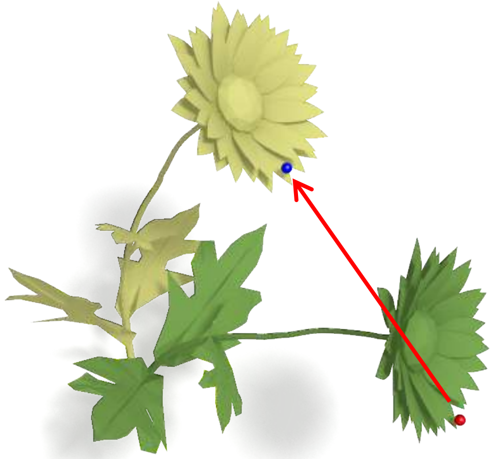
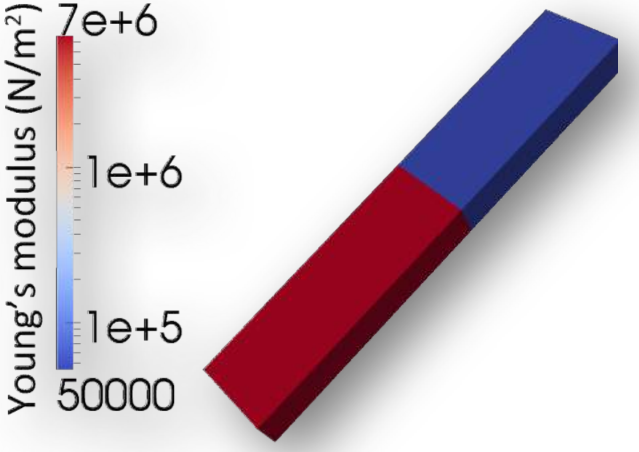


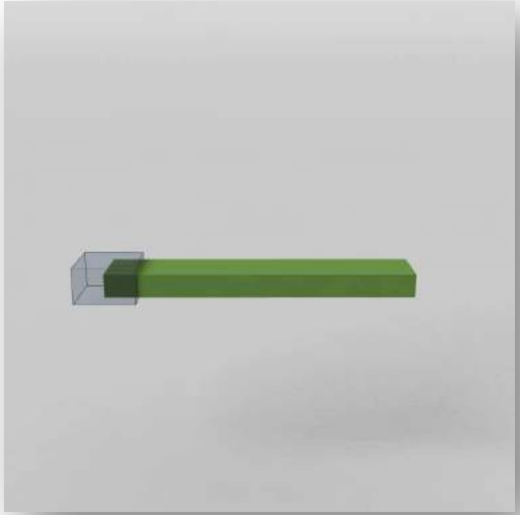
Our result

Recovering material parameters

We use input animation as keyframes in the space-time editing.



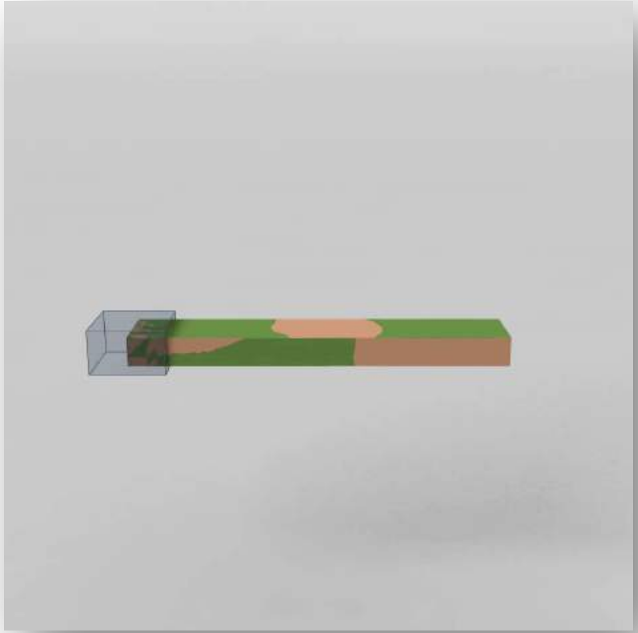
Non-uniform material for the experiments.



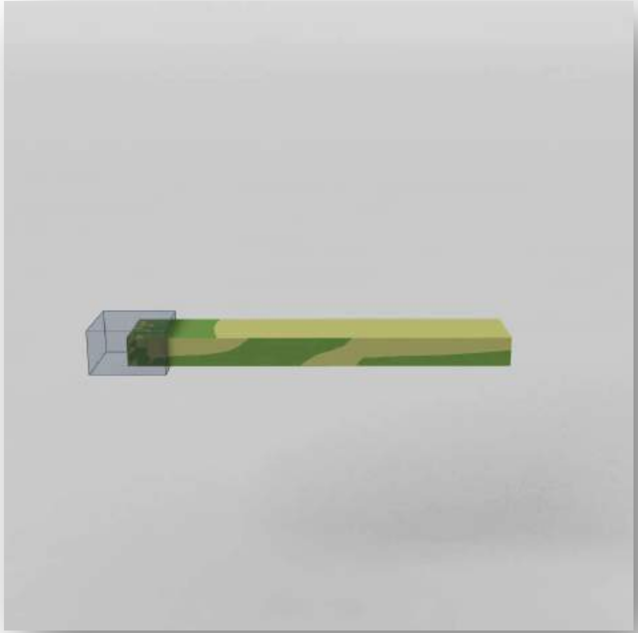
Input animation, with
first 150 frames as constraints,
last 150 frames for comparison.

Recovering material parameters

Compare the simulated results with the last 150 frames.



Recovered material



Uniform material

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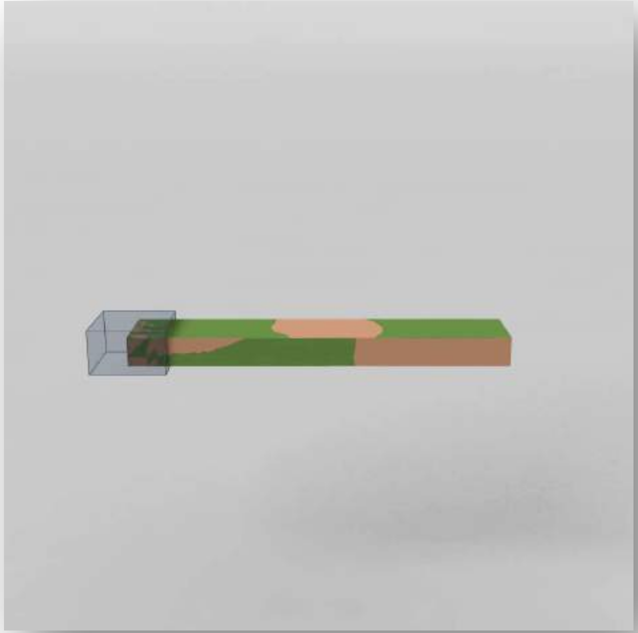
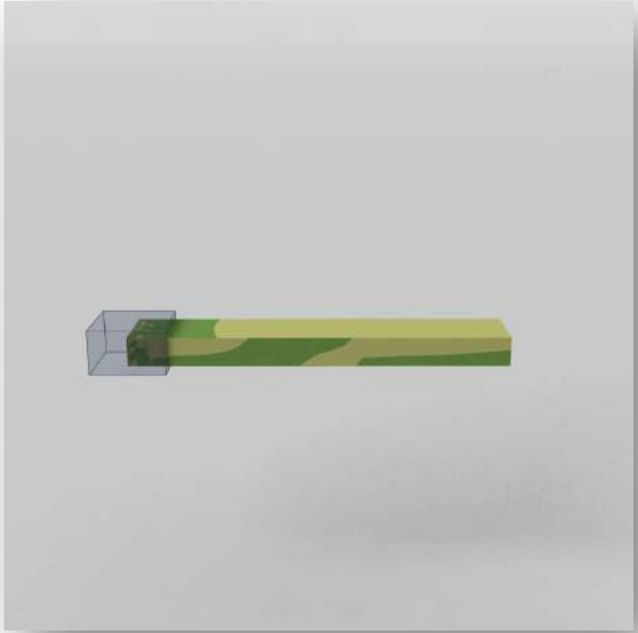


Recovered material



Uniform material

T.rex King Kong

General simulation [Pan 2015]

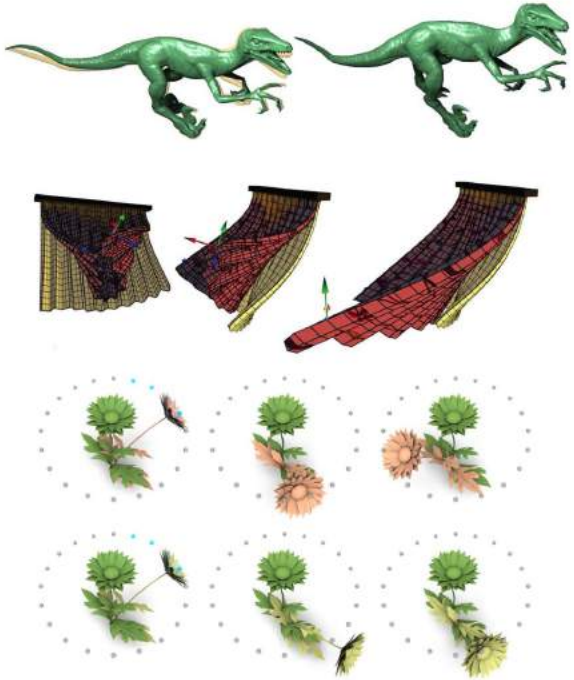
Rotation-Strain Space

- RS as general coordinate

$$y = (\omega, \epsilon)$$

- Potential energy $V(y) = V(\epsilon)$
  - Reduce the non-linearity a lot!
  - For linear elastic material:

$$V(y) = \frac{1}{2} y^T \tilde{K} y$$



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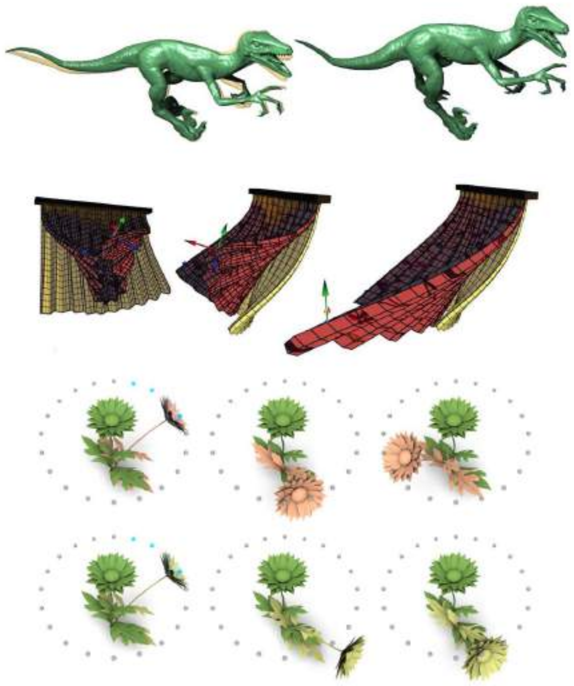
- Kinetic energy T?



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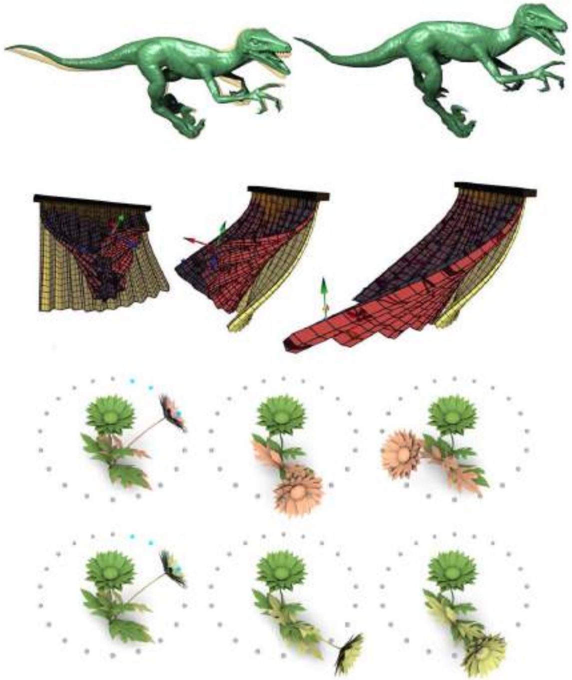
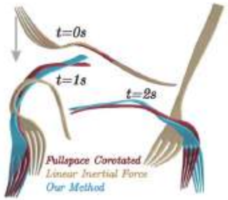
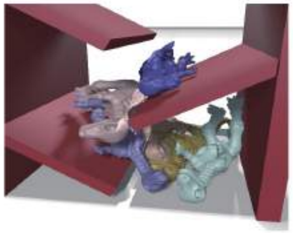
- Kinetic energy T?
- External force g?



Contributions of This Paper

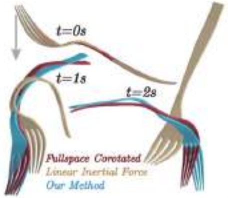
- A full featured dynamical model in RS space
  - Non-linear elastic material
  - Correct kinetic energy
  - External force and constraints

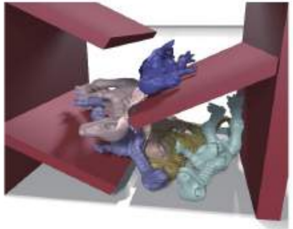


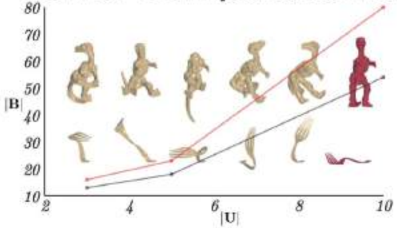


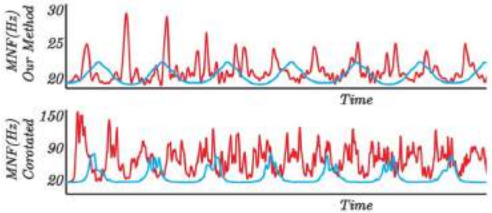
Contributions of This Paper

- A full featured dynamical model in RS space
  - Non-linear elastic material
  - Correct kinetic energy
  - External force and constraints
- Advantages
  - Lower dimensional subspace for dynamics
  - Lower computational complexity









Non-linear Elastic Material

- Directly apply to strain ϵ
  - StVK material using Green-Lagrange strain $\frac{1}{2}(J^T J - I) = \frac{1}{2}\epsilon^2 + \epsilon$
- Replace Green-Lagrange strain by Cauchy strain
  - Less non-linearity but still rotation invariant

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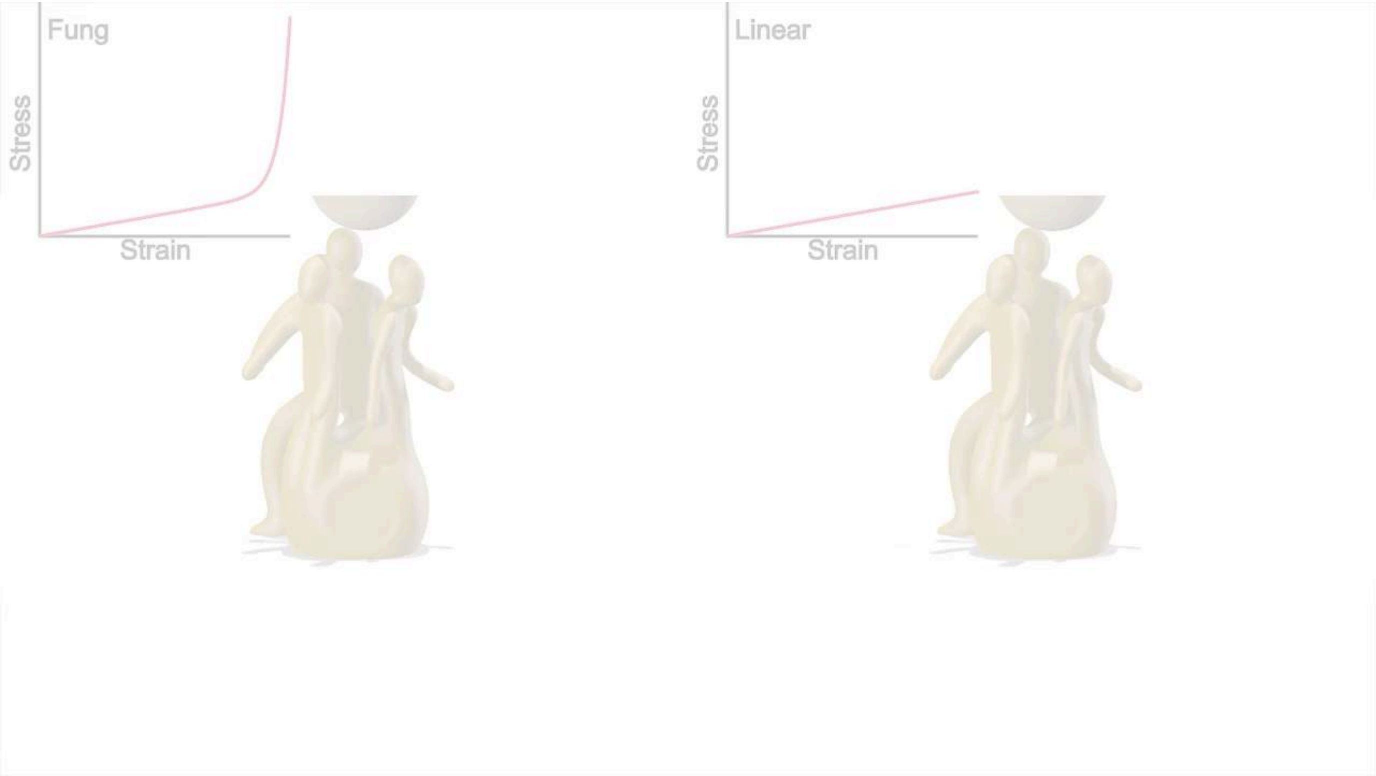
- Replace Green-Lagrange strain by Cauchy strain
  - Less non-linearity but still rotation invariant
- For example, Fung's model

$$W_{StVK}(J, \mu_1, \lambda_1) + c\left(e^{W_{StVK}(J,\mu_2,\lambda_2)} - 1\right)$$
$$\Longrightarrow$$
$$\frac{1}{2}y^TK_1y + c\left(e^{\frac{1}{2}y^TK_2y} - 1\right)$$

Video



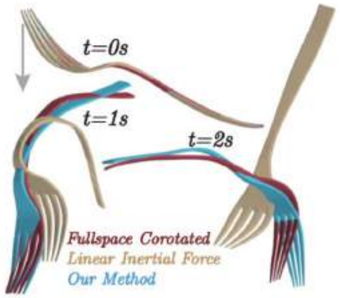
Correct Kinetic Energy

- Substitute y into $T(\dot{x})$

$$T(\dot{y}, y) = \frac{1}{2}\dot{x}(y)^T M \dot{x}(y) = \frac{1}{2}\dot{y}^T \left(\frac{\partial x}{\partial y}^T M \frac{\partial x}{\partial y} \right) \dot{y}$$

- From the Lagrange $L = T - V$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = 0 \Longrightarrow \boxed{\frac{\partial x}{\partial y}}^T M \ddot{x} + \frac{\partial V}{\partial y} = 0$$



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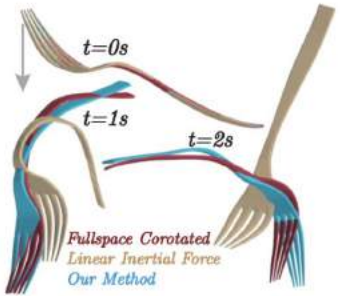

- More details in the paper
  - Coupling with floating frame
  - Temporal discretization

External Force and Constraints

- Force g: gradient of a potential energy V_g
  - E.g. gravity g: $V_g(x) = mg^T x$
- Constraints
  - Equality $c(x) = 0$
  - Inequality $c(x) > 0$

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  - Inequality $c(x) > 0$
- Involve the derivatives

$$\frac{\partial V_g}{\partial y} = \frac{\partial V_g}{\partial x}\frac{\partial x}{\partial y}, \quad \frac{\partial c}{\partial y} = \frac{\partial c}{\partial x}\frac{\partial x}{\partial y}$$

Accelerate the Integration

- Identifying bottleneck
  - Coordinates transformation

$$(\epsilon, \omega) \xrightarrow{\tilde{\nabla}} \nabla \mathbf{x} \xrightarrow{\Phi} (x_1, x_2, x_3)$$

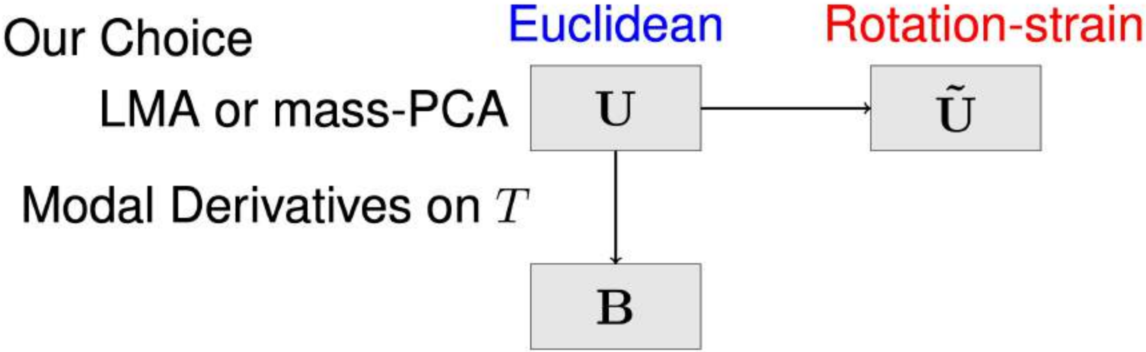
- Energy evaluation
  - Potential energy $V(\epsilon)$
  - Kinetic energy $T(x(y))$

Inspiration

- Existing methods for Euclidean space simulation
  - Linear modal analysis *[Pentland and Williams 1989]*
  - Modal derivatives *[Barbič and James 2005]*
  - Polynomial precomputation *[Barbič and James 2005]*
  - Cubature *[Ann et al. 2008]*
- Natural to transfer these methods from Euclidean to RS.

Choice of Basis

- Any set of basis can be used for x and y respectively
  - Set $\tilde{\mathbf{U}}, \mathbf{B}$ to identity matrix $\Rightarrow$ Full space
  - Fewer basis $\Rightarrow$ Linear subspace
  - Expressivity and number of basis?
- Our Choice



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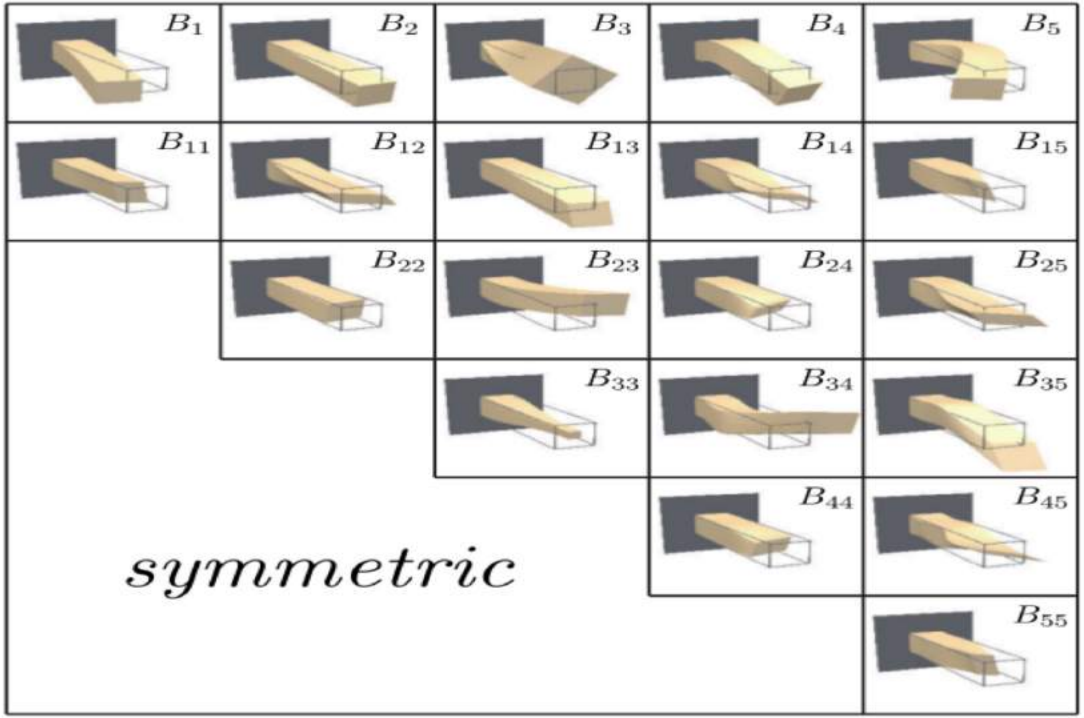
Euclidean → Rotation-strain

LMA or mass-PCA: $\mathbf{U}$ → $\tilde{\mathbf{U}}$

Modal Derivatives on T

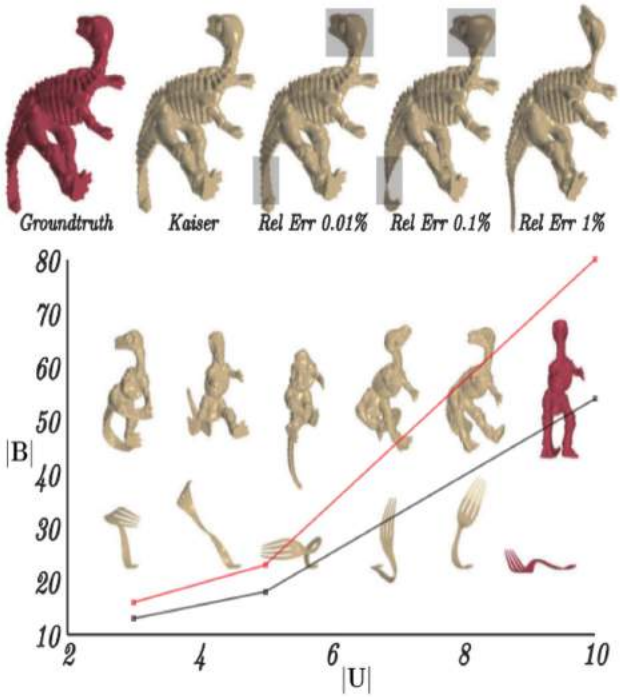
$$\left[(\nabla^T D \nabla)^{-1} \nabla^T D \frac{\partial \tilde{\nabla}}{\partial y_i}(\mathbf{0}), (\nabla^T D \nabla)^{-1} \nabla^T D \frac{\partial \tilde{\nabla}}{\partial y_i y_j}(\mathbf{0}),\right]$$

Choice of Basis



Validation

- Experiment Settings
  - Run simulation with $\mathbf{B} = \text{Id}$ and $\tilde{\mathbf{U}} = 3, 5, 10$
  - Reconstruct the shape using resonably small subspace $\mathbf{B}$
  - Truncation criterion
    - $\lambda_i > 1$
    - $\lambda_i > 0.01\%\lambda_{max}$



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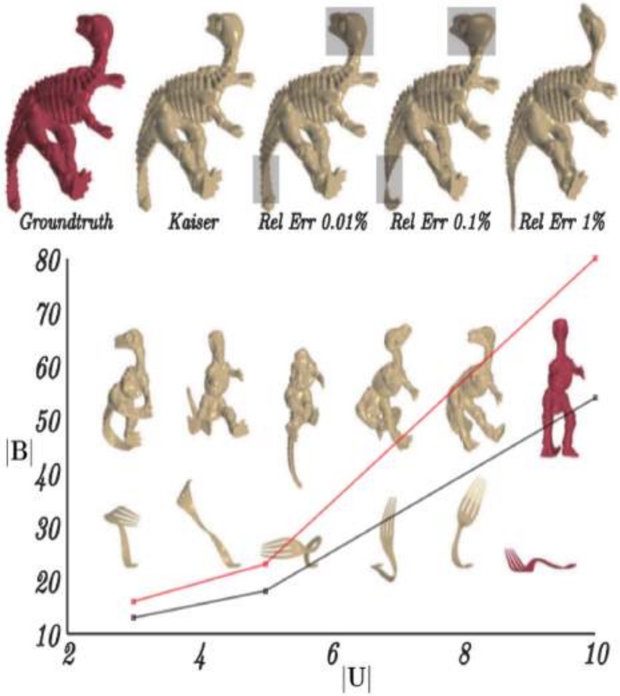
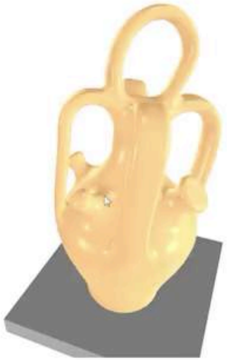
$$|\mathbf{B}| = \mathcal{O}(|\tilde{\mathbf{U}}|^2)$$

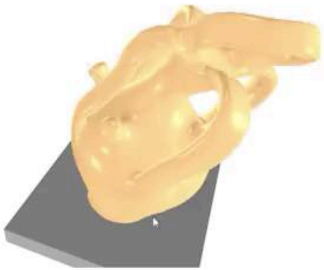


Realistic Dynamics

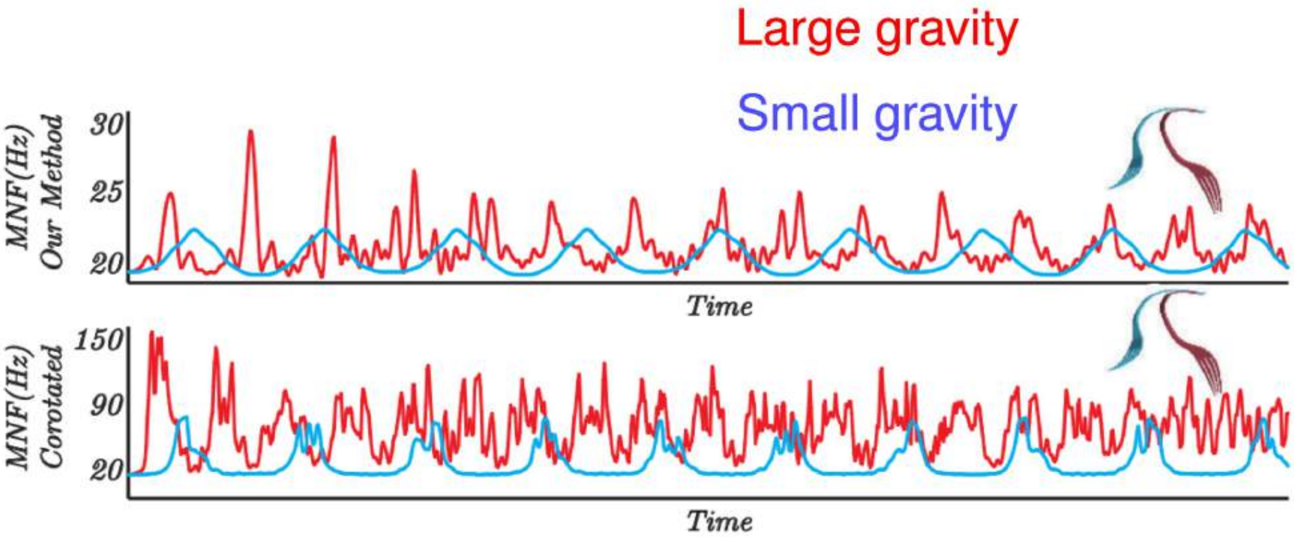
Our Method p=3 q=2
FPS: 25



Our Method Cubature
FPS: 88

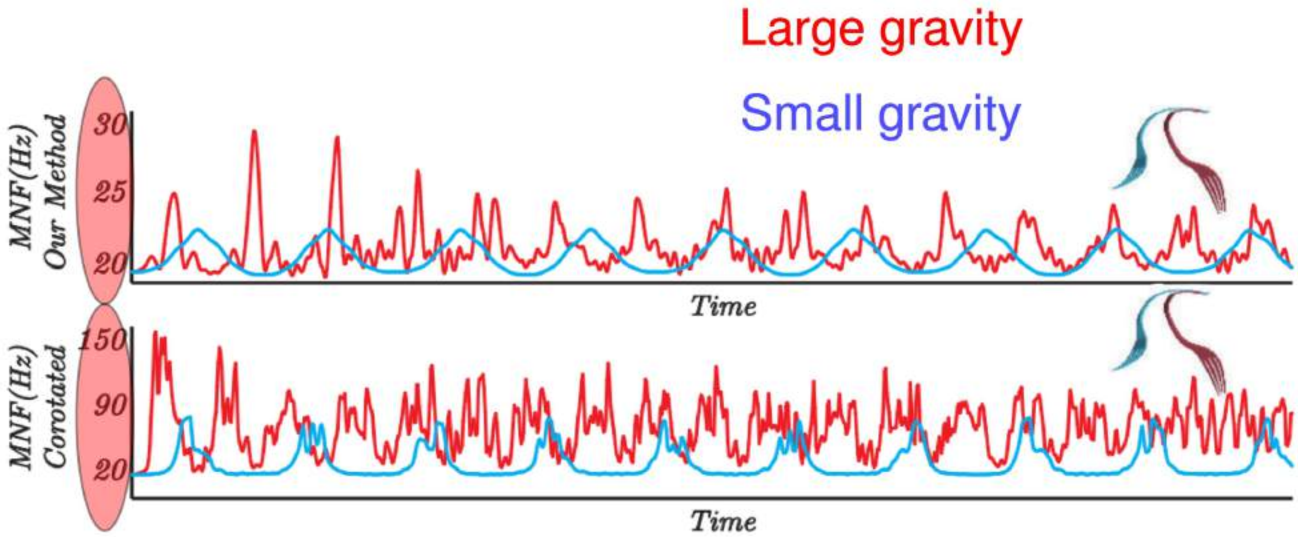


Results



- 2s simulation run on the fork model
- Numerical indicator for nonlinearity $\rightarrow$ MNF: Minimal Natural Frequency
- $\sqrt{\lambda^*},\ \lambda^* = \text{argmin}_\lambda \, \lambda \mathbf{M}\mathbf{v} = \mathbf{K}\mathbf{v}$

Results



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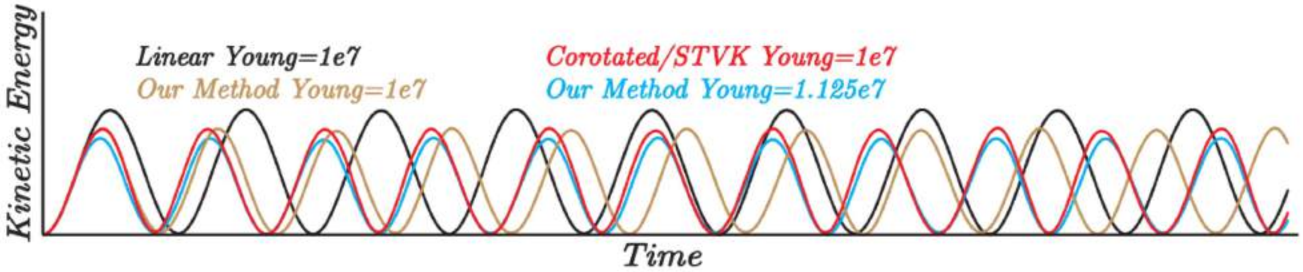
Linear Young=1e7
Our Method Young=1e7
Corotated/STVK Young=1e7
Our Method Young=1.125e7
Kinetic Energy
Time

Future work

- Intrinsic representation of elasticity
  - Redundant DOFs: 9|T| v.s 3|N|
  - Pure strain representation
  - Embeddable condition (integratable condition)
- Physically accurate warping
  - Change rotation extrapolation function, e.g. Cayley mapping
  - Introduce material-aware metric for Poisson construction

Thank you!