Fractional Gaussian Fields for Modeling and Rendering of Spatially-Correlated Media

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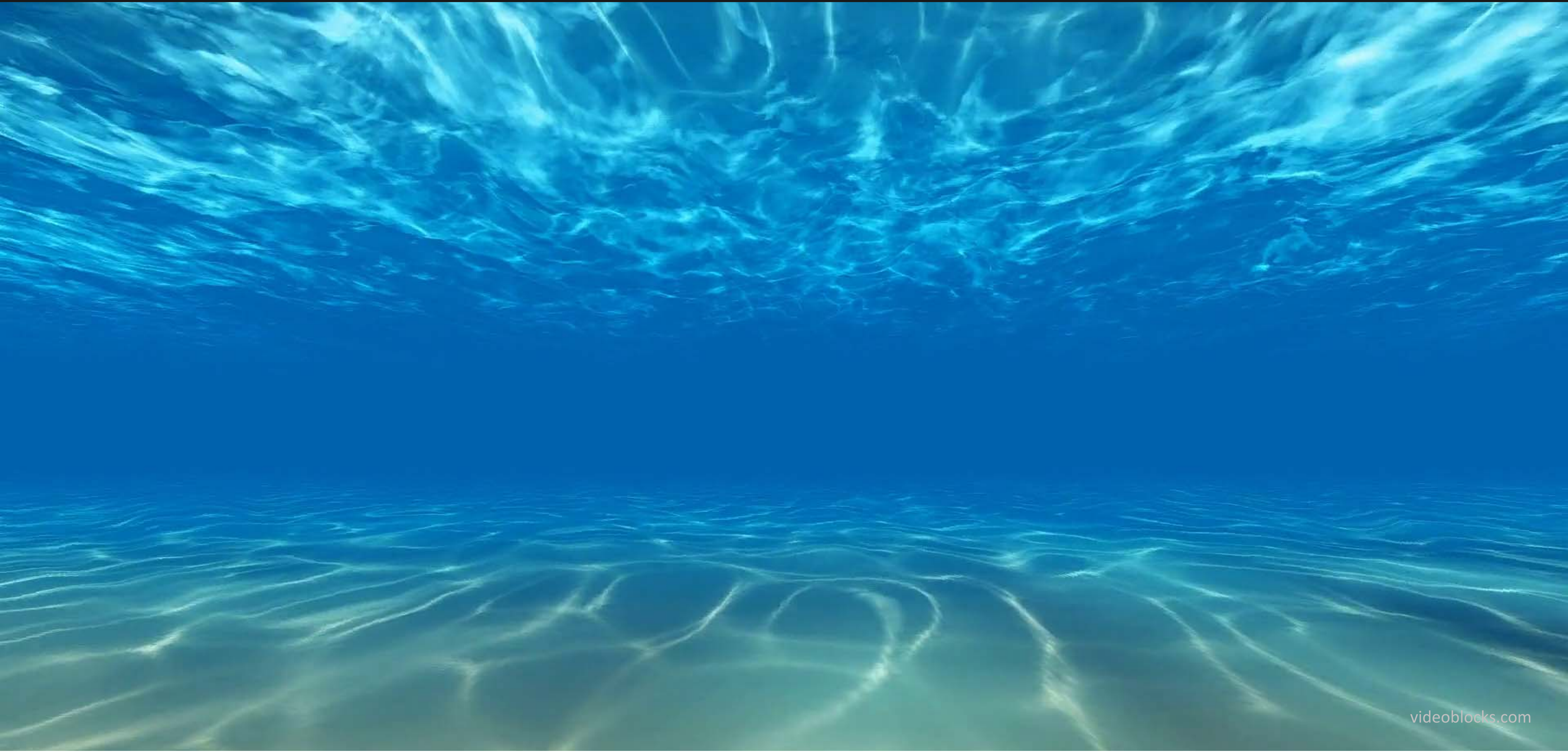
Participating Media

Participating Media



Participating Media

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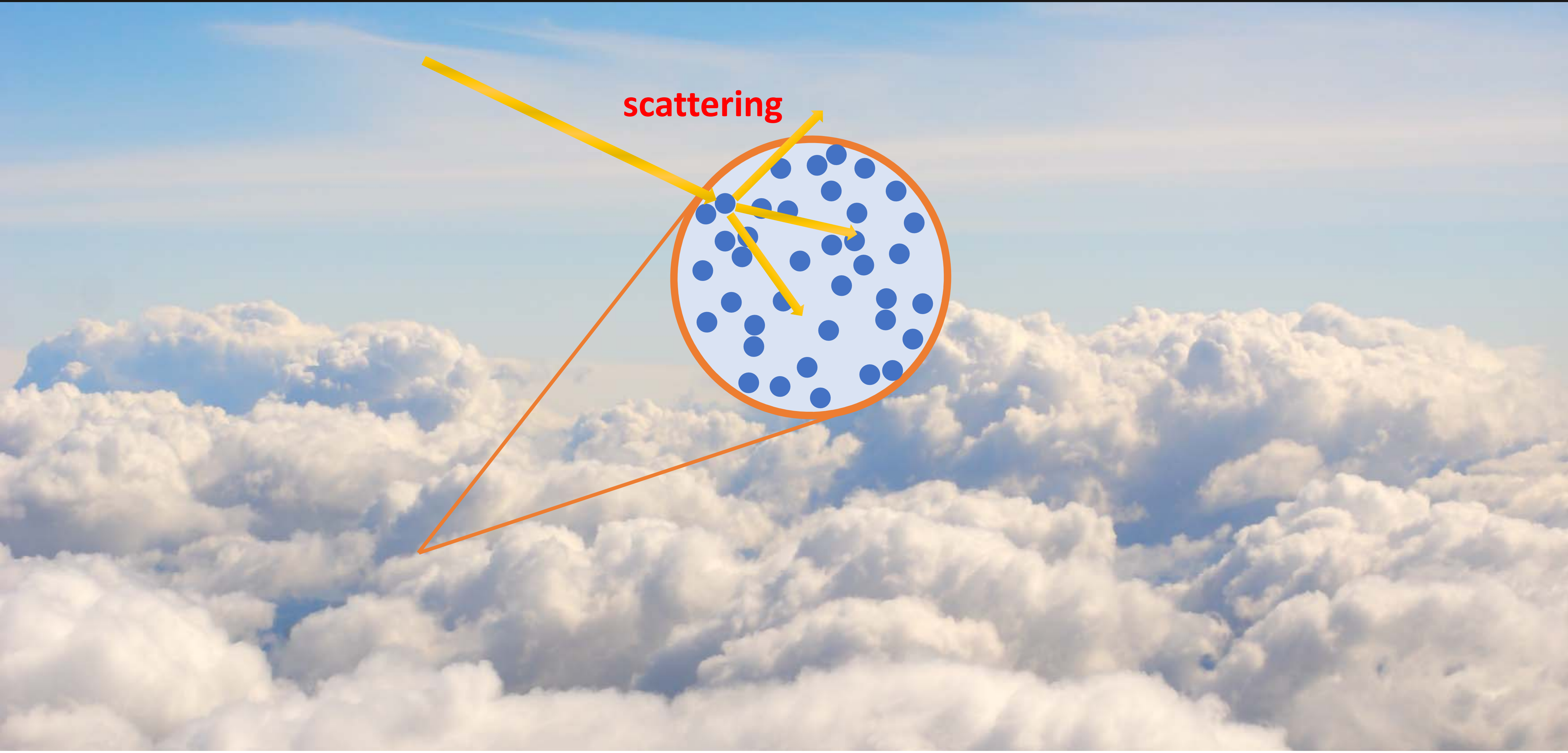
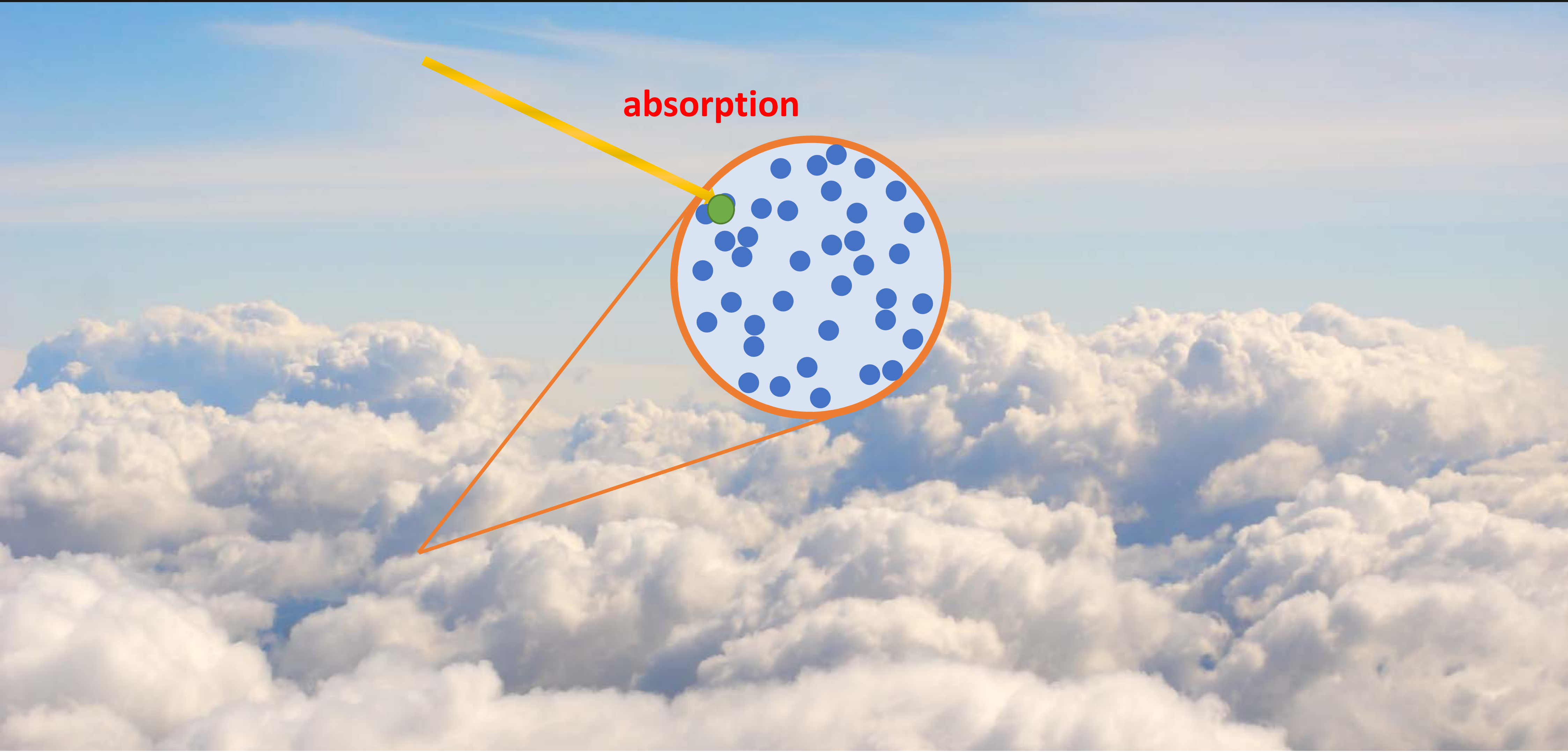
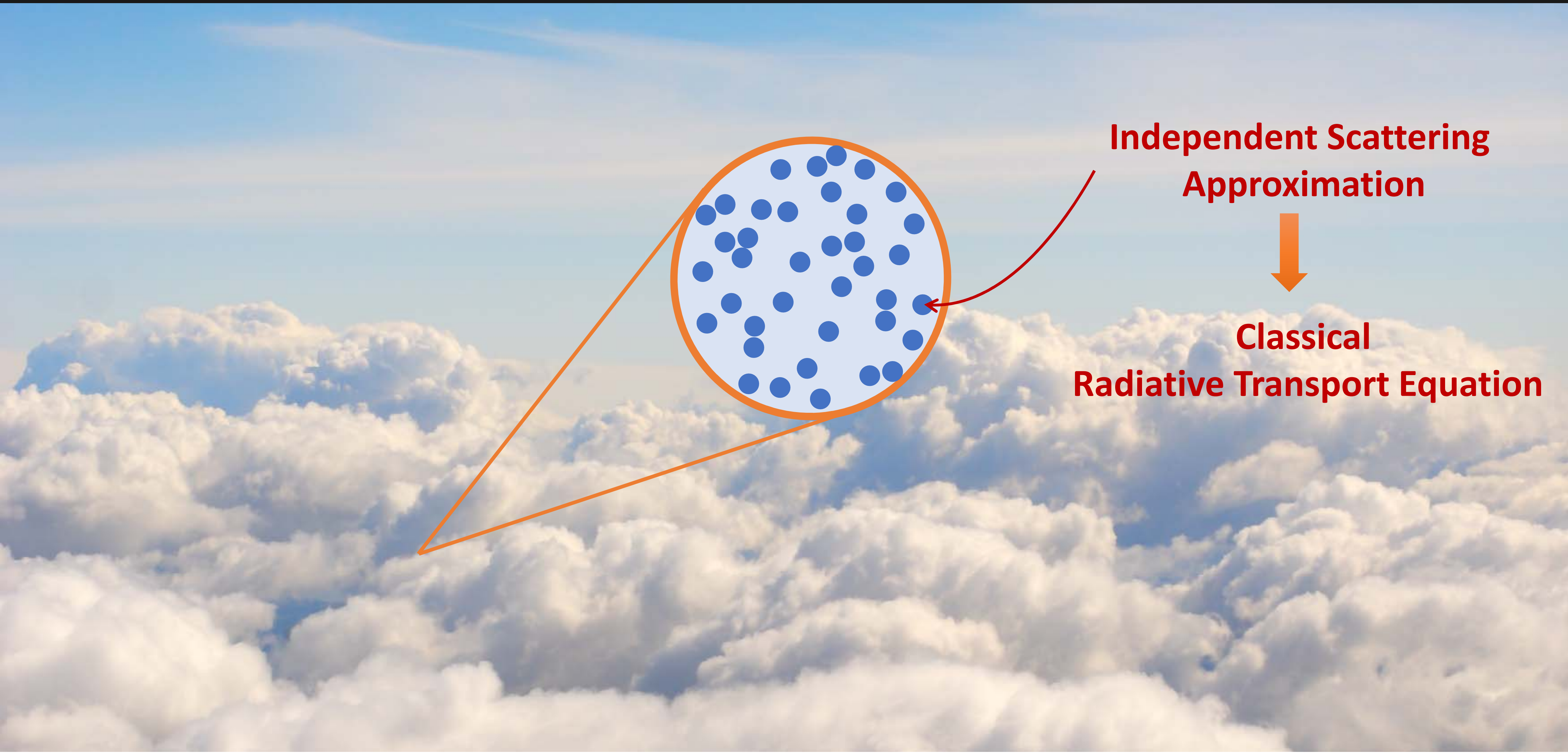


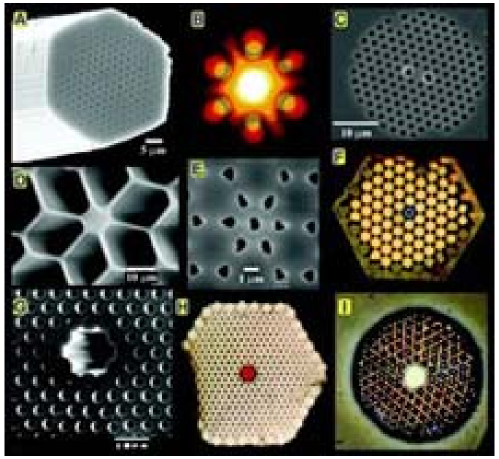
Participating Media

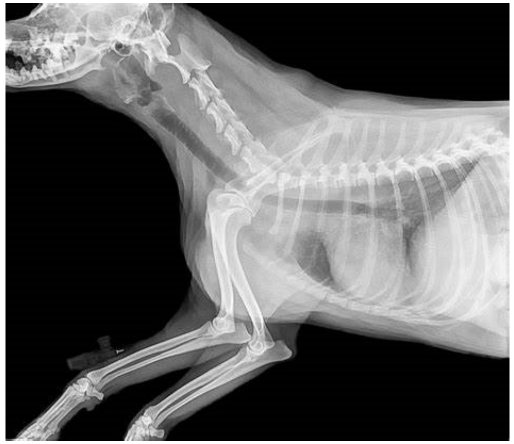


Participating Media

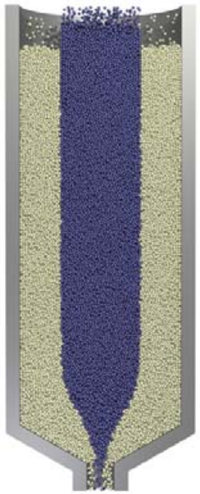


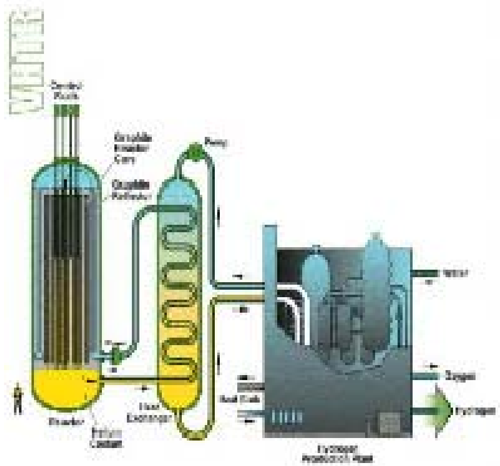
Related Work





(But, a lot of works have observed **correlations** in particle distribution)





Related Work

Related Work


[Meng et al. 2015]


[Müller et al. 2016]


[Jarabo et al. 2018]


[Bitterli et al. 2018]

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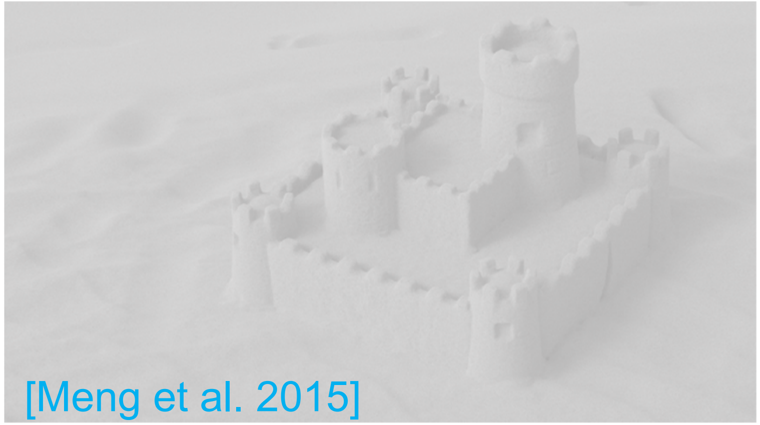
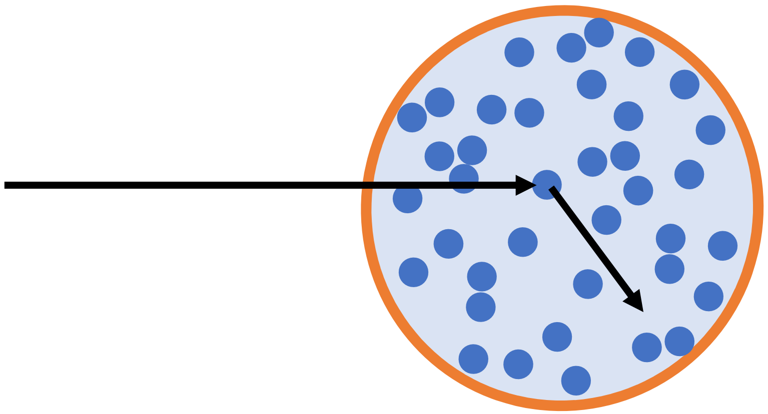


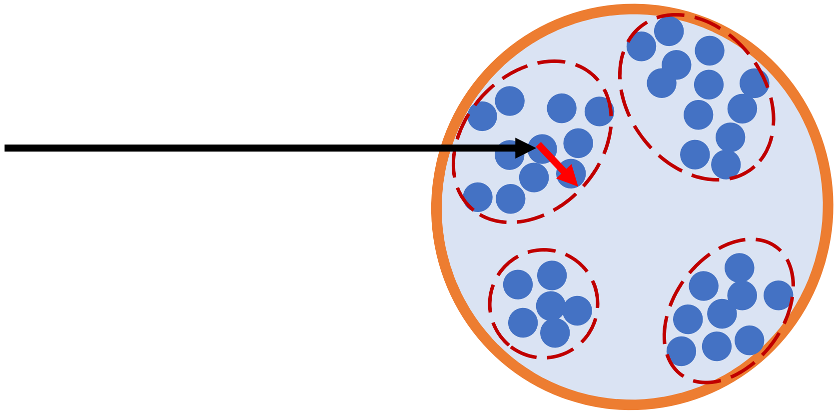
Our method

- ✓ General media
- ✓ Heterogeneity
- ✓ Long-range correlations

Spatially-Correlated Media

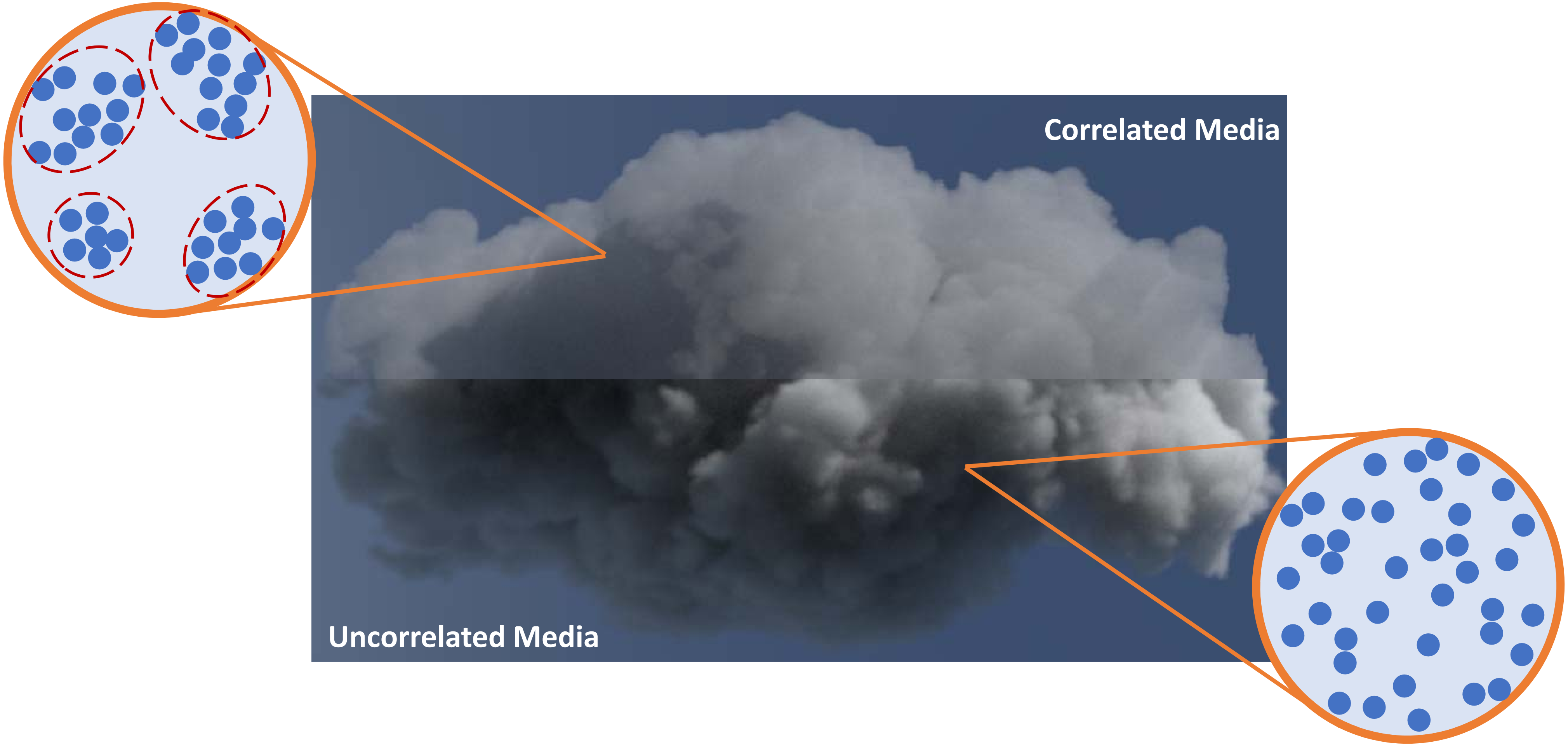


Uncorrelated Media

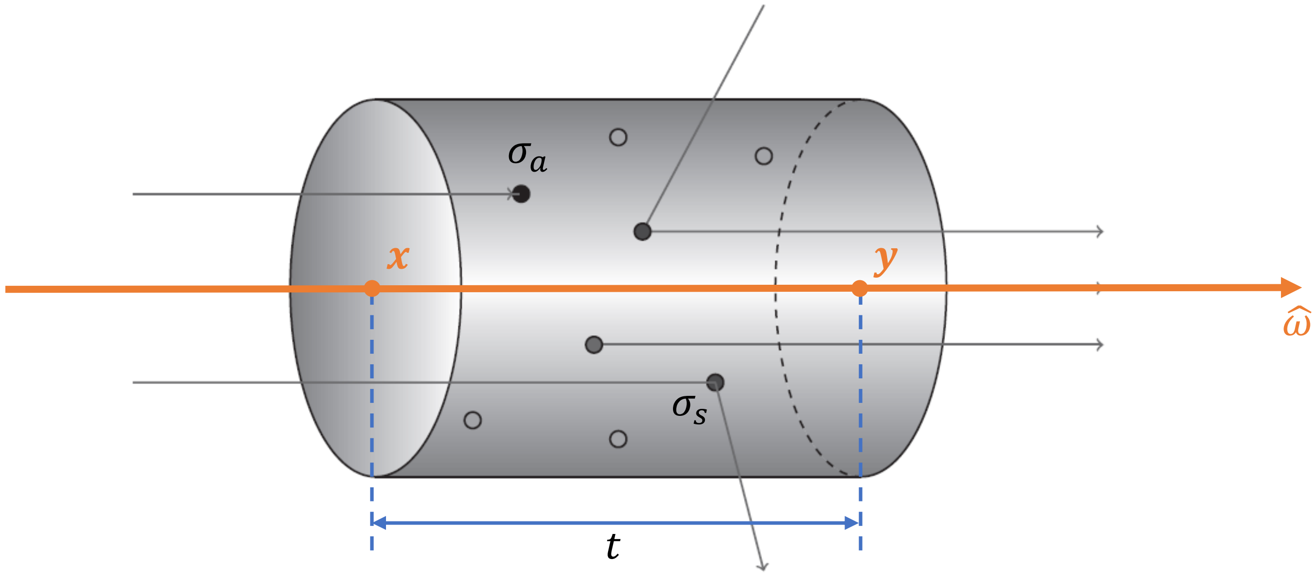


Correlated Media

Spatially-Correlated Media



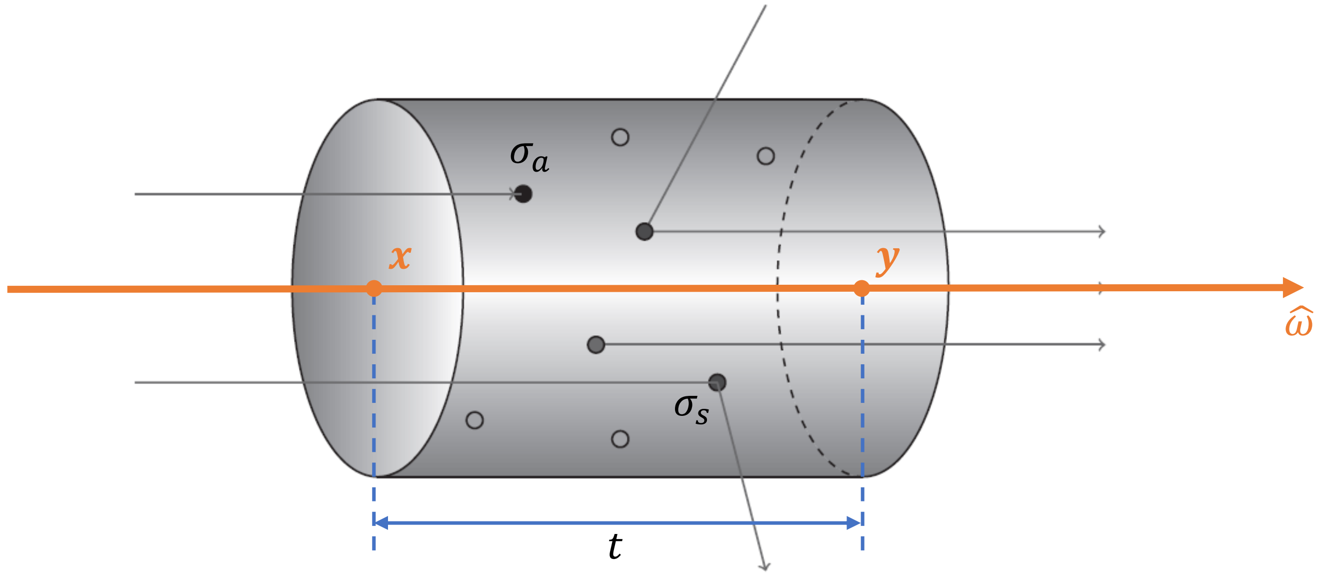
Spatially-Correlated Media



Transmittance: $T(\boldsymbol{x}, \boldsymbol{y}) = e^{-\int_0^t \sigma_t(\boldsymbol{x}+s\hat{\omega})\mathrm{d}s} = e^{-\tau(\boldsymbol{x},t)}$

Extinction field: $\sigma_t(\boldsymbol{x}) = \sigma_a(\boldsymbol{x}) + \sigma_s(\boldsymbol{x})$

Spatially-Correlated Media



Transmittance: $T(\boldsymbol{x}, \boldsymbol{y}) = e^{-\int_0^t \sigma_t(\boldsymbol{x}+s\hat{\omega})\mathrm{d}s} = e^{-\tau(\boldsymbol{x},t)}$

Extinction field: $\sigma_t(\boldsymbol{x}) = \sigma_a(\boldsymbol{x}) + \sigma_s(\boldsymbol{x})$

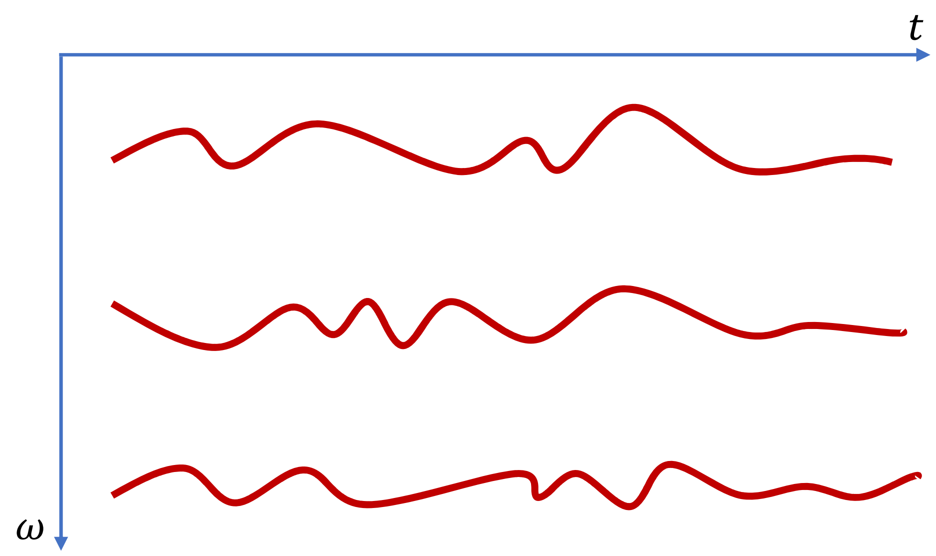
We model σ_t as a
Fractional Gaussian Field (FGF)

Fractional Gaussian Fields

Random field

A collection of random variables:

$$X = \{X(t,\omega), t \in \mathbb{R}^d, \omega \in \Omega\}$$



Fractional Gaussian Fields

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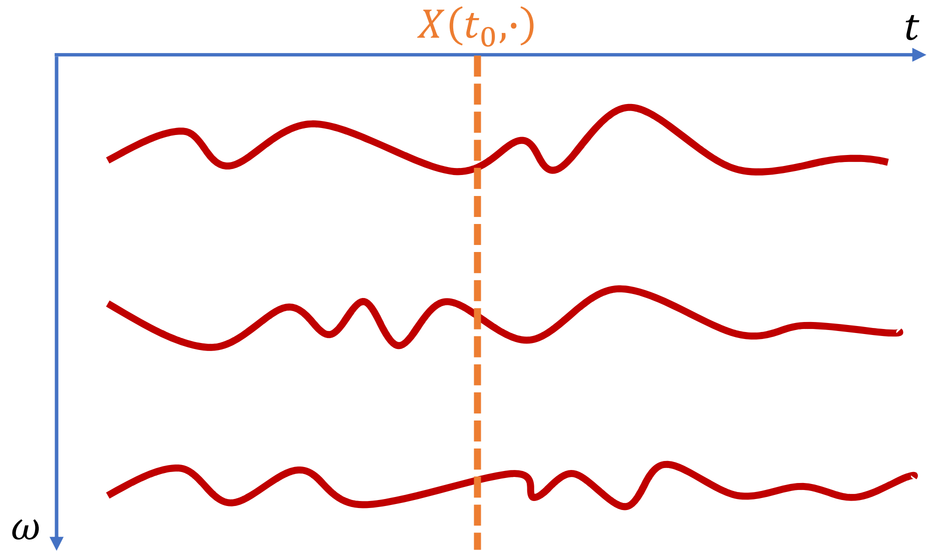

Fractional Gaussian Fields

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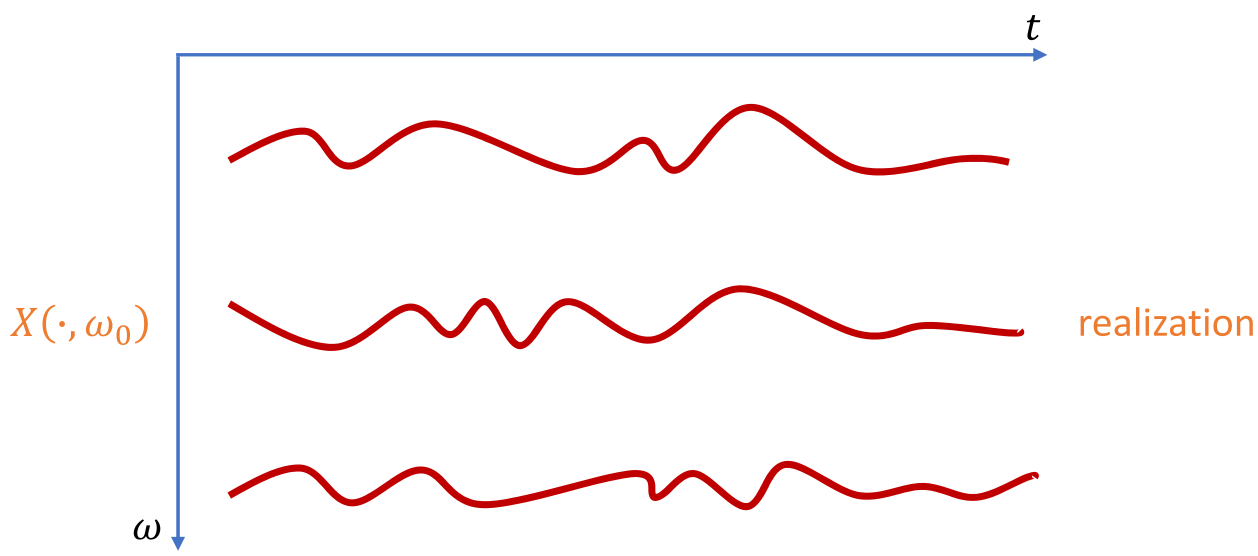


Fractional Gaussian Fields

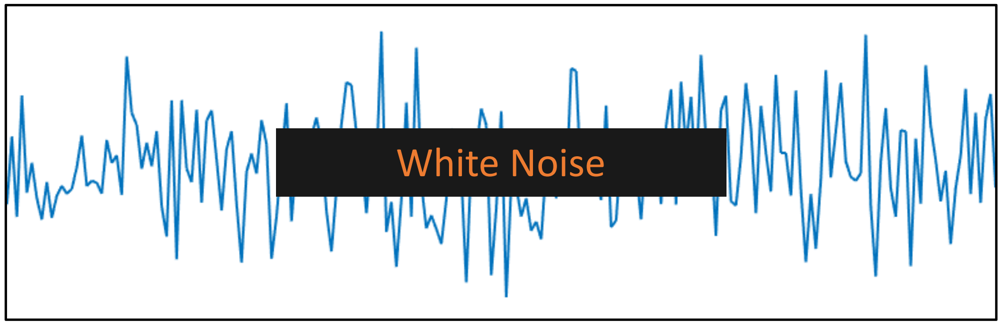
Random field

A collection of random variables:

$$X = \{X(t,\omega), t \in \mathbb{R}^d, \omega \in \Omega\}$$

Gaussian (random) field

A collection of Gaussian-distributed random variables.



Fractional Gaussian Fields

Then what is "**fractional**"?

Let I be the *integral operator*

$$I[1] = x \qquad I^2[1] = x^2/2 \qquad \sqrt{I}[x] = ?$$

$$I^a[x] = ?\,, a > 0$$

Fractional Gaussian Fields

Then what is "**fractional**"?

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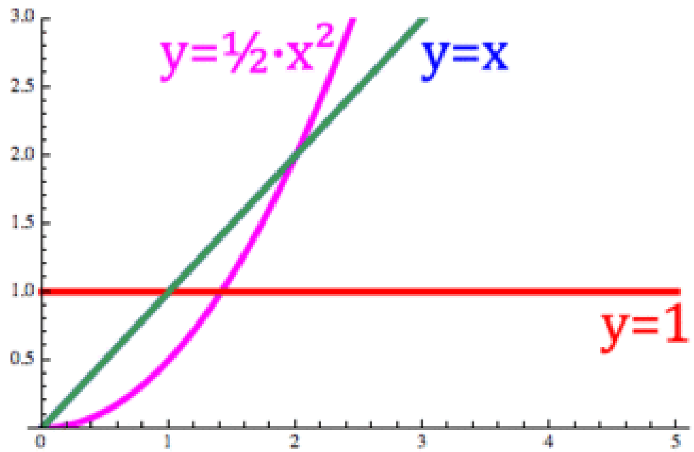
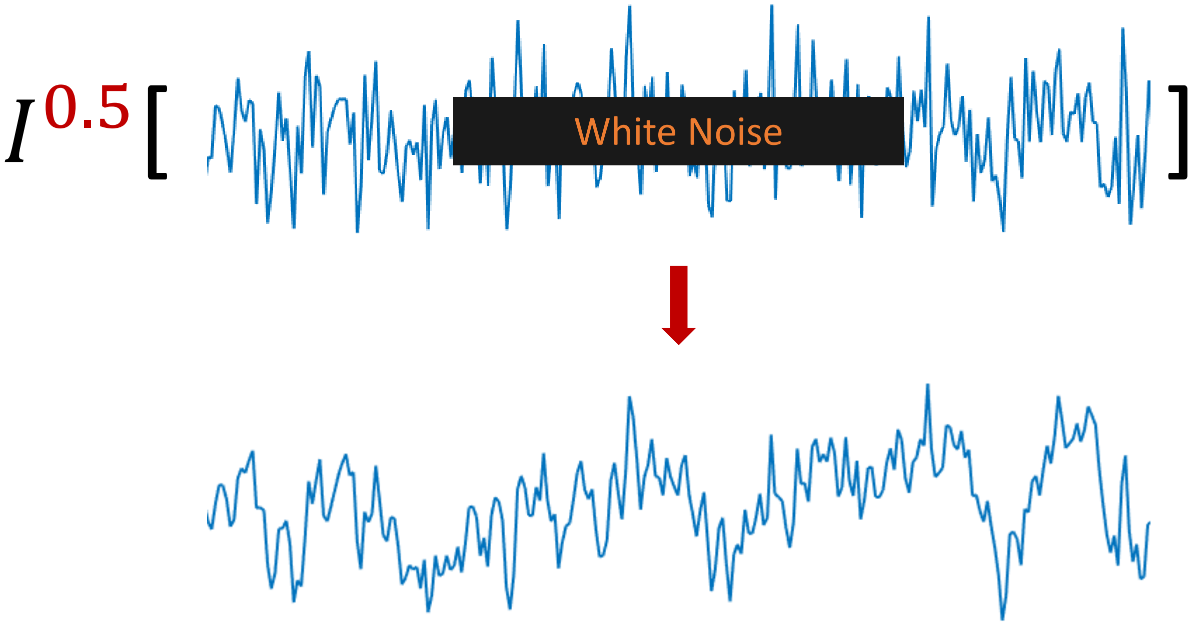


Fractional Gaussian Fields

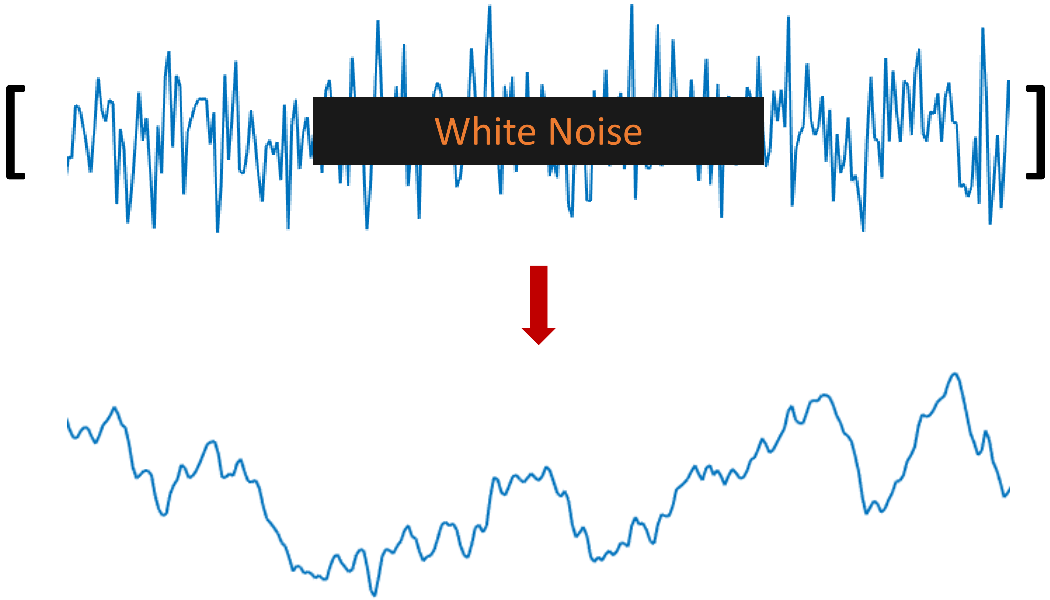
With the *fractional integral operator* I^{a}



Fractional Gaussian Fields

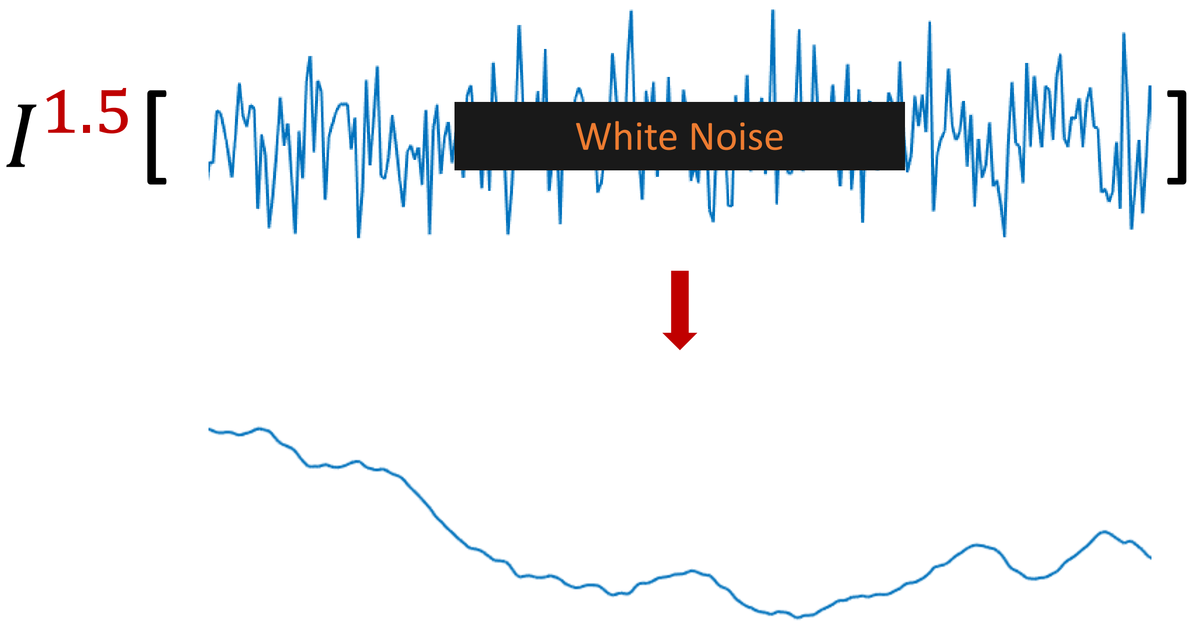
With the *fractional integral operator* I^a

$I^{1.0}$

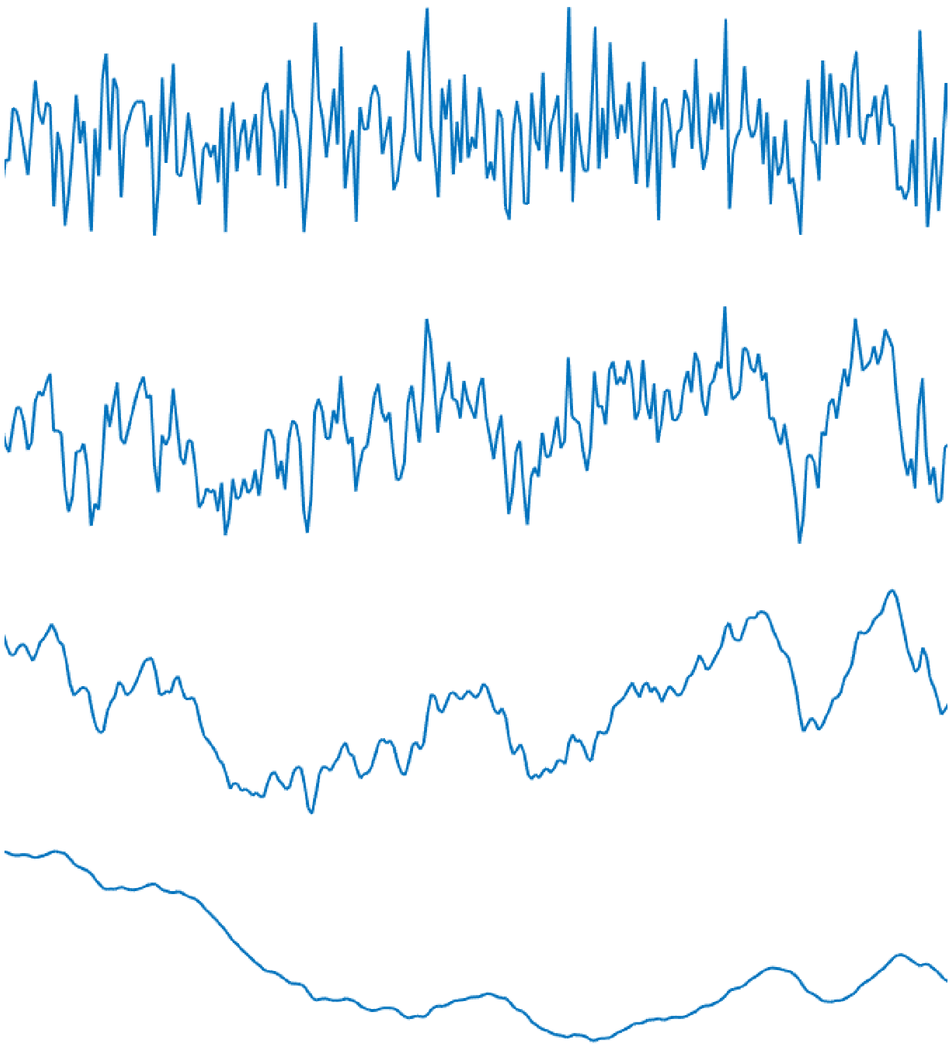


Fractional Gaussian Fields

With the *fractional integral operator* I^a



Fractional Gaussian Fields



Fractional Gaussian Fields

Fractional Gaussian Fields

d-dimensional FGF: $M = (-\Delta)^{-\frac{s}{2}} W$

W: white noise

fractional Laplacian

The FGF family

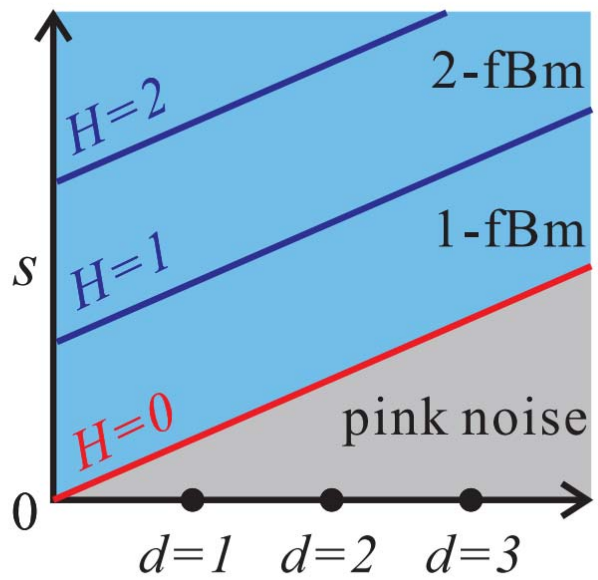
$s = 0$	white noise
$s = 1, d = 1$	Brownian motion
$\frac{1}{2} < s < \frac{3}{2}, d = 1$	fractional Brownian motion (fBm)

Fractional Gaussian Fields

d-dimensional FGF: $M = (-\Delta)^{-\frac{s}{2}} W$ W: white noise

Hurst parameter $H = s - \frac{d}{2}$

$H \nearrow$ Correlation $\nearrow$



Autocovariance Function

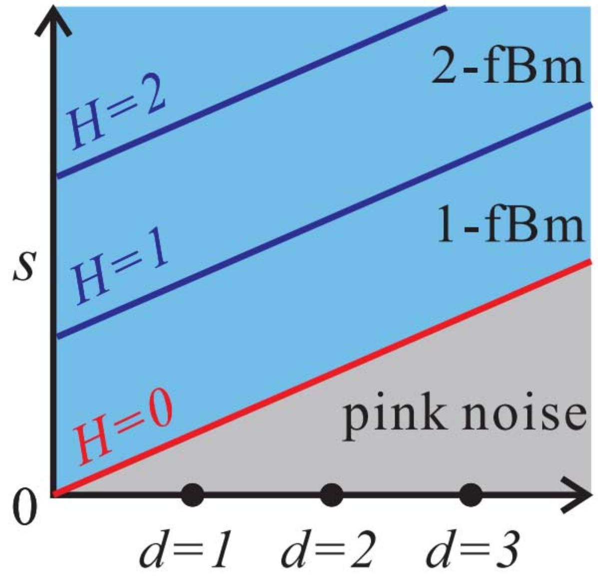
d-dimensional FGF: $M = (-\Delta)^{-\frac{s}{2}} W$ $\quad$ W: white noise

Autocovariance function of pink noise:

$$\mathrm{cov}(\mathbf{x}, \mathbf{y}) = C(H, d) S_w |\mathbf{x} - \mathbf{y}|^{2H}$$

$C(H, d)$ Scaling term

S_w Power spectral density of white noise

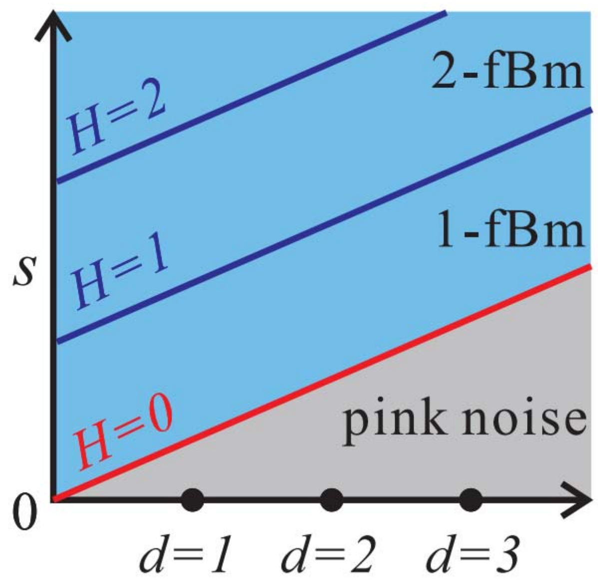


Autocovariance Function

d-dimensional FGF: $M = (-\Delta)^{-\frac{s}{2}} W$ $\quad$ W: white noise

Autocovariance function of k-fBm:

$$\mathrm{cov}(\mathbf{x},\mathbf{y}) = C(H,d) S_w \left\{ |\mathbf{x}-\mathbf{y}|^{2H} - \sum_{j=0}^{k-1} (-1)^j \binom{2H}{j} \cdot \left[\left(\frac{|\mathbf{x}|}{|\mathbf{y}|} \right)^j |\mathbf{y}|^{2H} + \left(\frac{|\mathbf{y}|}{|\mathbf{x}|} \right)^j |\mathbf{x}|^{2H} \right] \right\}$$

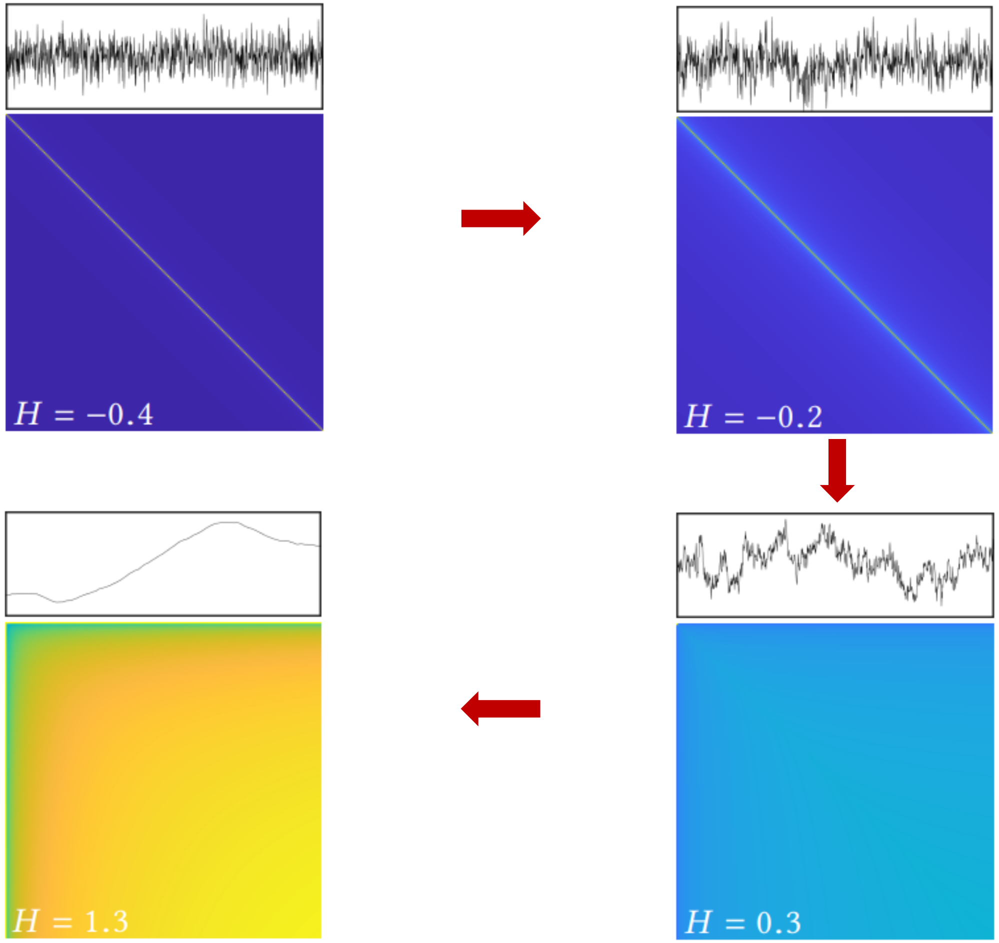


Autocovariance Function

covariance matrix (1D)

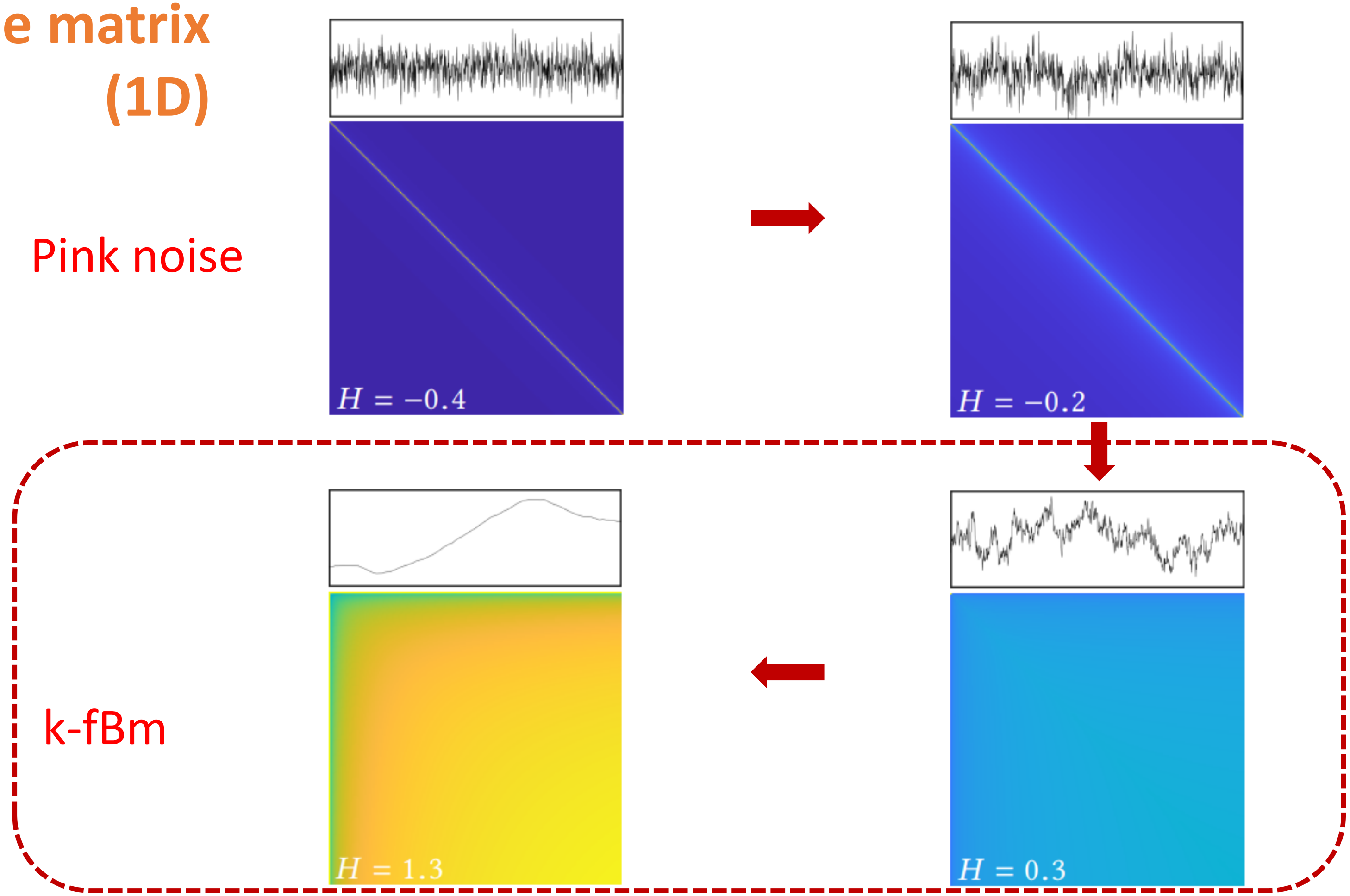
Pink noise
(H<0)

k-fBm
(H>0)



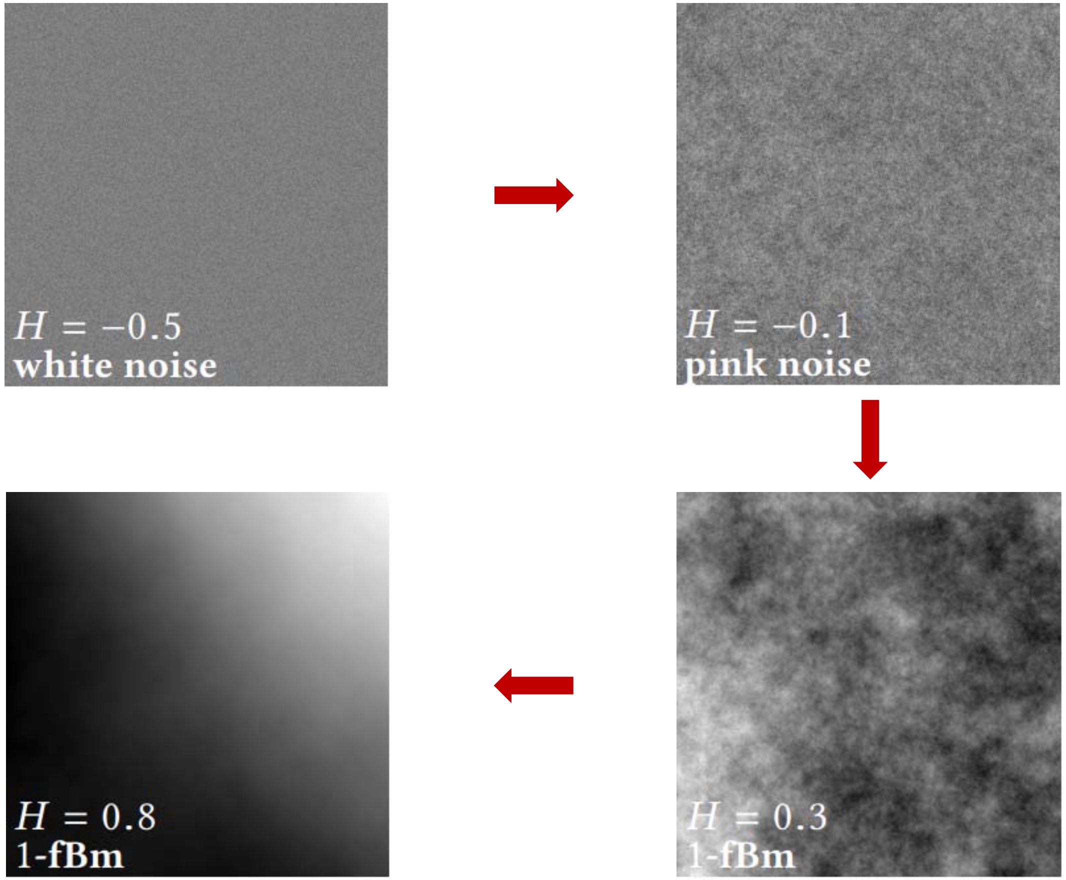
Autocovariance Function

covariance matrix (1D)



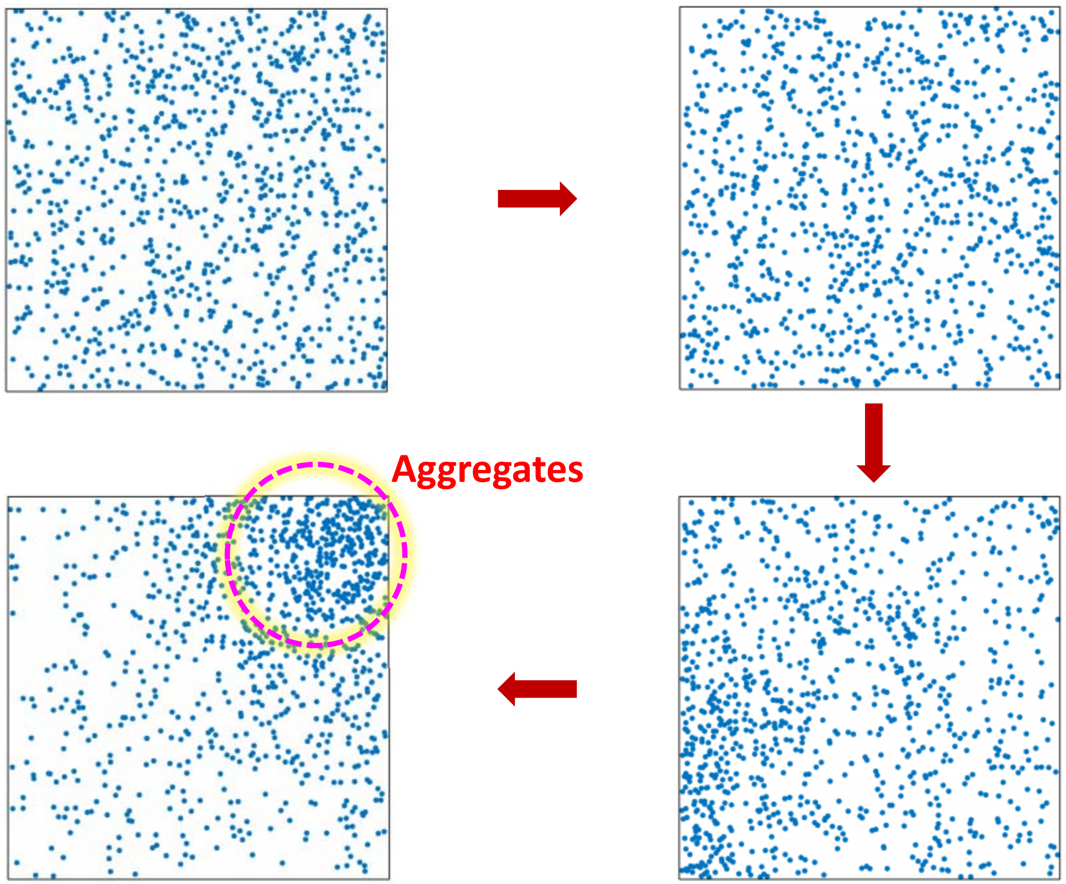
Long-range correlation

2D Fractional Gaussian Fields



2D Fractional Gaussian Fields

H ↗ Correlation ↗



Extinction Field

We model σ_t as a **FGF** :

$$\sigma_t(\boldsymbol{x}) = \sigma_m + \sigma_\mu(\boldsymbol{x})$$

$$\mathrm{var}[\sigma_t(\boldsymbol{x})] = ?$$

Macro-scale: **Constant**

Micro-scale: **FGF**

Fixing $\boldsymbol{x}$, $\sigma_t(\boldsymbol{x})$ is a random variable with mean σ_m and its variance is controlled by the FGF

Extinction Field

We model σ_t as a **FGF** :

The *line-averaged extinction* $\overline{\sigma_t}(\boldsymbol{x}) = \frac{\int_0^t \sigma_t(\boldsymbol{x}+s\widehat{\omega})\mathrm{d}s}{t} = \frac{\tau(\boldsymbol{x},t)}{t}$ is a Gaussian-distributed random variable.

$$\mathrm{var}[\overline{\sigma_t}] = \frac{1}{t^2}(\langle\tau(\boldsymbol{x},t)^2\rangle - \langle\tau(\boldsymbol{x},t)\rangle^2) = \frac{1}{t^2}\int_0^t\int_0^t \boxed{\mathrm{cov}(\boldsymbol{x}',\boldsymbol{x}'')}\,\mathrm{d}t'\mathrm{d}t''$$

autocovariance function of the **FGF**

Extinction Field

Pink noise (H<0)

$$\mathrm{cov}(\mathbf{x}, \mathbf{y}) = C(H, d) S_w |\mathbf{x} - \mathbf{y}|^{2H}$$

$$\mathrm{var}[\overline{\sigma_t}] = \frac{1}{t^2}(\langle \tau(\boldsymbol{x}, t)^2 \rangle - \langle \tau(\boldsymbol{x}, t) \rangle^2) = \frac{1}{t^2} \int_0^t \int_0^t \mathrm{cov}(\boldsymbol{x}', \boldsymbol{x}'') \mathrm{d}t' \mathrm{d}t''$$

$$\mathrm{var}_p[\bar{\sigma}_t] = -2C(H+1) S_w t^{2H}$$

Extinction Field

k-fBm (H>0)

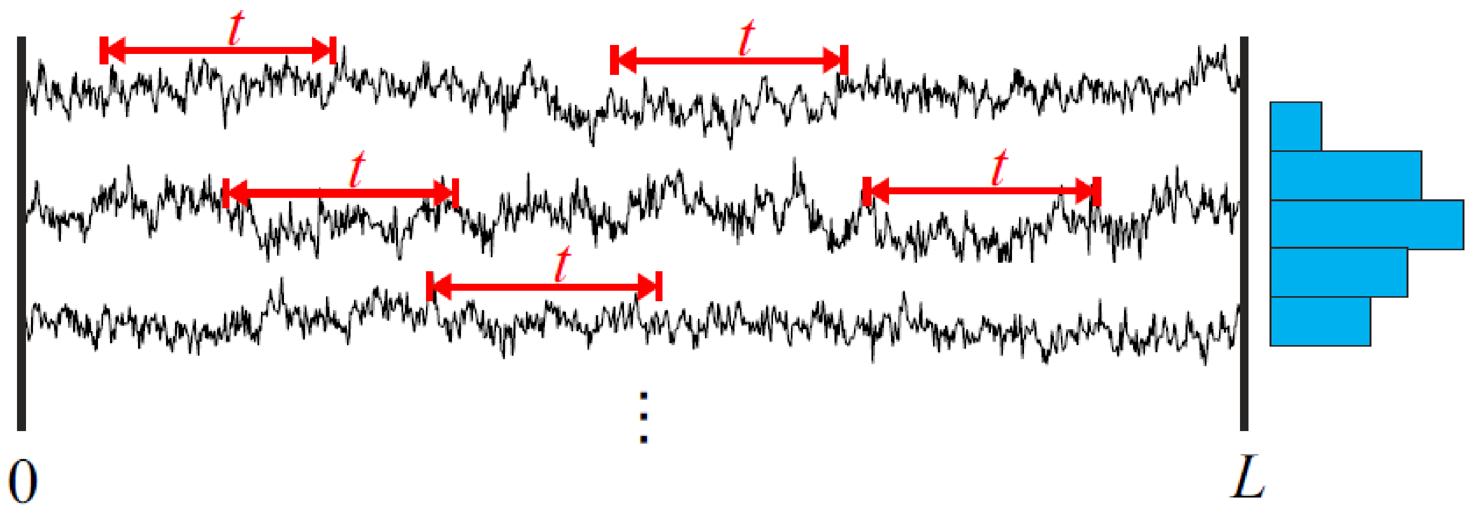
$$\mathrm{cov}(\mathbf{x}, \mathbf{y}) = C(H, d) S_w \left\{ |\mathbf{x} - \mathbf{y}|^{2H} - \sum_{j=0}^{k-1} (-1)^j \binom{2H}{j} \cdot \left[\left(\frac{|\mathbf{x}|}{|\mathbf{y}|} \right)^j |\mathbf{y}|^{2H} + \left(\frac{|\mathbf{y}|}{|\mathbf{x}|} \right)^j |\mathbf{x}|^{2H} \right] \right\}$$

$$\mathrm{var}[\overline{\sigma_t}] = \frac{1}{t^2} \left(\langle \tau(\boldsymbol{x}, t)^2 \rangle - \langle \tau(\boldsymbol{x}, t) \rangle^2 \right) = \frac{1}{t^2} \int_0^t \int_0^t \mathrm{cov}(\boldsymbol{x}', \boldsymbol{x}'') \mathrm{d}t' \mathrm{d}t''$$

$$\mathrm{var}_{kf}[\bar{\sigma}_t] = \frac{2C(H) S_w (-1)^k}{2H + 1} \binom{2H - 1}{k - 1} L^{2H}$$

Extinction Field

Numerical verification:



Extinction Field

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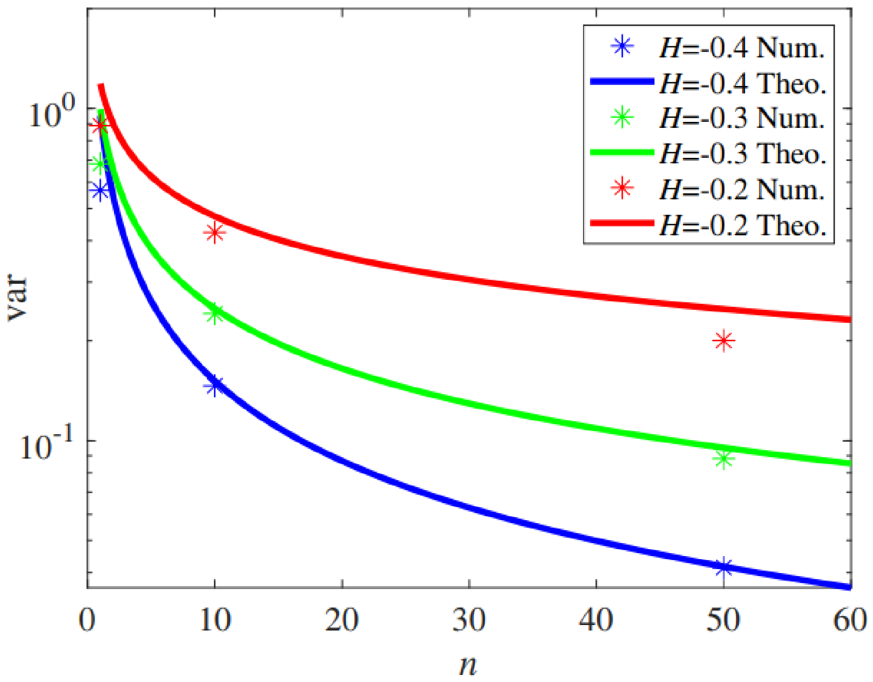
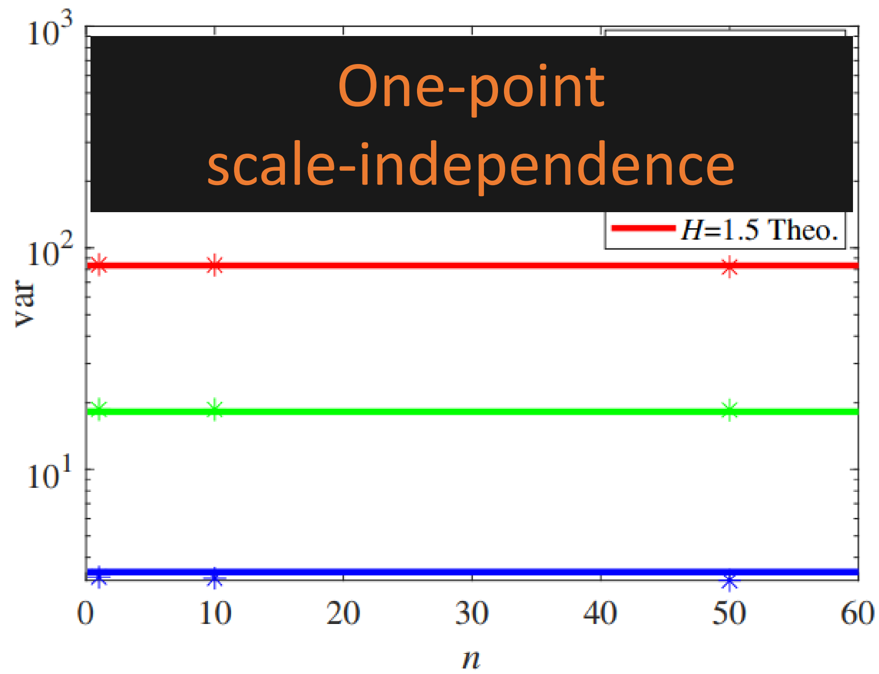


Pink noise



k-fBm

Extinction Field

The variance of $\overline{\sigma_t}(\boldsymbol{x})$:

$$\mathrm{var}[\bar{\sigma}_t] = \begin{cases} S_w t^{-1} & H = -1/2 \\ S_p(H) t^{2H} & H \in (-1/2, 0) \\ S_{kf}(H) L^{2H} & H \in (k-1, k) \end{cases}$$

White noise

Pink noise

K-fBm

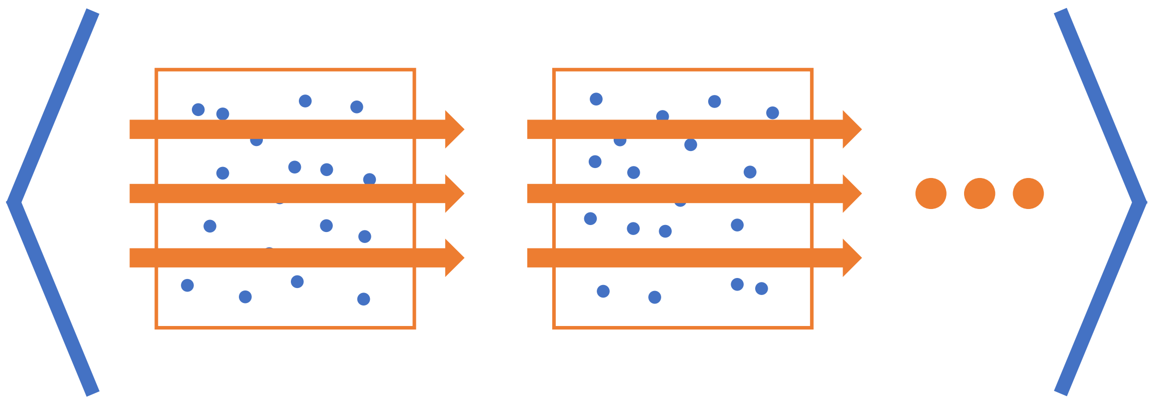
$$S_p(H) = \frac{S_w}{\Gamma(2H+3)|\sin(\pi H)|}$$

$$S_{kf}(H) = \frac{S_w}{\Gamma(2H+2)|\sin(\pi H)|}\binom{2H-1}{k-1}$$

Transmittance

Ensemble-averaged transmittance

$$\langle \mathrm{Tr}(t) \rangle = \left\langle e^{-t\overline{\sigma_t}} \right\rangle = \int_0^\infty e^{-t\overline{\sigma_t}} pdf\,(\overline{\sigma_t})\,\mathrm{d}\overline{\sigma_t}$$



Transmittance

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$$\langle \mathrm{Tr}(t) \rangle = \left\langle e^{-t\overline{\sigma_t}} \right\rangle = \int_0^{\infty} e^{-t\overline{\sigma_t}} pdf\,(\overline{\sigma_t})\mathrm{d}\overline{\sigma_t}$$

characteristic function $\varphi_{\overline{\sigma_t}}(\mathbf{i}t)$

$$\overline{\sigma_t} \sim \Gamma\left(\frac{\langle\overline{\sigma_t}\rangle^2}{\mathrm{var}[\overline{\sigma_t}]}, \frac{\langle\overline{\sigma_t}\rangle}{\mathrm{var}[\overline{\sigma_t}]}\right)$$

Use gamma distribution for non-negative extinction

Transmittance

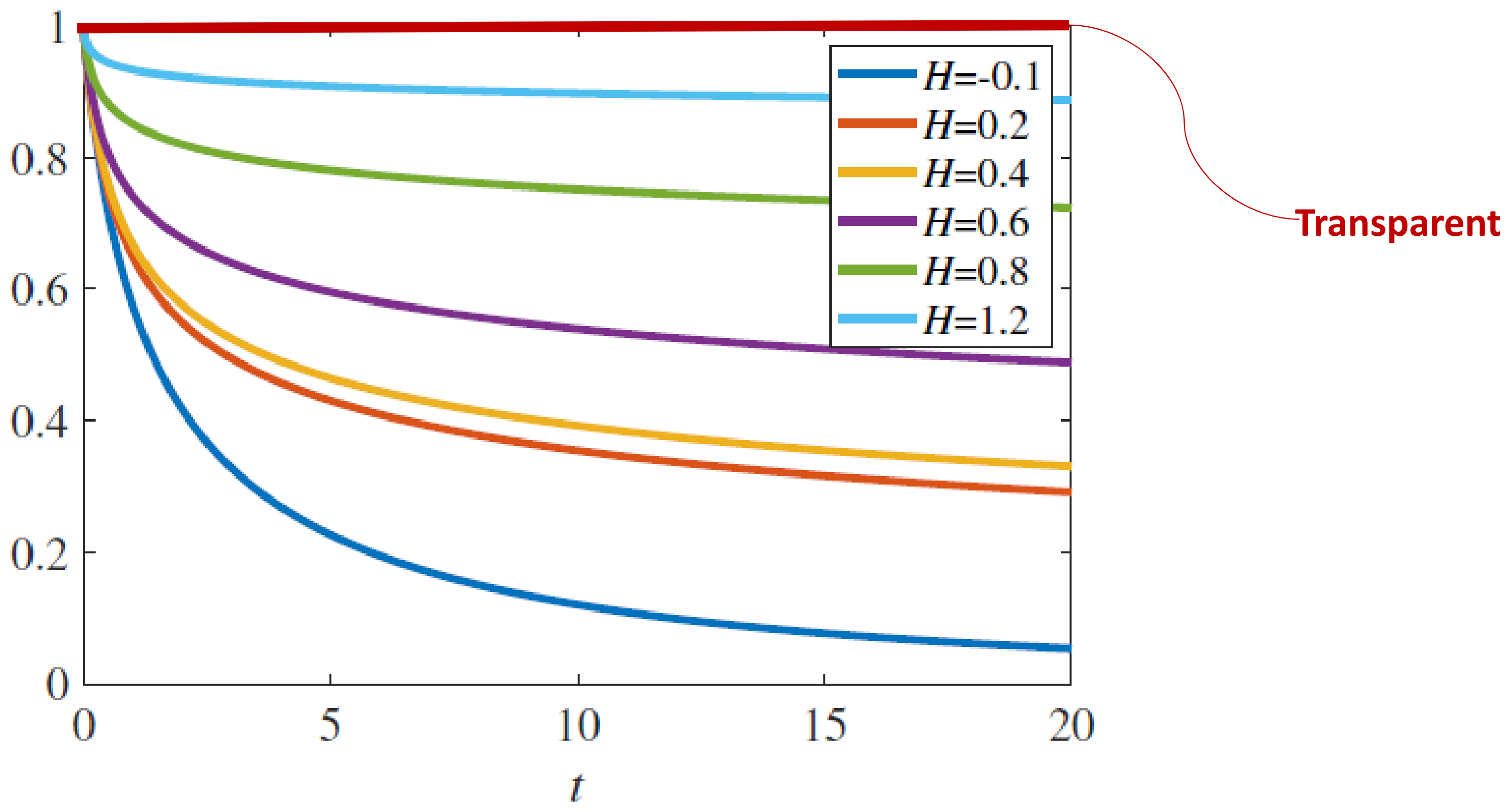
Ensemble-averaged transmittance

$$\langle \mathrm{Tr}(t) \rangle = \langle e^{-t\overline{\sigma_t}} \rangle = \int_0^{\infty} e^{-t\overline{\sigma_t}} pdf(\overline{\sigma_t}) \mathrm{d}\overline{\sigma_t}$$

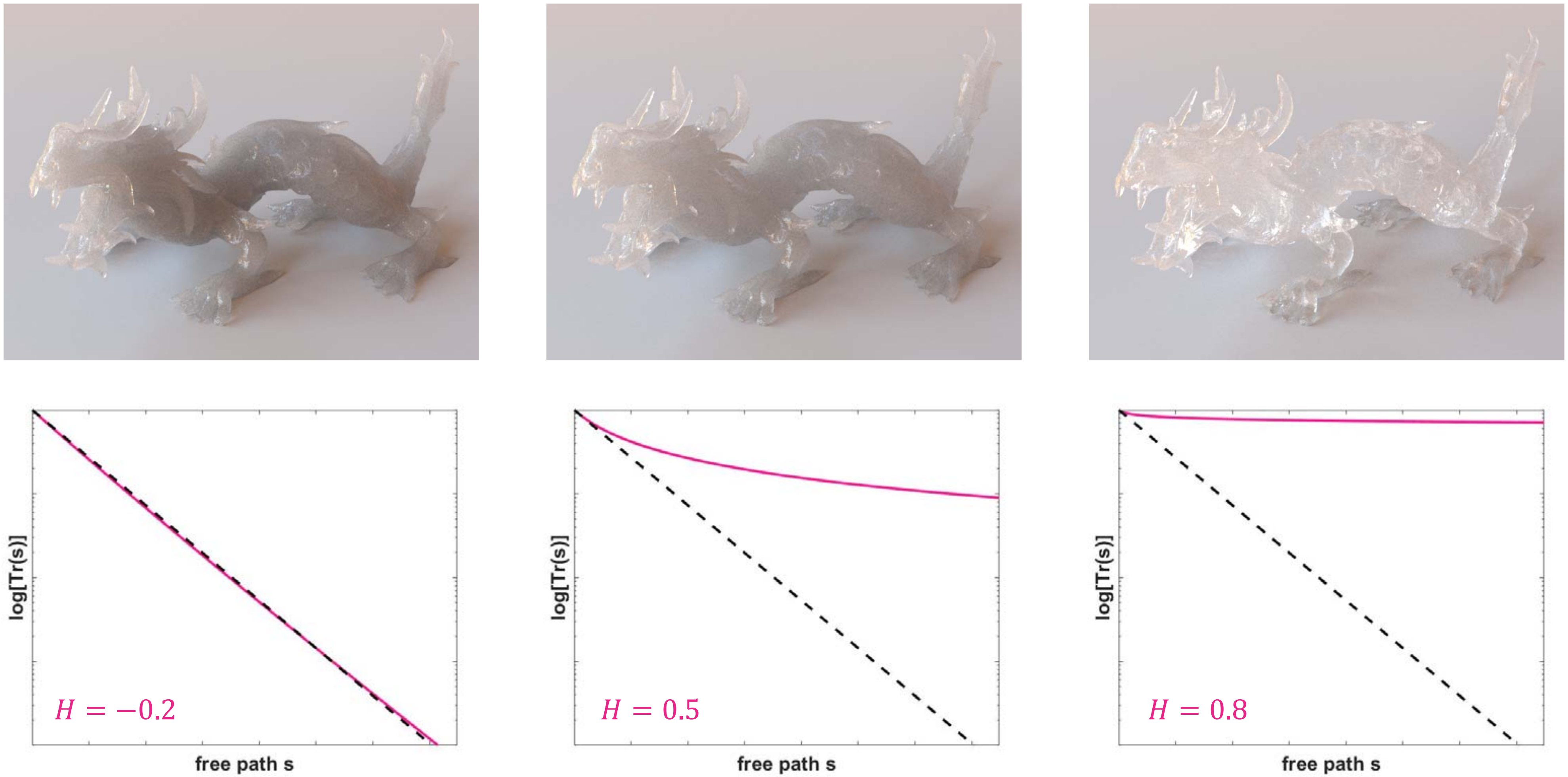
$$\mathrm{Tr}(t) = \varphi_{\bar{\sigma}_t}(\mathrm{i}t) = \left(1 + \frac{\langle \bar{\sigma}_t \rangle}{\alpha(t)} t\right)^{-\alpha(t)}$$

$$\alpha(t) = \frac{\langle \bar{\sigma}_t \rangle^2}{\mathrm{var}[\bar{\sigma}_t]}$$

Transmittance



Transmittance



Rendering Techniques

Energy-Conserving Volumetric Rendering Equation

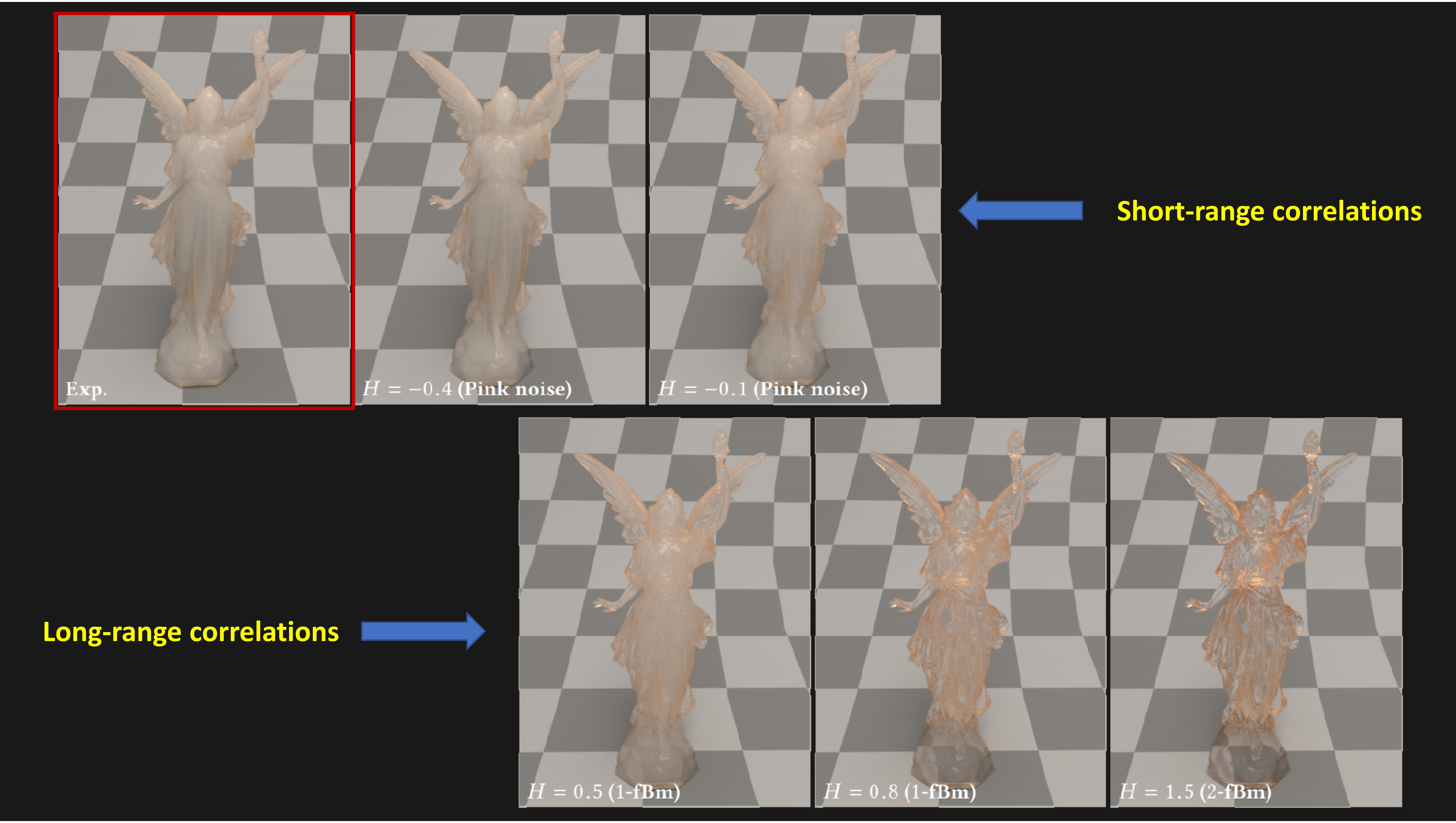
$$L_o(\mathbf{x}, \boldsymbol{\omega}) = \int_0^t T(\mathbf{x}, \mathbf{x}')\Lambda\sigma_t(\mathbf{x}')L_i(\mathbf{x}', \boldsymbol{\omega})\mathrm{d}t' + T(\mathbf{x}, \mathbf{x}_s)L_s(\mathbf{x}_s, \boldsymbol{\omega})$$

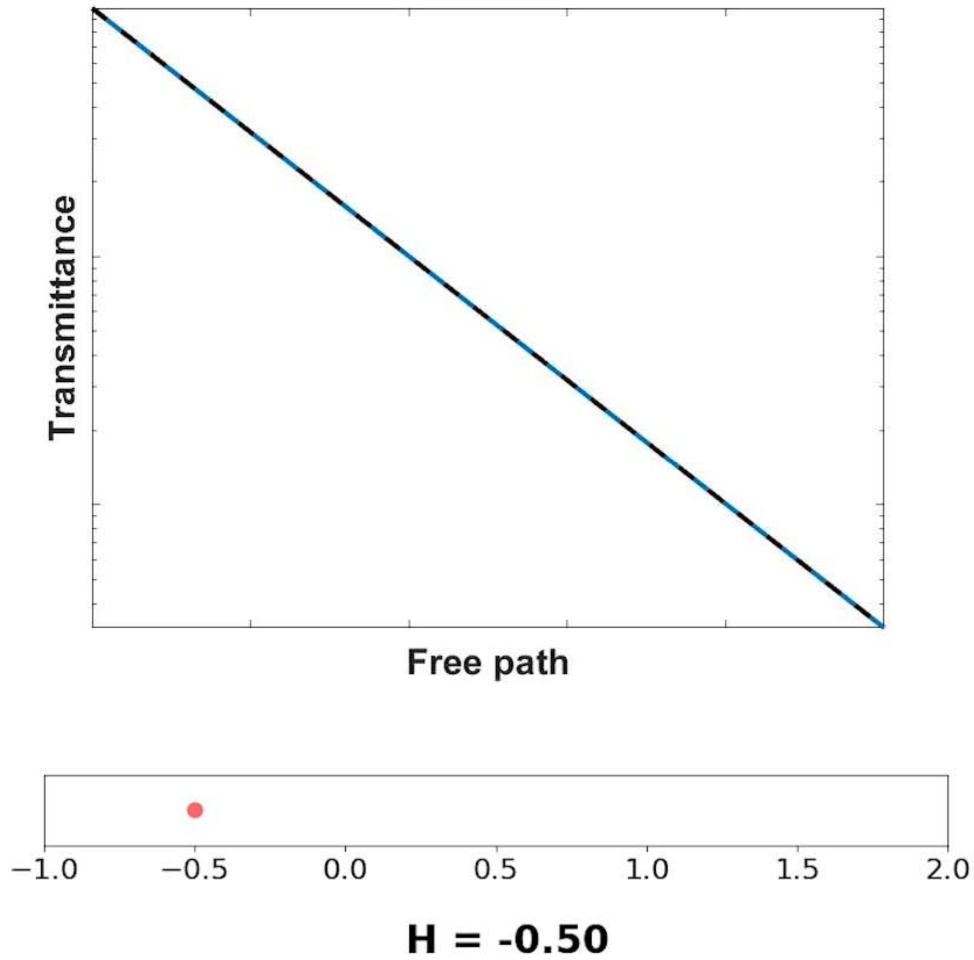
Transport kernel:

$$T(\mathbf{x}, \mathbf{x}') = \begin{cases} \mathrm{Tr}(\mathbf{x}, \mathbf{x}') & \text{if } \mathbf{x}' \text{ on surface} \\ -\frac{1}{\sigma_t(\mathbf{x}')}\frac{\partial \mathrm{Tr}(\mathbf{x},\mathbf{x}')}{\partial t} & \text{if } \mathbf{x}' \text{ in media.} \end{cases}$$

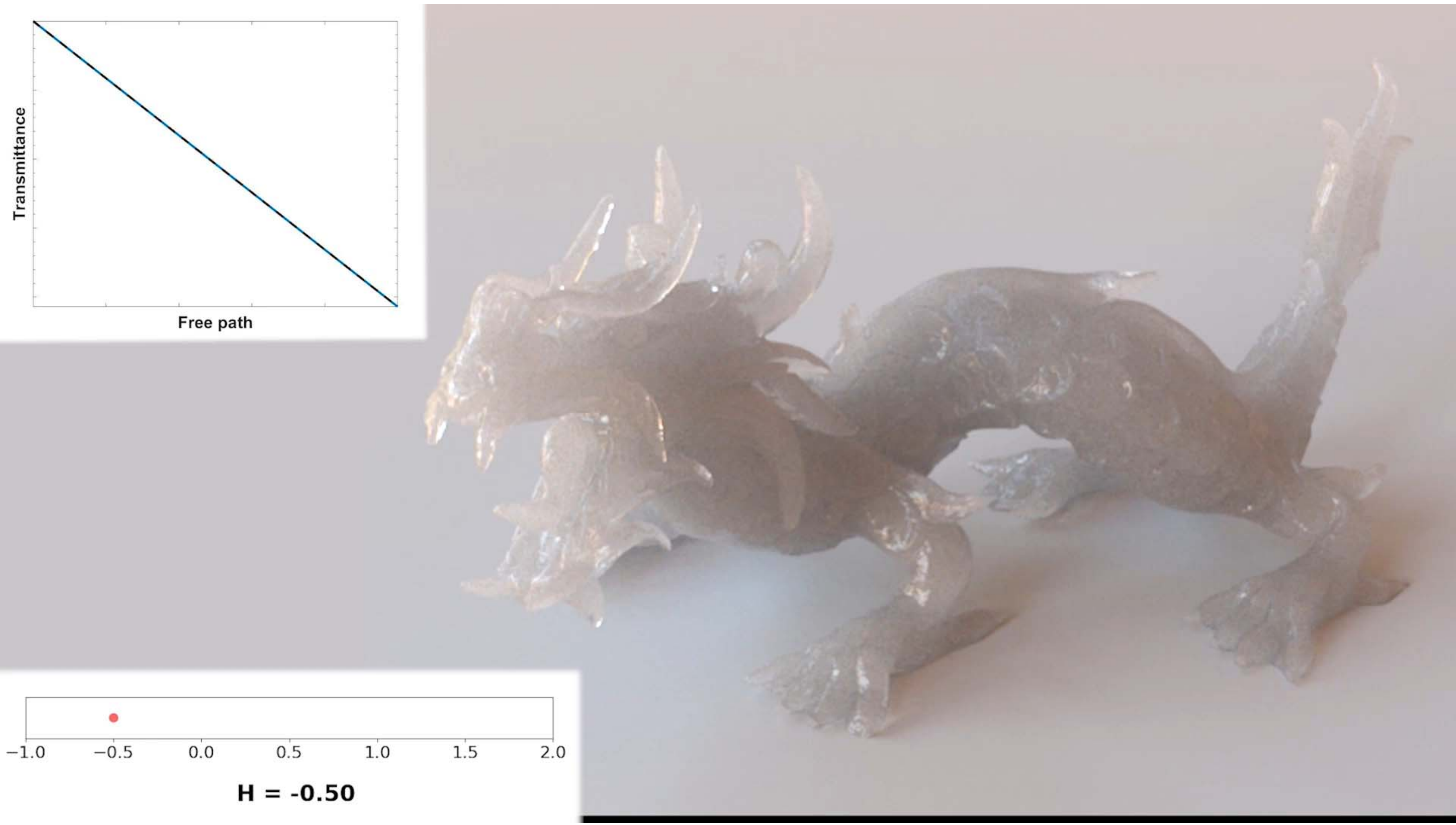
$$\frac{\partial \mathrm{Tr}(t)}{\partial t} = \mathrm{Tr}(t)\left[-\frac{\partial \alpha(t)}{\partial t}\ln\beta(t) - \frac{\partial \beta(t)}{\partial t}\frac{\alpha(t)}{\beta(t)}\right]$$

Results

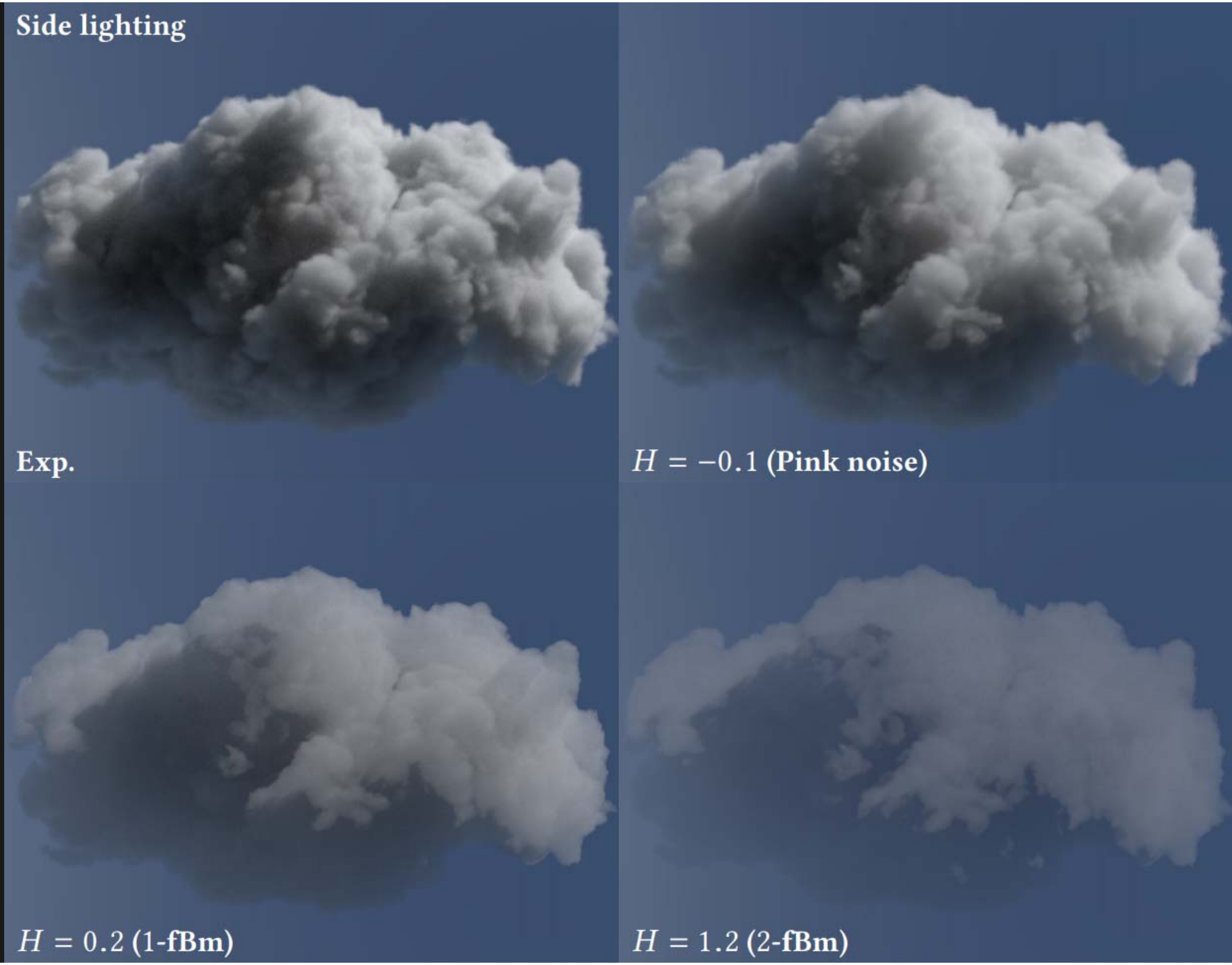

Exp.
$H = -0.4$ (Pink noise)
$H = -0.1$ (Pink noise)
Short-range correlations
Long-range correlations
$H = 0.5$ (1-fBm)
$H = 0.8$ (1-fBm)
$H = 1.5$ (2-fBm)

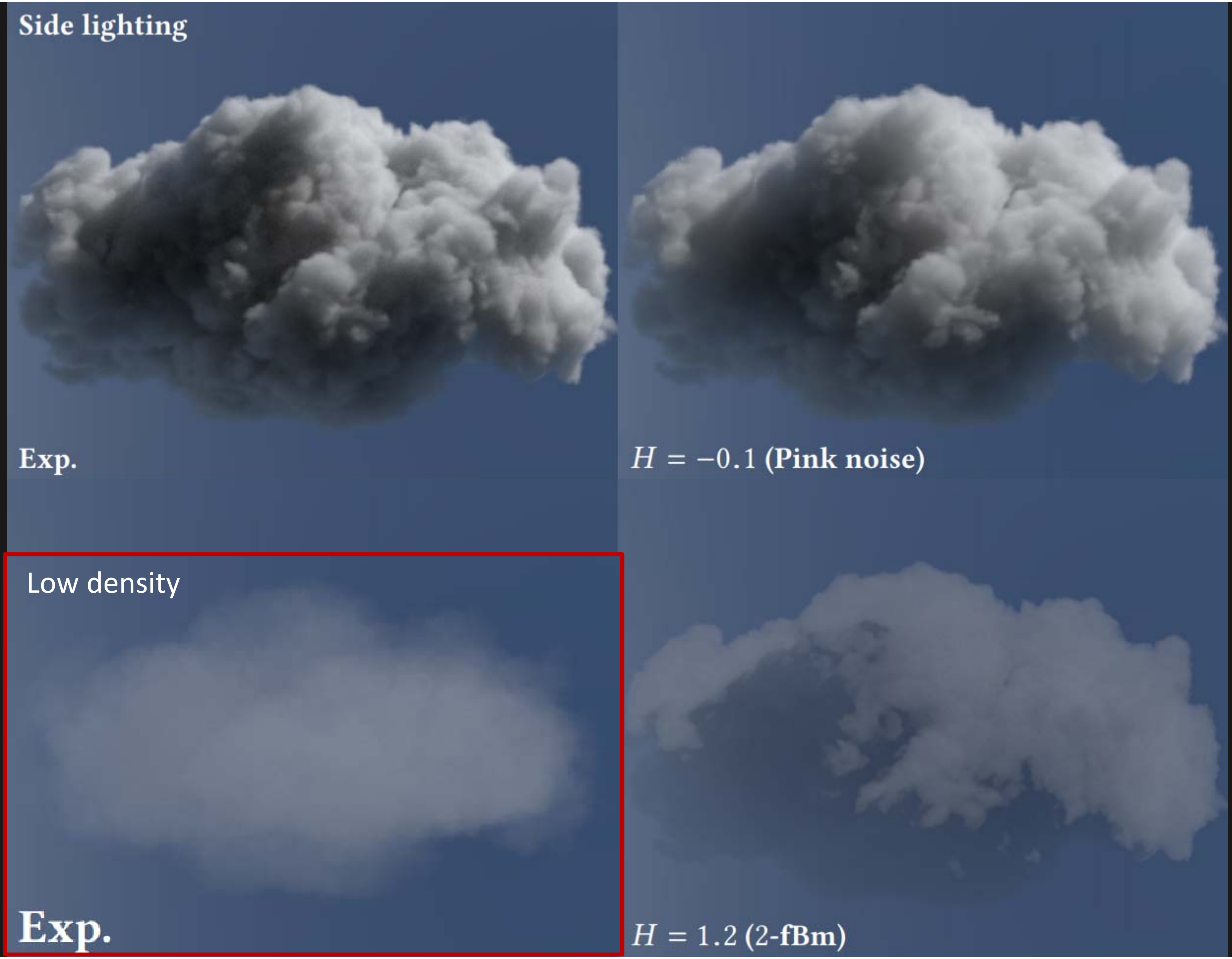


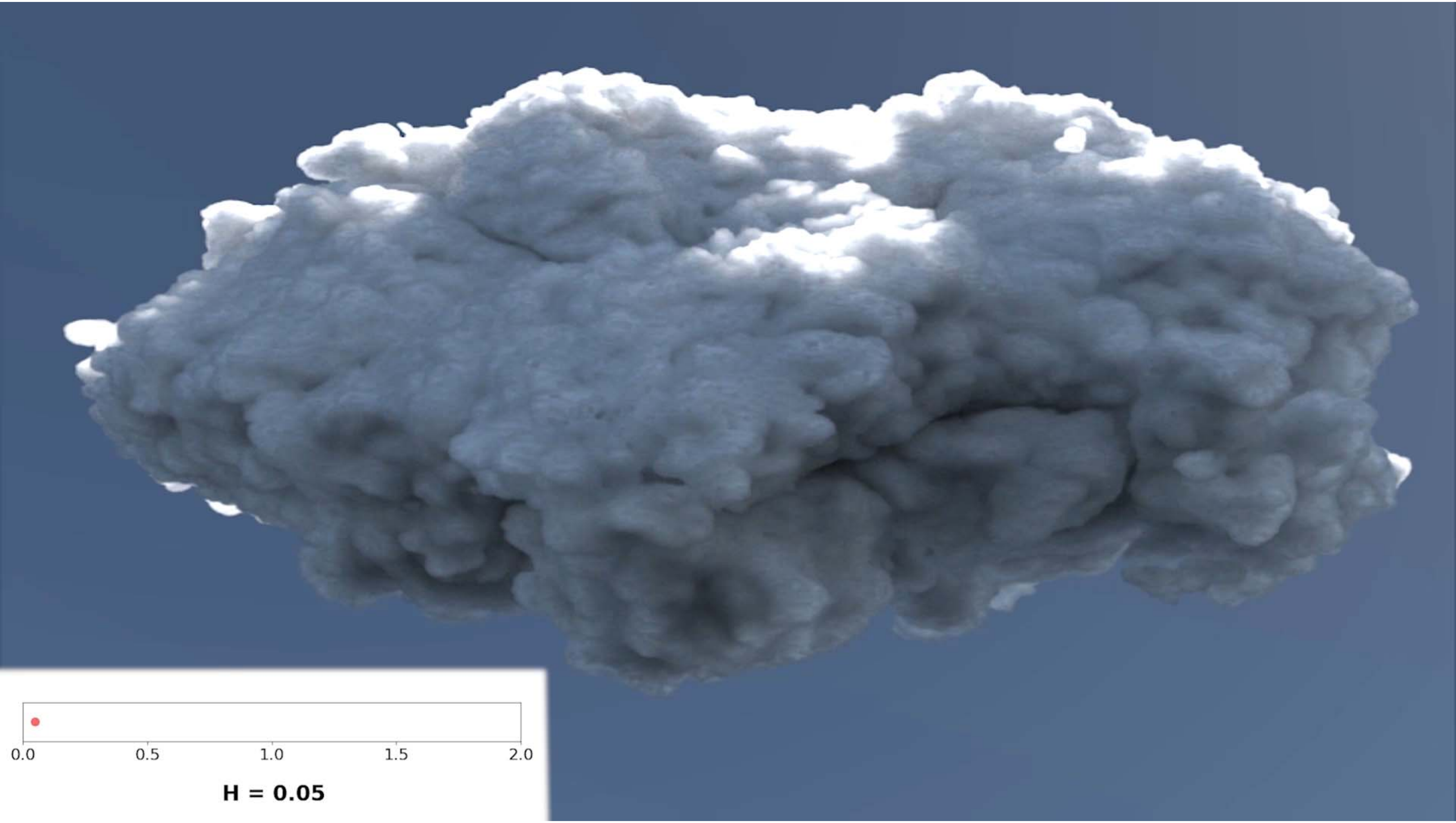
Transmittance
Free path
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
H = -0.50



Transmittance
Free path
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
H = -0.50

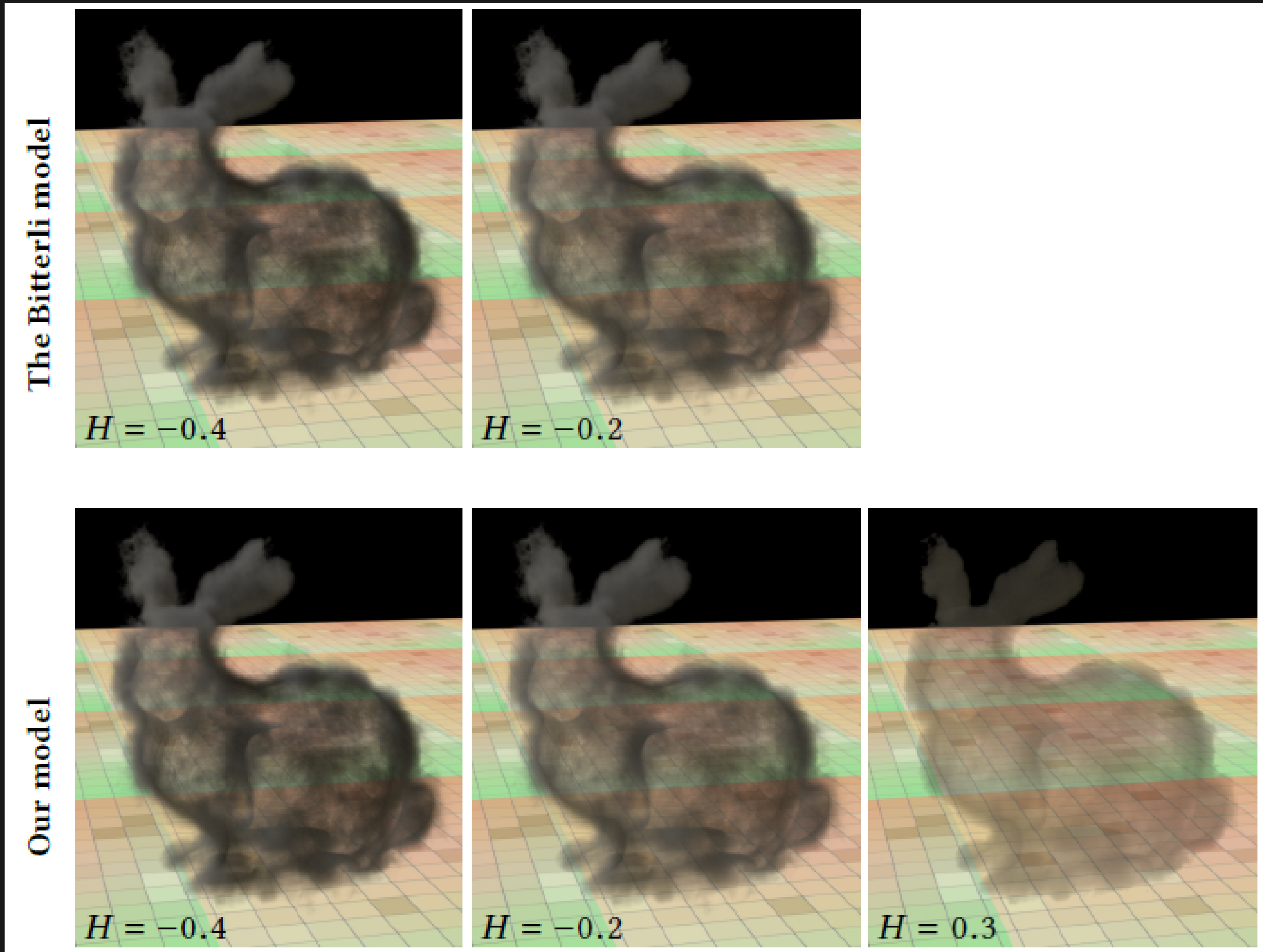

Side lighting
Exp.
$H = -0.1$ (Pink noise)
$H = 0.2$ (1-fBm)
$H = 1.2$ (2-fBm)


Side lighting
Exp.
H = −0.1 (Pink noise)
Low density
Exp.
H = 1.2 (2-fBm)


0.0
0.5
1.0
1.5
2.0
H = 0.05

Comparing to the GBE [Jarabo et al. 2018]

Comparing to the Bitterli model [Bitterli et al. 2018]



Uncorrelated media

Correlated media

Conclusion

- A mathematical tool for physically-based modeling spatial correlations in random media.
- Using k-th order fBm to generate long-range correlations.
- The usage of the non-exponential transmittance functions in an energy-conserving RTE framework.


Thank you!
Q&A
H=0.2
H=-0.4
H=1.8
H=1.2
H=0.85