Quadrilateral Mesh Generation: Meromorphic Quartic Differential and Abel-Jacobi Condition

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Thanks

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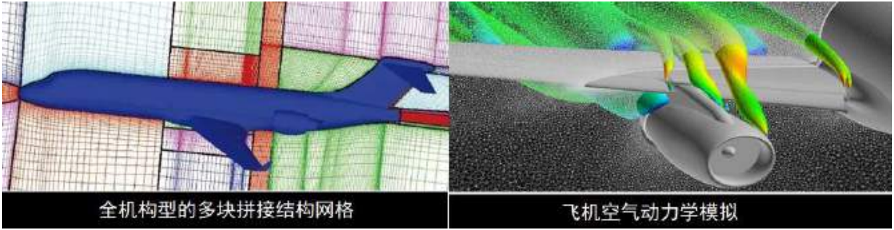
and many students.

References

- W. Chen, X. Zheng, J. Ke, N. Lei, Z. Luo, X. Gu. "Quadrilateral Mesh Generation I : Metric Based Method", *Computer Methods in Applied Mechanics and Engineering*, V356:652-668, 2019.
- N. Lei, X. Zheng, Z. Luo, F. Luo, X. Gu "Quadrilateral Mesh Generation II : Meromorphic Quartic Differentials and Abel-Jacobi Condition", https://arxiv.org/pdf/1907.00216.pdf

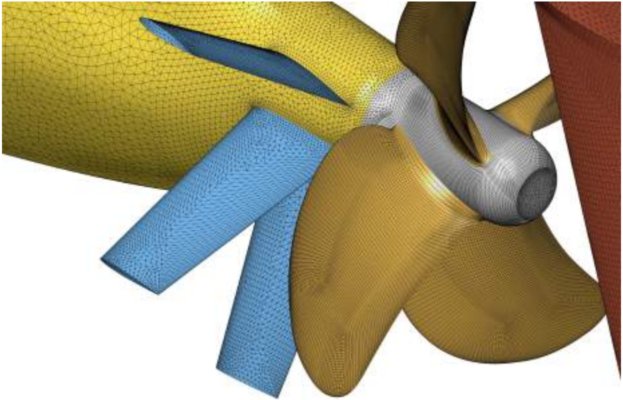
Motivation

Simulation



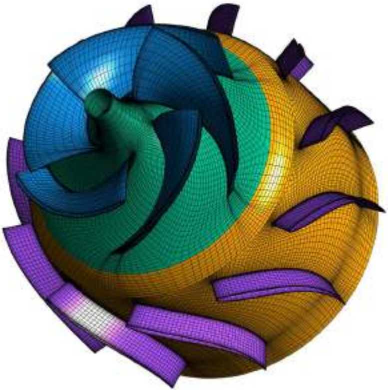
Numerical simulation is very important in flight vehicle design and engineering. After we have a designed CAD model, the first step is to convert the designed models and the external flow fields into meshes and then to use computational fluid dynamics software to simulate. In the whole procedure, meshing step cost 70% time and human power for manufacture industry, such as Boeing.

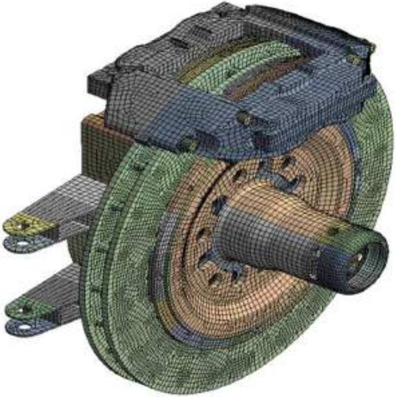
Simulation



Triangular/Tetrahedral meshes and quadrilateral/hexahedral meshes have been widely used in simulation.

Simulation



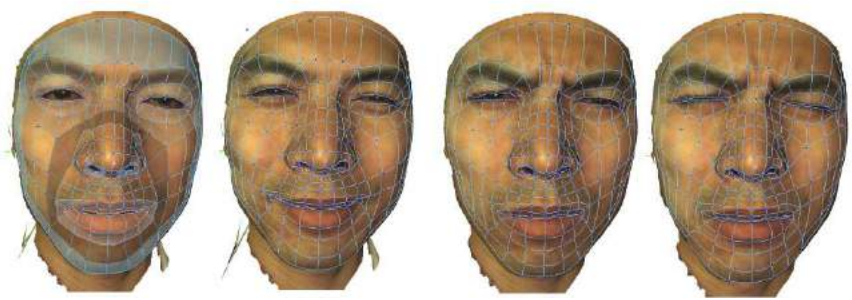


Comparing to triangular meshes, quadrilateral meshes have many advantages.

Advantages

Advantages of Quad-mesh

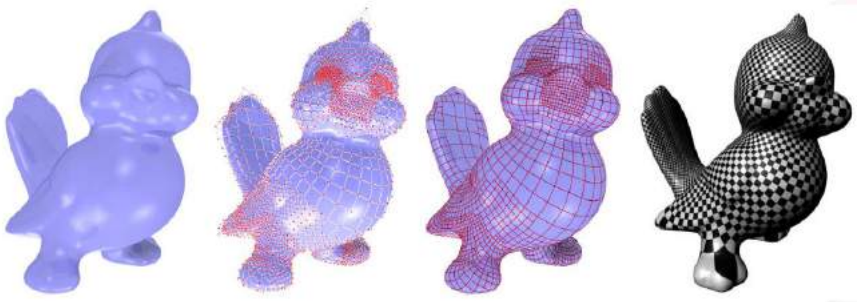
- Quad-mesh can better capture the local principle curvature directions or sharp features, as well as the semantics of modeled objects, therefore it is widely used in animation industry.



Advantages

Advantages of Quad-mesh

- Quad-mesh has tensor product structure, it is suitable for fitting splines or NURBS. Therefore it is applied for high-order surface modeling, such as CAD/CAM for Splines and NURBS, and the entertainment industry for subdivision surfaces.

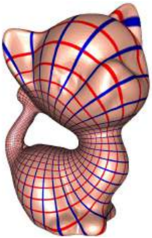


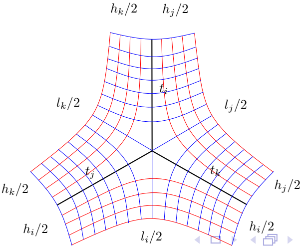
Quad-Mesh Generation

Categories

Categories of Quad-meshes

1. **Regular quad-mesh**: all the interior vertices are with topological valence 4, there are no singularities, such as torus.
2. **Semi-regular quad-mesh**: The separatrices divide the quad-mesh into several topological rectangles, the interior of each topological rectangle is regular grids.
3. **Unstructured quad-mesh**: A large fraction of its vertices are irregular.





Regular vs Semi-regular Quad-mesh

Regular vs Semi-regular Quad-mesh

- Regular quad-meshes have strong topological requirements for the surfaces, such as topological torus or annulus.
- Semi-regular quad-meshes can be realized for surfaces with any topologies, but the number of singularities, the behavior of separatrices are difficult to control.

Quad-Mesh

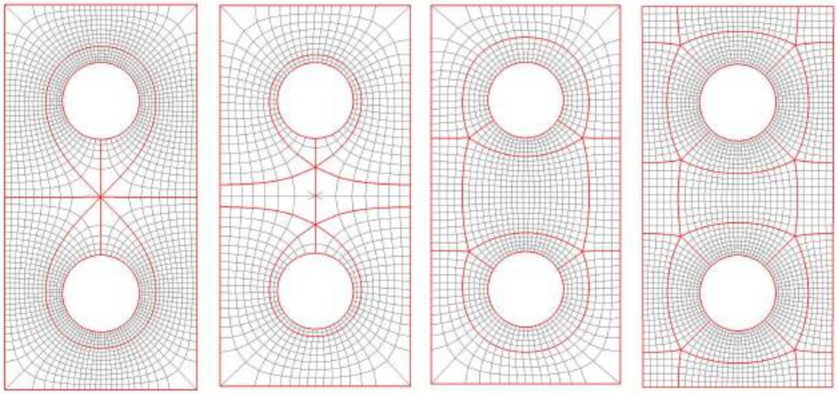


Figure: Quad-meshes with different number of singularities.

Quad-Mesh

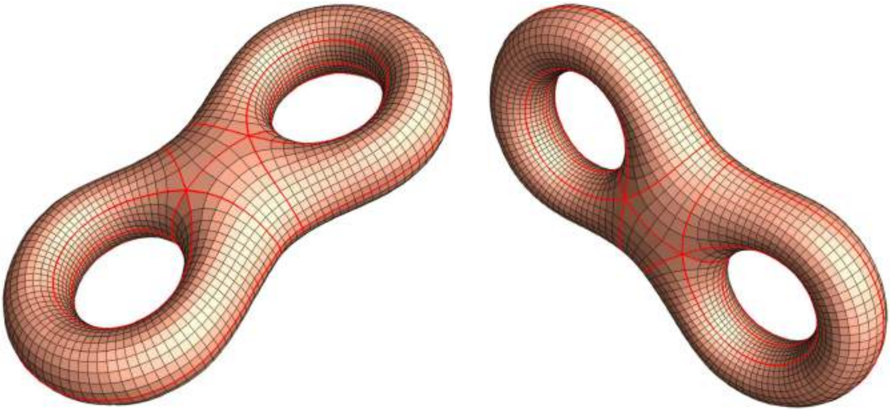


Figure: Quad-meshes with different number of singularities.

Quad-Mesh

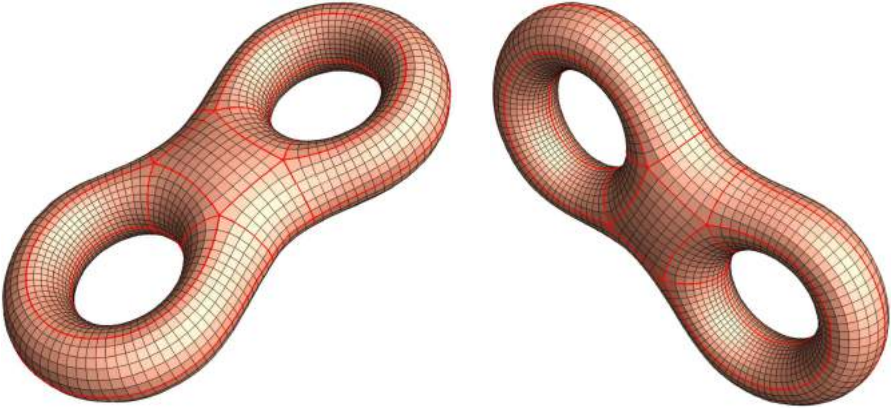
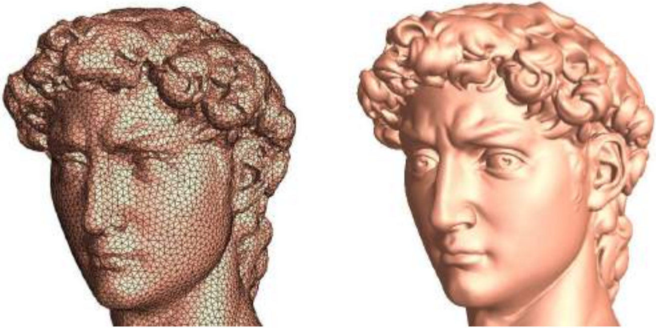


Figure: Quad-meshes with different number of singularities.

Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.

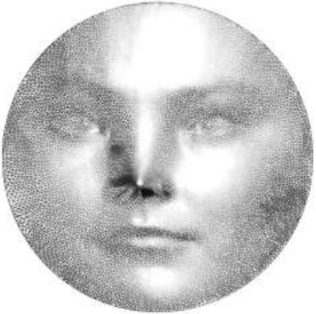


Discrete Metrics

Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, $l : E = \{all\ edges\} \rightarrow \mathbb{R}^+$, satisfies triangular inequality.

A mesh has infinite metrics.



Discrete Curvature

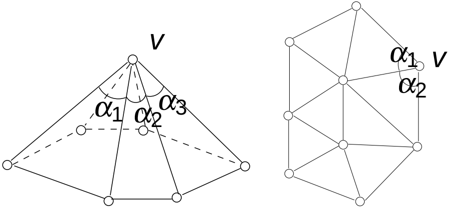
Definition (Discrete Curvature)

Discrete curvature: $K : V = \{vertices\} \to \mathbb{R}^1$.

$$K(v) = 2\pi - \sum_i \alpha_i, v \notin \partial M; K(v) = \pi - \sum_i \alpha_i, v \in \partial M$$

Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$



Quad-Mesh Metric

Quad-Mesh Metric

Definition (Quad-Metric)

Given a quad-mesh $\mathscr{Q}$, each face is treated as the unit planar square, this will define a Riemannian metric, the so-called quad-mesh metric $\mathbf{g}_{\mathscr{Q}}$, which is a flat metric with cone singularities.

Theorem (Quad-Mesh Metric Conditions)

Given a quad-mesh $\mathscr{Q}$, the induced quad-mesh metric is $\mathbf{g}_{\mathscr{Q}}$, which satisfies the following four conditions:

1. *Gauss-Bonnet condition;*
2. *Holonomy condition;*
3. *Boundary Alignment condition;*
4. *Finite geodesic lamination condition.*

Gauss-Bonnet condition

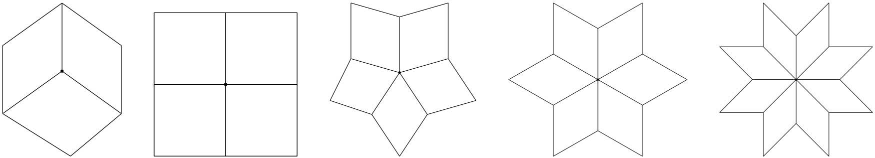
Gauss-Bonnet Condition

Definition (Curvature)

Given a quad-mesh $\mathcal{Q}$, for each vertex v_i, the curvature is defined as

$$K(v) = \begin{cases} \frac{\pi}{2}(4 - k(v)) & v \notin \partial\mathcal{Q} \\ \frac{\pi}{2}(2 - k(v)) & v \in \partial\mathcal{Q} \end{cases}$$

where $k(v)$ is the topological valence of v, i.e. the number of faces adjacent to v.



$k = \pi/2$ $\quad$ $k = 0$ $\quad$ $k = -\pi/2$ $\quad$ $k = -\pi$ $\quad$ $k = -2\pi$

Gauss-Bonnet Condition

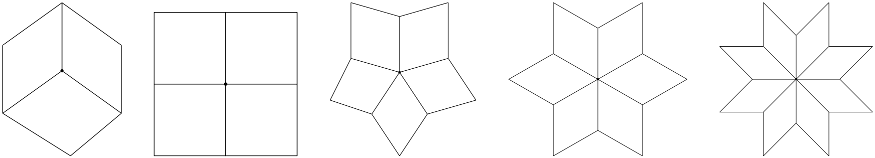
Theorem (Gauss-Bonnet)

Given a quad-mesh $\mathscr{Q}$, the induced metric is $\mathbf{g}_{\mathscr{Q}}$, the total curvature satisfies

$$\sum_{v_i \in \partial \mathscr{Q}} K(v_i) + \sum_{v_i \notin \partial \mathscr{Q}} K(v_i) = 2\pi\chi(\mathscr{Q}).$$

Namely

$$\sum_{v_i \in \partial \mathscr{Q}} (2 - k(v_i)) + \sum_{v_i \notin \partial \mathscr{Q}} (4 - k(v_i)) = 4\chi(\mathscr{Q}).$$



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Quad-Mesh

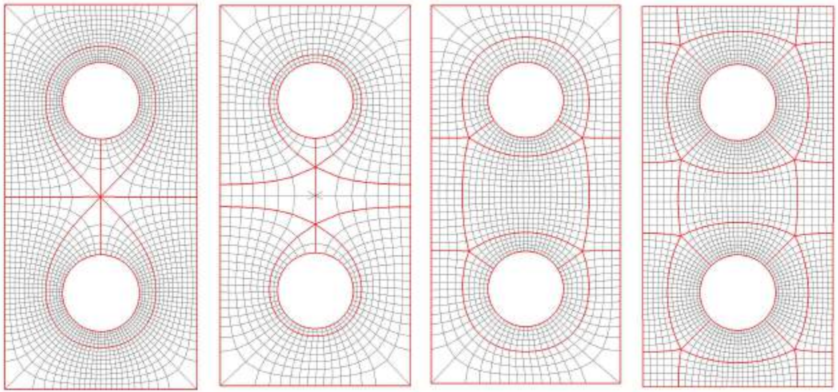


Figure: Quad-meshes with different number of singularities.

Quad-Mesh

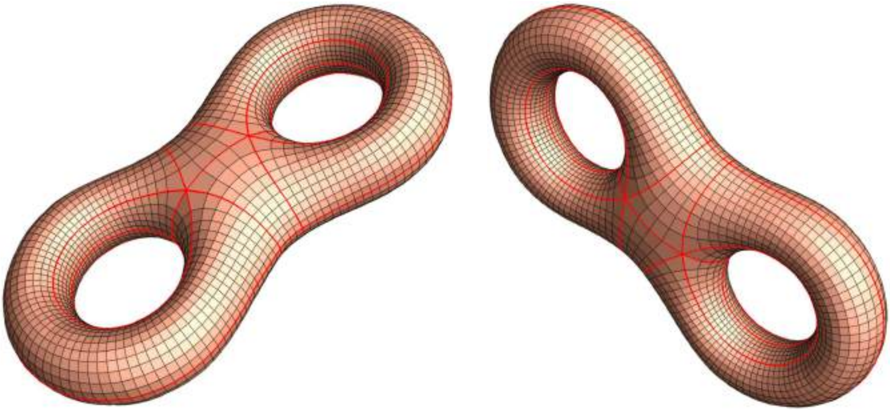


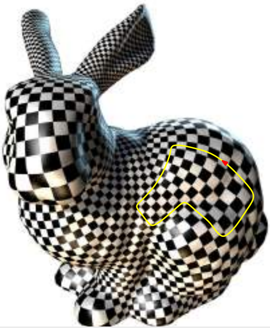
Figure: Quad-meshes with different number of singularities.

Holonomy Condition

Holonomy Condition

Definition (Holonomy)

Given a quad-mesh $\mathcal{Q}$, the induced flat metric is $\mathbf{g}_{\mathcal{Q}}$, the set of singular vertices is $S_{\mathcal{Q}}$. Suppose $\gamma : [0,1] \to \mathcal{Q} \setminus S_{\mathcal{Q}}$ is a closed loop not through singularities, choose a tangent vector $\mathbf{v}(0) \in T_{\gamma(0)}\mathcal{Q}$, parallel transport $\mathbf{v}(0)$ along $\gamma(t)$ to obtain $\mathbf{v}(t)$. The rotation angle from $\mathbf{v}(0)$ to $\mathbf{v}(1)$ in $T_{\gamma(0)}\mathcal{Q}$ is the holonomy of γ, denoted as $\rho(\gamma)$.



Face Loop

Definition (face path)

A sequence of faces, $\{f_0, f_1, \cdots, f_n\}$, such that f_i and f_{i+1} share an edge. If f_0 equals to f_n, then the face path is called a face loop.

Figure: A face path and a face loop.

Holonomy

Definition (Holonomy of a face loop)

Given a face loop γ through σ_0, fix a frame on σ_0, parallel transport the frame along γ. When we return to σ_0, the frame is rotated by an angle $k\pi/2$, which is called the holonomy of γ, and denoted as $\langle\gamma\rangle$.

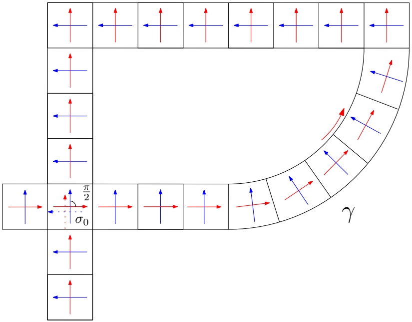


Figure: Parallel transportation along a face loop.

Holonomy

Theorem (Holonomy Condition)

Suppose $\mathscr{Q}$ is a closed quad-mesh, then the holonomy group induced by $\mathbf{g}_{\mathscr{Q}}$ is a subgroup of the rotation group $\{e^{i\frac{k}{2}\pi}, k \in \mathbb{Z}\}$.

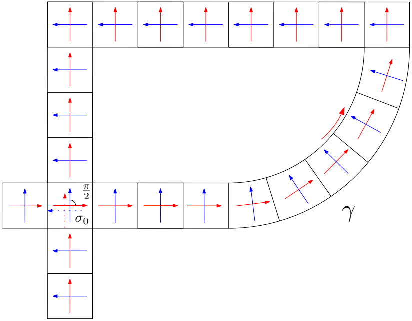


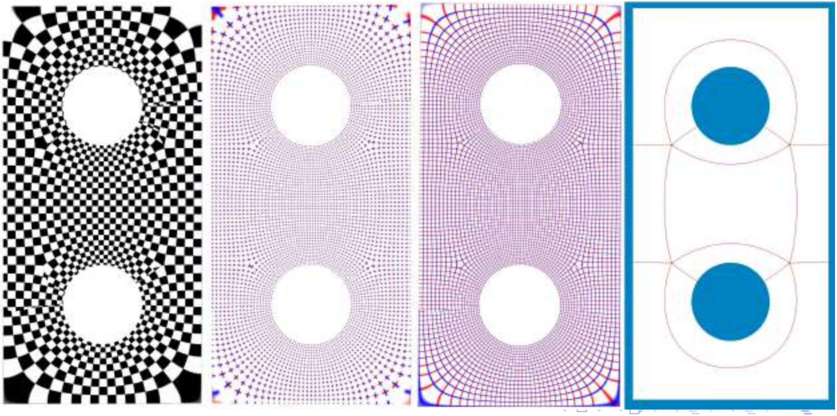
Figure: Parallel transportation along a face loop.

Boundary Alignment Condition

Boundary Alignment Condition

Definition (Boundary Alignment Condition)

Given a quad-mesh $\mathscr{Q}$, with induced metric $\mathbf{g}_{\mathscr{Q}}$, one can define a global cross field by parallel transportation, which is aligned with the boundaries.



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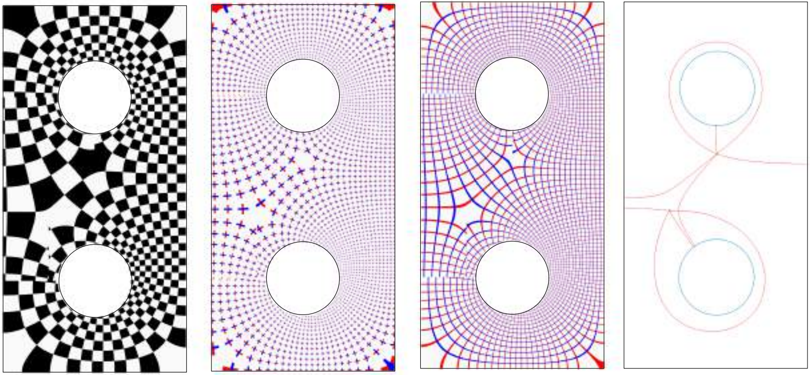


Finite Geodesic Lamination Condition

Finite Geodesic Lamination Condition

Open Trajectories

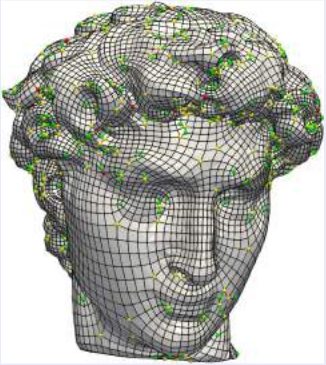
Finite Geodesic Lamination Condition

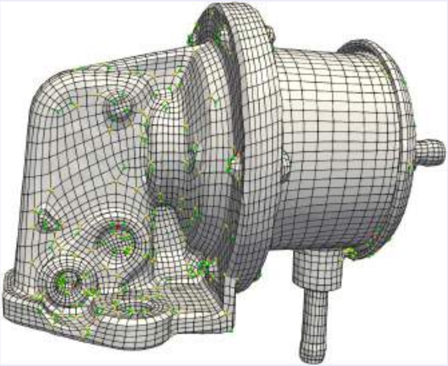
Definition (Finite Geodesic Lamination Condition)

The stream lines parallel to the cross field are finite geodesic loops. This is the finite geodesic lamination condition.

Quad-Mesh Singularity Abel condition

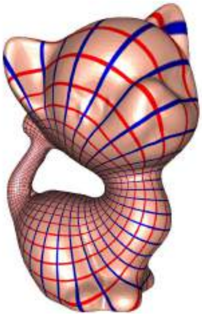
Singularities on Quadrilateral Meshes





Green, Yellow and Red vertices are with topological valence 3, 5 and 6 respectively.

Singularities on Quadrilateral Meshes



Topological Torus

$$\chi = 2 - 2g = 0,$$

$$\sum K = 2\pi\chi = 0.$$

Is it possible to construct a quad mesh on a topological torus with one valence 3 singular point and one valence 5 singular point?

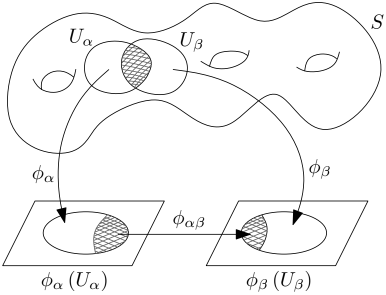
Main Problem

Central Question

Given a surface M and the singularities (including positions and indicies) on M satisfying the Gauss-Bonnet condition, whether there exists a high quality quad-mesh Q on M?

Riemann Surface Theory

Manifold



Definition (Manifold)

M is a topological space, $\{U_\alpha\}$ $\alpha \in I$ is an open covering of M, $M \subset \cup_\alpha U_\alpha$. For each U_α, $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism. The pair (U_α, ϕ_α) is a chart. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, the transition function $\phi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is continuous $\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$ then M is called a topological manifold, $\{(U_\alpha, \phi_\alpha)\}$ is called an atlas.

Holomorphic Function

Definition (Holomorphic Function)

Suppose $f:\mathbb{C}\to\mathbb{C}$ is a complex function, $f: x+iy \to u(x,y)+iv(x,y)$, if f satisfies Riemann-Cauchy equation

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},$$

then f is a holomorphic function.

Denote

$$dz=dx+idy, d\bar{z}=dx-idy,\frac{\partial}{\partial z}=\frac{1}{2}(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}),\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)$$

then if $\frac{\partial f}{\partial \bar{z}}=0$, then f is holomorphic.

Biholomorphic Function

Definition (biholomorphic Function)

Suppose $f : \mathbb{C} \to \mathbb{C}$ is invertible, both f and f^{-1} are holomorphic, then f is a biholomorphic function.

Conformal Atlas

Definition (Conformal Atlas)

Suppose S is a topological surface, (2 dimensional manifold), $\mathfrak{A}$ is an atlas, such that all the chart transition functions $\phi_{\alpha\beta}:\mathbb{C}\to\mathbb{C}$ are bi-holomorphic, then A is called a conformal atlas.

Definition (Compatible Conformal Atlas)

Suppose S is a topological surface, (2 dimensional manifold), $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are two conformal atlases. If their union $A_1 \cup A_2$ is still a conformal atlas, we say $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are compatible.

Riemann Surface

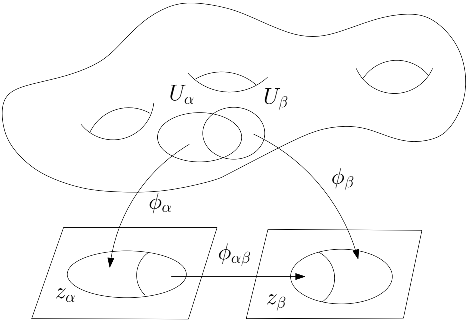


Figure: Riemann Surface.

Definition (Riemann Surface)

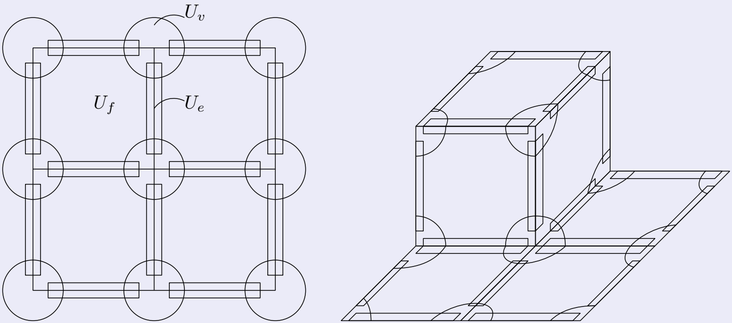
A surface with a conformal structure (a conformal $\mathscr{A}$) is called a Riemann surface.

Quad-Mesh Riemann Surface

Theorem (Quad-Mesh Riemann Surface)

Suppose Q is a closed quadrilateral mesh, then Q induces a conformal structure and can be treated as a Riemann surface M_Q.

Proof.



(a) conformal atlas (b) singularities

Riemann Surface

Definition (Meromorphic Function)

Suppose $f: M \to \mathbb{C} \cup \{\infty\}$ is a complex function defined on the Riemann surface M. If for each point $p \in M$, there is a neighborhood $U(p)$ of p with local parameter $z(p) = 0$, f has Laurent expansion

$$f(z) = \sum_{i=k}^{\infty} a_i z^i,$$

then f is called a meromorphic function.

Meromorphic Differential

Definition (Meromorphic Differential)

Given a Riemann surface $(M, \{z_\alpha\})$, ω is a meromorphic differential of order n, if it has local representation,

$$\omega = f_\alpha(z_\alpha)(dz_\alpha)^n,$$

where $f_\alpha(z_\alpha)$ is a meromorphic function, n is an integer; if $f_\alpha(z_\alpha)$ is a holomorphic function, then ω is called a holomorphic differential of order n.

Quad-Mesh Meromorphic Differential

Theorem (Quad-Mesh Meromorphic Differential)

Suppose Q is a closed quadrilateral mesh, then Q induces a meromorphic quartic differential.

Proof.

On each face f, define dz_f, $\omega_Q = (dz_f)^4$; vertex face transition

$$z_v^{\frac{k}{4}} = e^{i\frac{n\pi}{2}} z_f + \frac{1}{2}(\pm 1 \pm i) \tag{1}$$

where k is the vertex valence, therefore

$$\left(\frac{k}{4}\right)^4 z_v^{k-4} (dz_v)^4 = (dz_f)^4 = \omega_Q. \tag{2}$$

□

Zeros and Poles

Definition (Zeros and Poles)

Suppose $f : M \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function. For each point p, there is a neighborhood $U(p)$ of p with local parameter $z(p) = 0$, f has Laurent expansion

$$f(z) = \sum_{i=k}^{\infty} a_i z^i,$$

if $k > 0$, then p is a zero with order k; if $k = 0$, then p is a regular point; if $k < 0$, then p is a pole with order k. The assignment of p with respect to f is denoted as $v_p(f) = k$.

Quad-Mesh singularities vs. Zeros and poles

Theorem

The singularities with valence 3 are the poles of the meromorphic quartic differential induced by the quad mesh. The singularities with valence 5 are the simple zeros of the meromorphic quartic differential and the singularities with valence more than 5 are the multiple zeros of the meromorphic quartic differential.

Divisor

Definition (Divisor)

The Abelian group freely generated by points on a Riemann surface is called the divisor group, every element is called a divisor, which has the form $D = \sum_p n_p p$. The degree of a divisor is defined as $deg(D) = \sum_p n_p$. Suppose $D_1 = \sum_p n_p p$, $D_2 = \sum_p m_p p$, then $D_1 \pm D_2 = \sum_p (n_p \pm m_p) p$; $D_1 \leq D_2$ if and only if for all p, $n_p \leq m_p$.

Definition (Meromorphic Function Divisor)

Given a meromorphic function f defined on a Riemann surface S, its divisor is defined as $(f) = \sum_p v_p(f) p$, where $v_p(f)$ is the assignment of p with respect to f.

The divisor of a meromorphic function is called a principle divisor.

Divisor

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Definition (Quad-Mesh Divisor)

Suppose Q is a closed quadrilateral mesh, then Q induces a divisor

$$D_Q = \sum_{v_i \in Q} (k(v_i) - 4) v_i,$$

where v_i is a vertex with valence $k(v_i)$.

Principle Divisor

Theorem

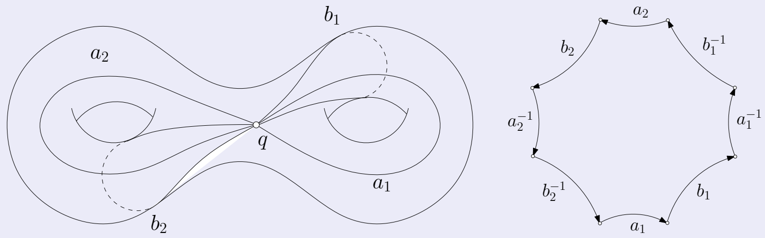
Suppose M is a compact Riemann surface with genus g, f is a meromorphic function, then

$$\deg((f)) = 0,$$

ω is a meromorphic differential, then

$$\deg((\omega)) = 2g - 2.$$

Canonical Fundamental Group Generators



Algebraic intersection numbers satisfy the conditions:

$$a_i \cdot b_j = \delta_{ij}, a_i \cdot a_j = 0, b_i \cdot b_j = 0.$$

A continuous map $\gamma : [0,1] \to S$ is a path, if $\gamma(0) = \gamma(1) = q$, then it is a loop. All the loops are classified by homotopy, all the homotopic equivalence classes form the fundamental group. The fundamental group is non-Abelian and has $2g$ generators, where g is the genus.

Holomorphic Differential Group Basis

Given a Riemann surface S with local conformal parameter $\{z_i\}$, a holomorphic one-form ω is globally well defined, and has the local representation:

$$\omega = f_i(z_i)dz_i = f_j(z_j)dz_j,$$

where f_i's are holomorphic functions, and satisfy the relation:

$$f_j(z_j(z_i))\frac{dz_j}{dz_i} = f_i(z_i).$$

Holomorphic Differential Group Basis

The holomorphic one-form basis $\{\varphi_1, \varphi_2, \cdots, \varphi_g\}$ satisfy the dual condition

$$\int_{a_j} \varphi_i = \delta_{ij}.$$

Period Matrix

Definition (Period Matrix)

Suppose M is a compact Riemann surface of genus g, with canonical fundamental group basis

$$\{a_1, a_2, \cdots, a_g, b_1, b_2, \cdots, b_g\}$$

and holomorphic one form basis

$$\{\varphi_1, \varphi_2, \cdots, \varphi_g\}$$

The period matrix is defined as $[A, B]$

$$A = \left(\int_{a_j} \varphi_i \right), B = \left(\int_{b_j} \varphi_i \right).$$

Matrix B is symmetric, $Img(B)$ is positive definite.

Jacobi Variety

Definition (Jacobi Variety)

Suppose the period matrix

$$A=(A_1,A_2,\cdots,A_g),\quad B=(B_1,B_2,\cdots,B_g),$$

the lattice Γ is

$$\Gamma=\left\{\sum_{i=1}^{g}\alpha_i A_i+\sum_{j=1}^{g}\beta_j B_j\right\},$$

the Jacobi variety of M is defined as

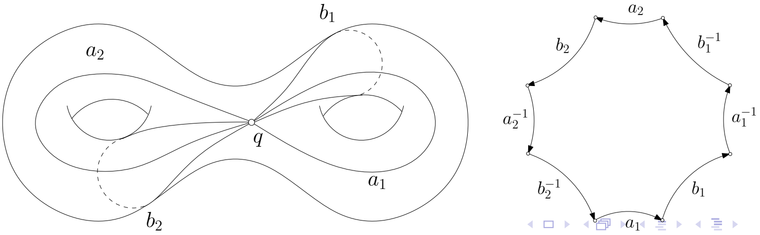
$$J(M)=\mathbb{C}^g/\Gamma.$$

Jacobi Map

Definition (Jacobi Map)

Given a compact Riemann surface M, choose a set of canonical fundamental group generators $\{a_1,\cdots,a_g,b_1,\cdots,b_g\}$, and obtain a fundamental domain Ω, $\partial\Omega = a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$. Choose a base point p_0, the Jacobi map $\mu : M \to J(M)$ is defined as follows: for any point $p \in M$, choose a path γ from p_0 to p inside Ω,

$$\mu(p) = \left(\int_\gamma \varphi_1, \int_\gamma \varphi_2, \cdots, \int_\gamma \varphi_g\right)^T.$$



Abel Theorem

Theorem (Abel)

Suppose M is a compact Riemann surface with genus g, D is a divisor, $\deg(D) = 0$*. D is principle if and only if*

$$\mu(D) = 0 \quad \text{in } J(M).$$

Quad-Mesh Abel Condition

Theorem (Quad-Mesh Abel Condition)

Suppose Q is a closed quadrilateral mesh, then for any holomorphic differential φ

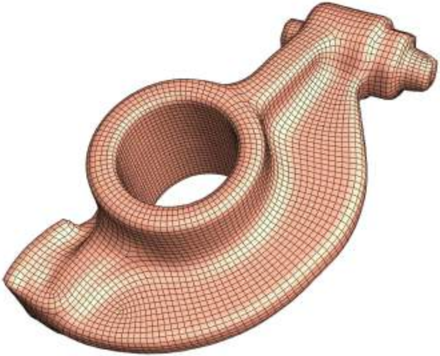
$$\mu(D_Q - 4(\varphi)) = 0 \quad \text{in } J(M_Q). \tag{3}$$

Quartic Differential

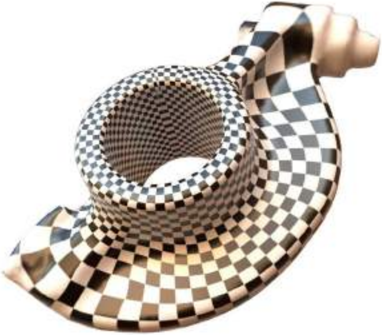
Theorem (Quartic Differential to Quad-Mesh)

Suppose M is a Riemann surface, ω is a meromorphic quartic differential with finite trajectories, then ω induces a quadrilateral mesh Q, such that the poles or zeros with order k of ω corresponds to the singular vertices of Q with valence $k+4$.

Genus One Case



a) quad-mesh



b) holomorphic 1-form

The induced meromorphic quartic differential has 18 poles and 18 zeros.

$$D_Q = \sum_{i=1}^{18} (p_i - q_i),$$

Genus One Case

The results of the Abel-Jacobi map are as follows:

$$\mu\left(\sum_{j=1}^{18} p_j\right) = 2.61069 + i0.588368,$$

and

$$\mu\left(\sum_{i=1}^{18} q_i\right) = 2.61062 + i0.588699.$$

Hence, $\mu(D_Q)$ is the difference between them, which equals to $6.967e-05 \; - \; i3.3064e-4$, very close to the origin in $J(S_Q)$.

Figure Eight Case

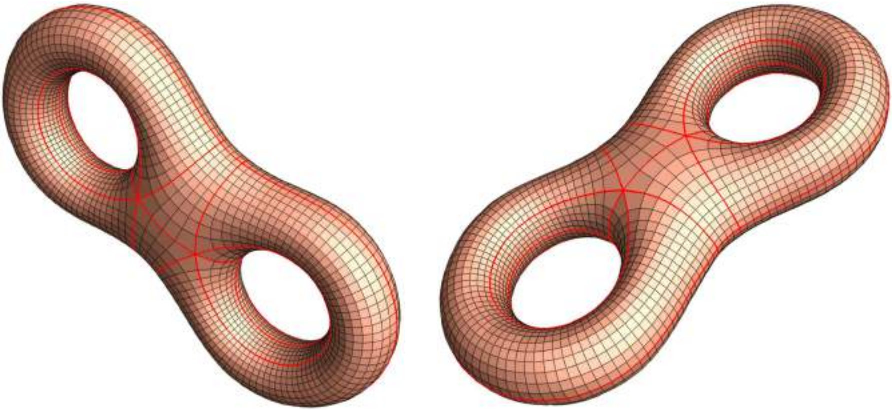


Figure: The input genus two quad-mesh.

Figure Eight Case

(a) tunnel loops (b) handle loops

Figure: The homology group basis.

Figure Eight Case

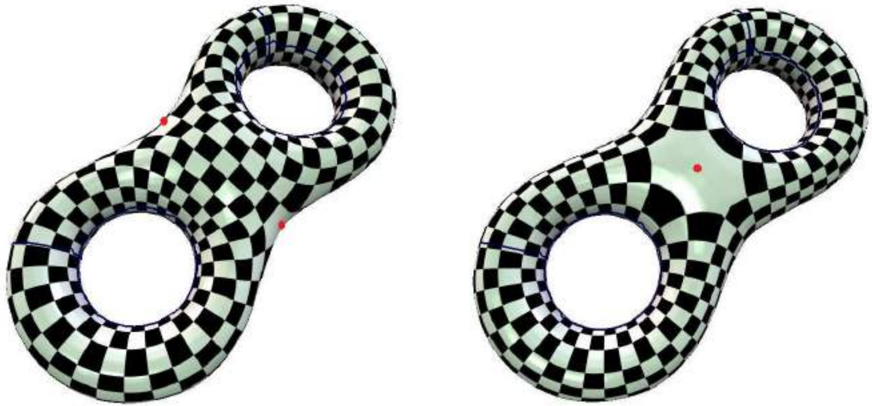


Figure: The holomorphic differential basis.

Abel Condition

We set φ_0 as ω_0 and verify the Abel-Jacobi condition by computing the Abel-Jacobi map $\mu(D_Q - 4(\omega_0))$. The period matrix A of the Riemann surface S_Q is

$$\begin{pmatrix} 0.99999999 - i1.4209e-09 & -0.99999989 + i6.01812e-08 \\ 0.99999998 + i5.12829e-09 & 0.99999992 - i2.88896e-08 \end{pmatrix}$$

The period matrix B is

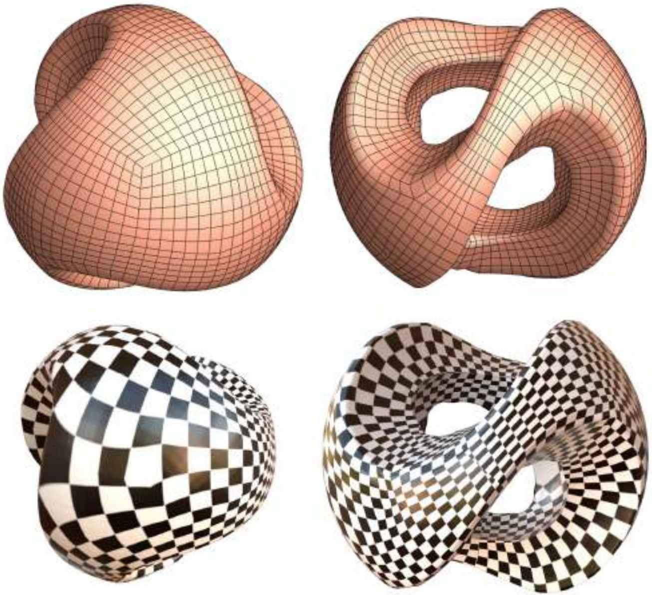
$$\begin{pmatrix} 3.18e-08 + i0.38191542 & 4.7433845e-20 + i0.3861979 \\ 1.433e-08 + i0.44392235 & -2.3716923e-20 - i0.44820492 \end{pmatrix}$$

The Abel-Jacobi image of the divisor,

$$\mu(D_Q - 4(\omega_0)) = \begin{pmatrix} 1e-06 \\ 2e-07 - i1.6e-06 \end{pmatrix},$$

which is very close to 0.

Genus Two Case



Genus Two Case: Abel Condition

We set φ_0 as ω_0 and verify the Abel-Jacobi condition by computing the Abel-Jacobi map $\mu(D_Q - 4(\omega_0))$. The period matrix A of the Riemann surface S_Q is

$$\begin{pmatrix} 0.99999997 - i2.8e-09 & -0.24999994 + i2.745e-08 \\ 0.99999999 + i1.13e-08 & 0.50000015 + i4.1e-08 \end{pmatrix}.$$

The period matrix B is

$$\begin{pmatrix} -4.8789098e-19 + i0.50669566 & 7.5894152e-19 + i0.15720634 \\ -7.5894152e-19 + i0.73261918 & 4.8789098e-19 + i0.589281 \end{pmatrix}.$$

The Abel-Jacobi map image of the divisor is

$$\mu(D_Q - 4(\omega_0)) = \begin{pmatrix} -1.568599999979e-05 + i3.69999999994e-06 \\ 4.28899999998e-05 - i4.400000000182e-07 \end{pmatrix},$$

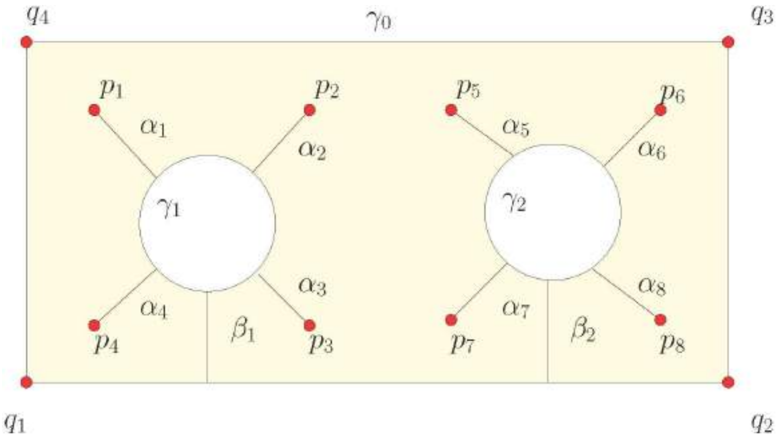
which is very close the 0.

Quad Mesh Algorithm

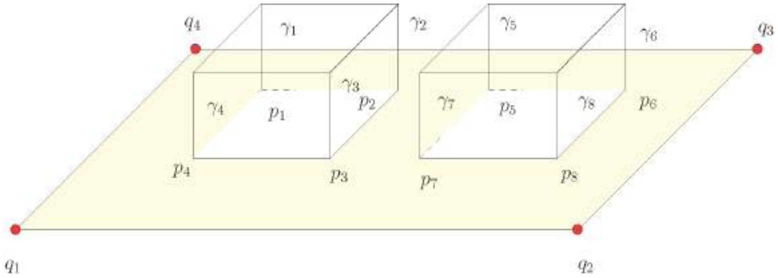
Algorithm Pipeline

1. Determine the positions and indices of singularities Γ;
2. Compute a flat metric $\mathbf{g}_0$ with cone singularities using discrete surface Ricci flow algorithm;
3. Compute a cut graph L of the surface, such that $S - L$ is a topological disk;
4. Isometrically immerse $(S - L, \mathbf{g}_0)$, the image is a planar immersed polygon P. Each pair of dual boundary segments of the polygon differ by a planar rigid motion.
5. Conformal structure deformation;
6. Compute geodesics under $\mathbf{g}$ align the cross field ω. The geodesics through the singularities defines the skeleton, further subdivisions of the skeleton gives the quad-mesh.

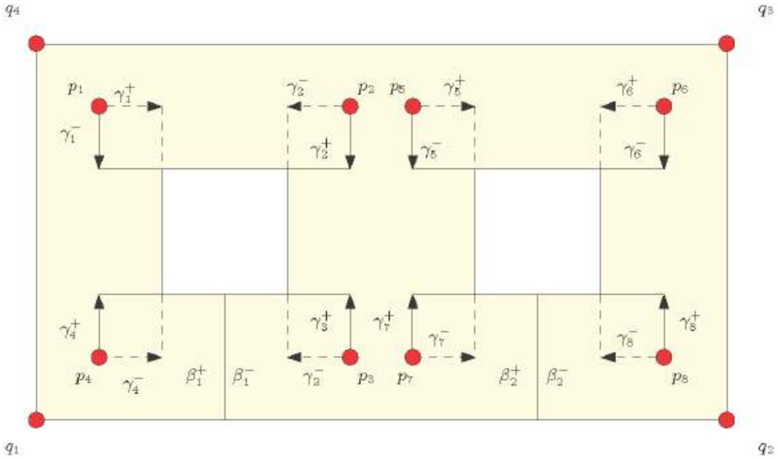
Singularity Allocation by Abel-Jabobi Condition



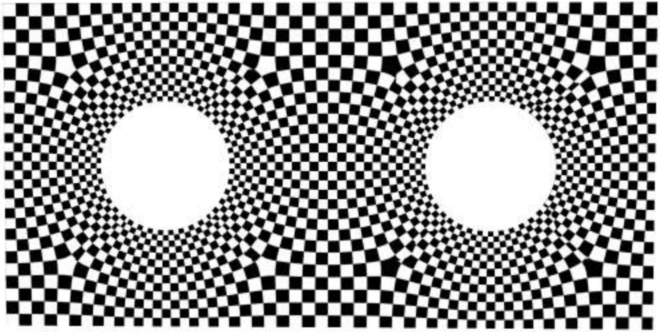
Desired Flat Metric with Cone Singularities



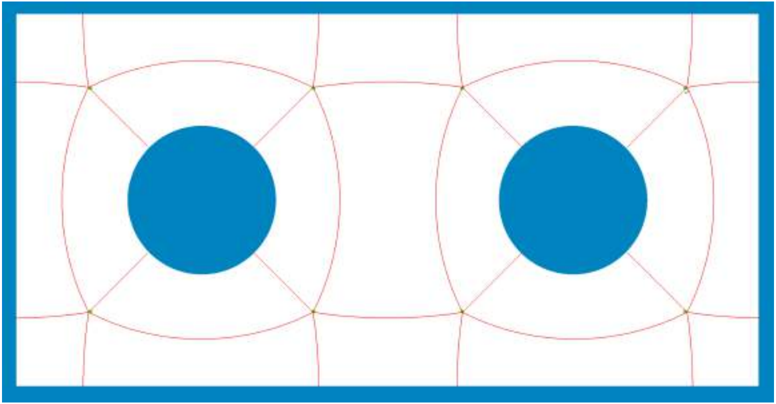
Ricci flow for Flat Metric With Cone Singularities



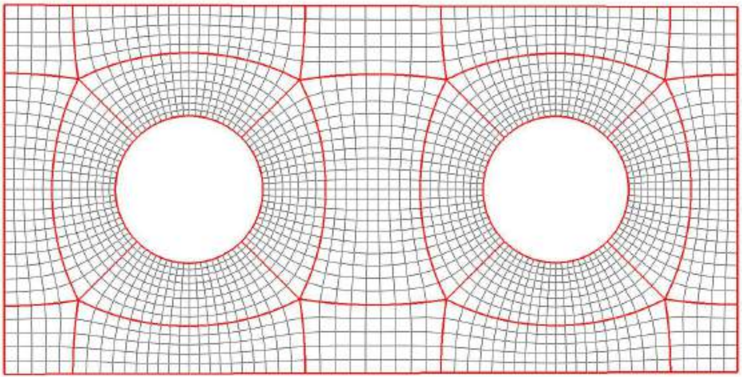
Parameterization



Tracing Separatrices



Quad Mesh



Conclusion

1. The quad-mesh metric satisfies conditions: Gauss-Bonnet, holonomy, boundary alignment and finite geodesic lamination;
2. Equivalence between meromorphic quartic differentials and quadrilateral meshes;
3. Singularities of a quad-mesh correspond to the divisor of the differential, which satisfies the Abel condition;
4. Ricci flow is an important tool to compute the quad-mesh metric.

Thanks

For more information, please email to nalei@dult.edu.cn.

Thank you!