SIGGRAPH THINK BEYOND

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FAST AND SCALABLE TURBULENT FLOW SIMULATION WITH TWO-WAY COUPLING

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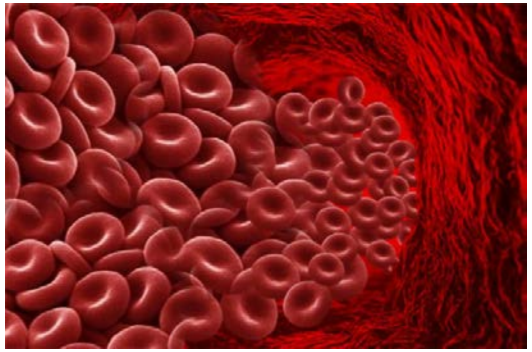
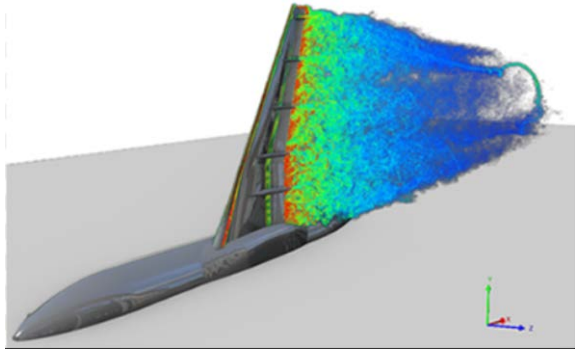
Motivation

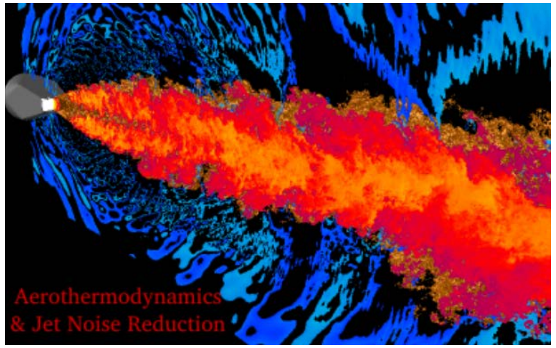
- **Large-scale simulation**
- **Capture turbulence**
- **Two-way coupling**

Applications

- Feature film special effects
- Computer games
- Medicine (e.g. blood flow in heart)
- Designing aircraft, ship, train…
- Turbulence study







Fluid model

Incompressible Navier-Stokes (N-S) equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \upsilon \nabla \cdot \nabla \boldsymbol{u} + \boldsymbol{F}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

non-linear advection

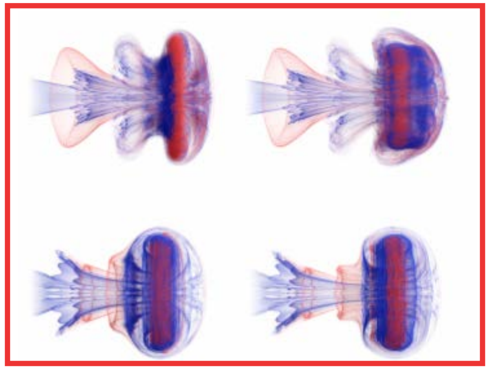
Compute pressure p by Poisson equation

$$\nabla^2 p = \rho(\nu \nabla \cdot \nabla^2 \boldsymbol{u} - \nabla \cdot (\boldsymbol{u} \cdot \nabla \boldsymbol{u})) + \nabla \cdot \boldsymbol{F}$$

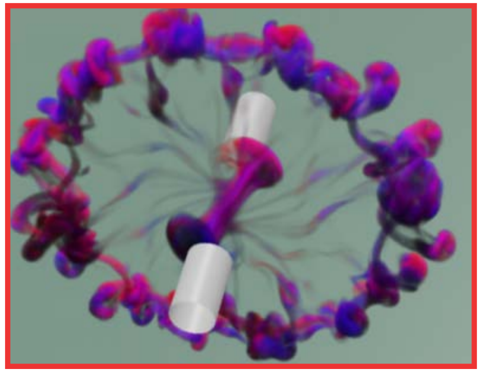
Related work

Fluid flow simulations

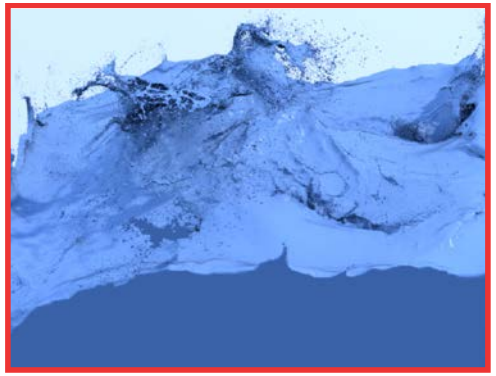
- Grid-based
- Mesh-based
- Particle-based
- Hybrid methods
- Data-driven approaches



Reflection-advection solver
Zehnder et al. SIG ’18

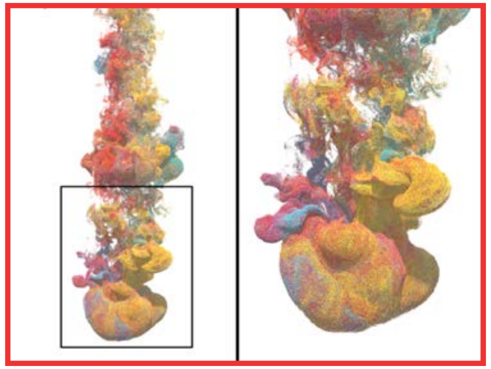


BiMocq^2 Solver
Qu et al. SIG ’19



Tetrahedral Meshes
Ando et al. SIG ’13

Implicit SPH method
Peer et al. SIG ’15



PolyPIC
Fu et al. SIG ’17

Vorticity-based
Zhang et al. SIG ’15

tempoGAN: super-resolution
Xie et al. SIG ’18

Related work

Fluid flow simulations

- Grid-base
- Mesh-bas
- Particle-based
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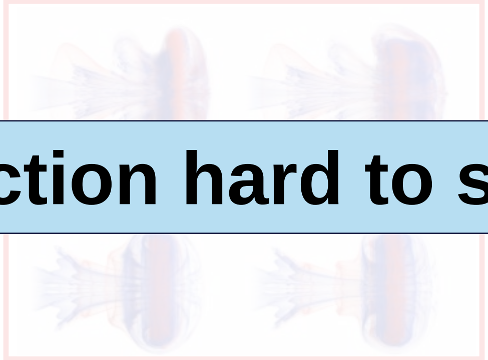
- Data-driven



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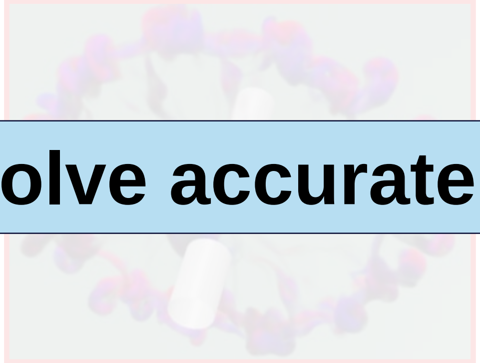


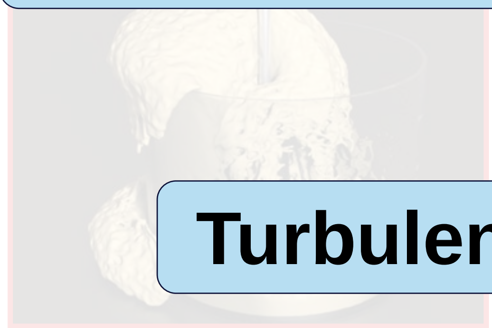
BiMocq^2 Solver

Tetrahedral Meshes
Ando et al. SIG ’13

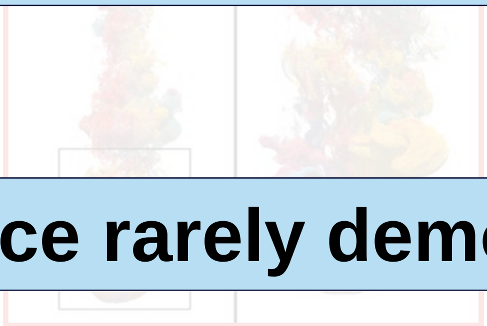
Nonlinear advection hard to solve accurately

Global linear systems need to be solved

Turbulence rarely demonstrated



Implicit SPH method
Peer et al. SIG ’15



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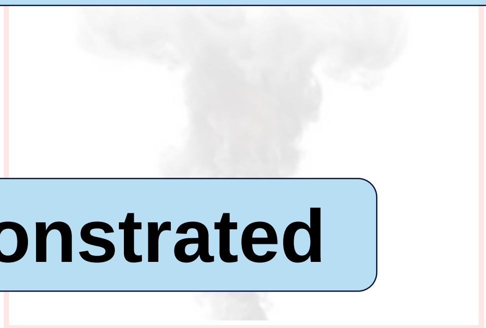
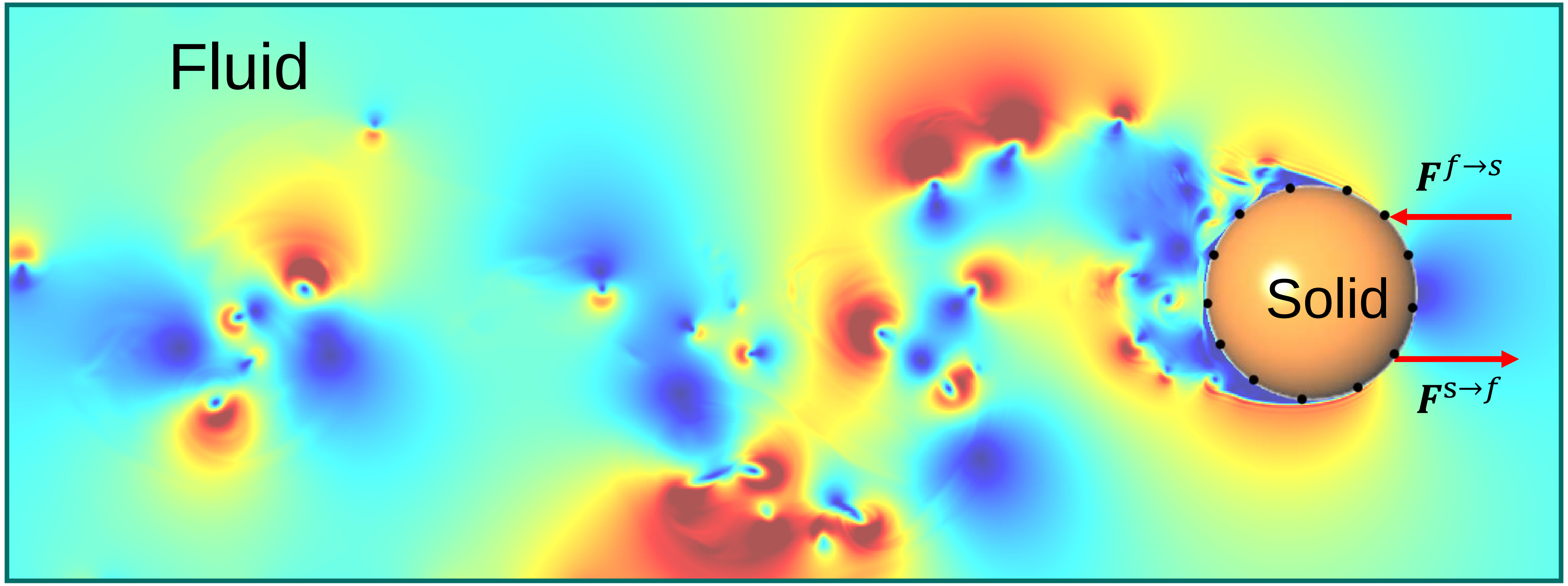


Vorticity-based
Zhang et al. SIG ’15

tempoGAN: super-resolution
Xie et al. SIG ’18

Fluid-solid coupling



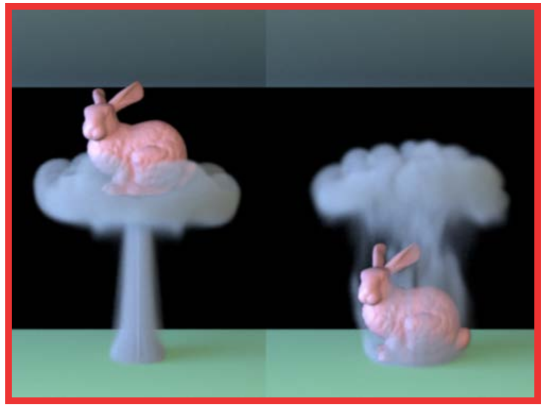
Boundary forces

Mutual interaction

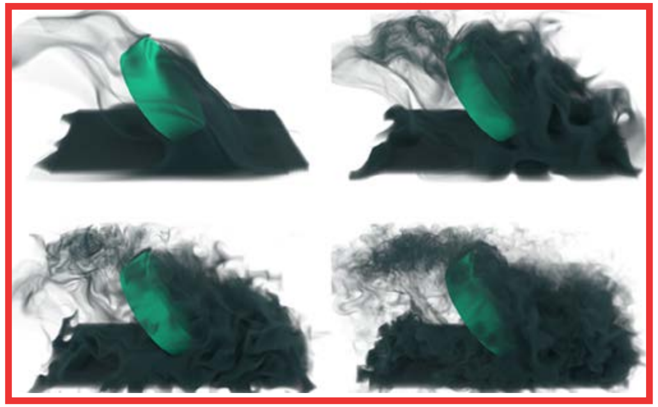
Related work

Fluid-solid coupling

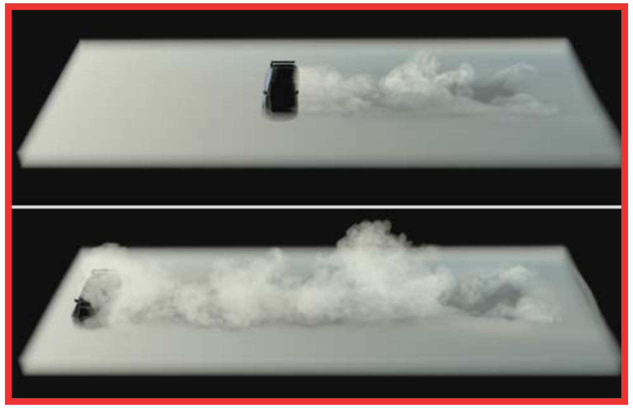
- Voxelized boundaries
- Hybrid grid-particle methods
- Cut-cell based



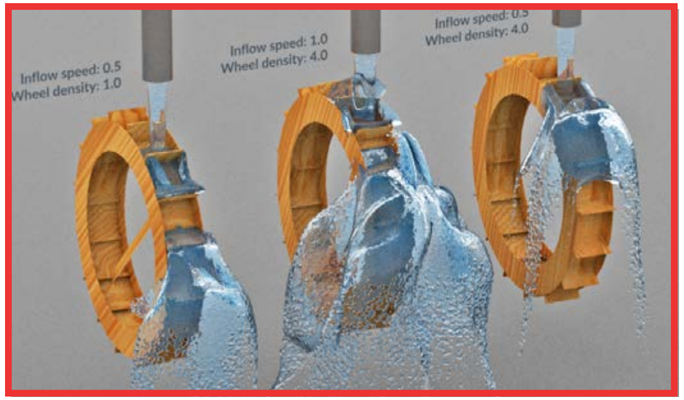
Eulerian Solid-Fluid Coupling
Teng et al. SIG ’16



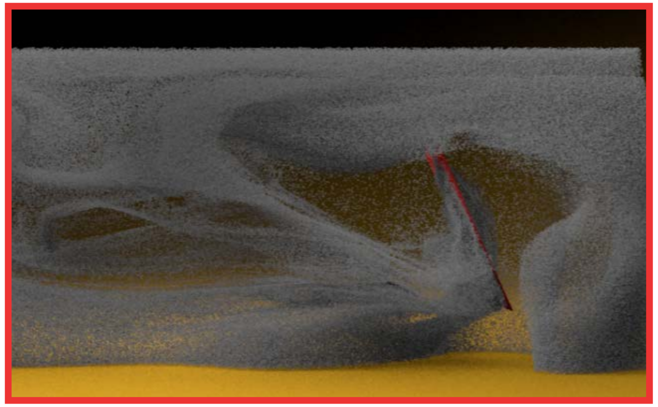
Scalable Laplacian Eigenfluids
Cui et al. SIG ’18



BiMocq^2 Solver
Qu et al. SIG ’19

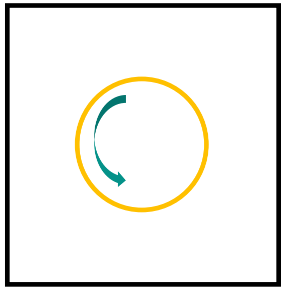


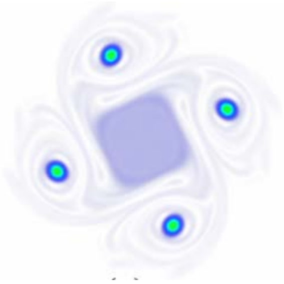
Moving least squares MPM
Hu et al. SIG ’16



Cut-cell method
Azevedo et al. SIG ’16

Dispersion errors at small time steps

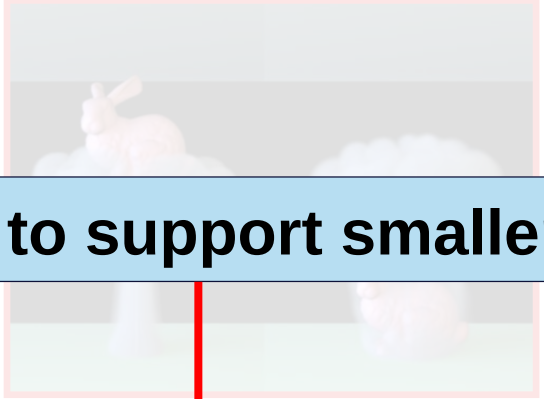




Related work

Fluid-solid coupling

- Voxelized boundaries
- Hybrid grid-particle methods
- Cut-cell based

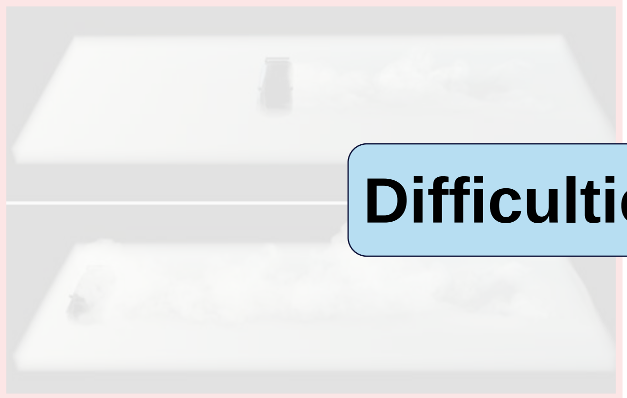


Eulerian Solid-Fluid Coupling
Teng et al. SIG '16

Scalable Laplacian Eigenfluids
Cui et al. SIG '18

Unable to support smaller Δt

Difficulties for two-way coupling turbulent flows



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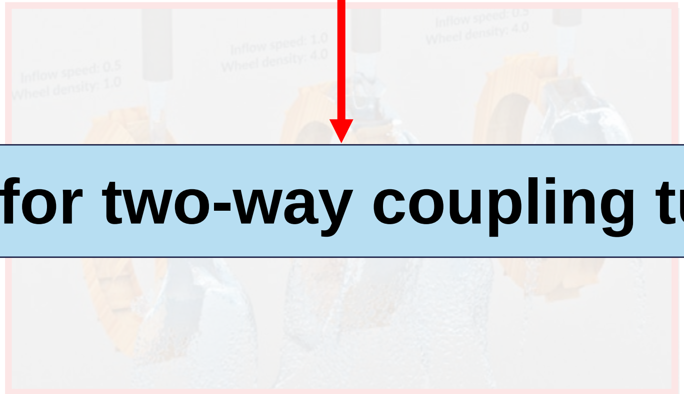

Moving least squares MPM
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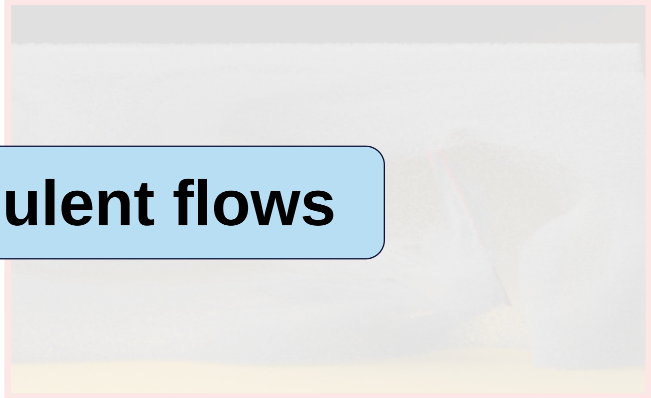


Cut-cell method
Azevedo et al. SIG '16

Macroscopic model

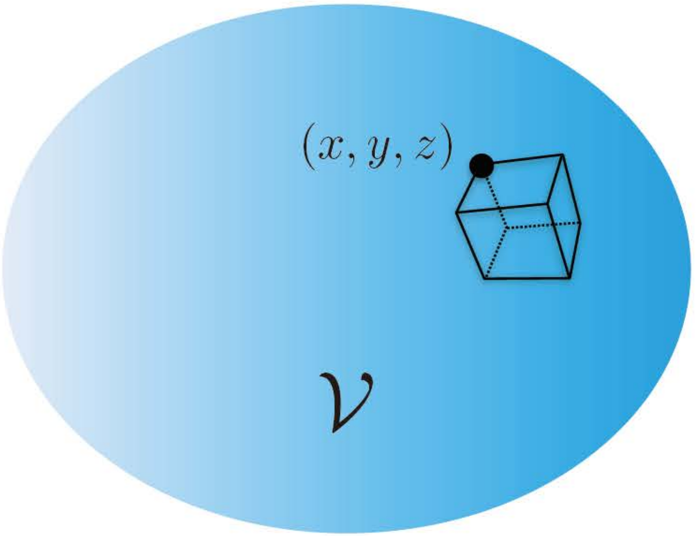
Well-known macroscopic model

- Incompressible Navier-Stokes (N-S) equations
- **Non-linear advection**
- **Solving Poisson equation**

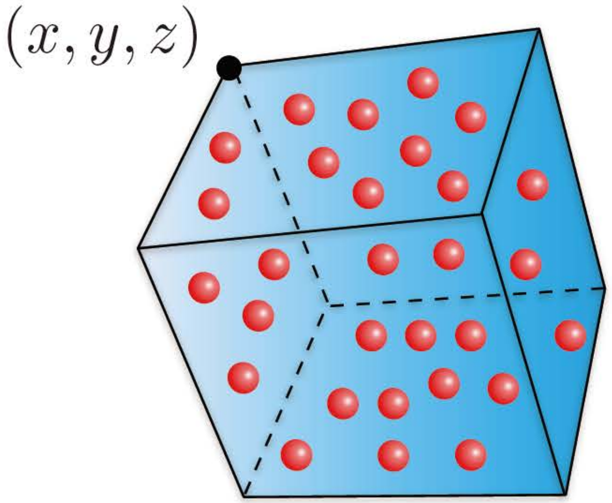
$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \upsilon \nabla \cdot \nabla \boldsymbol{u} + \boldsymbol{F}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

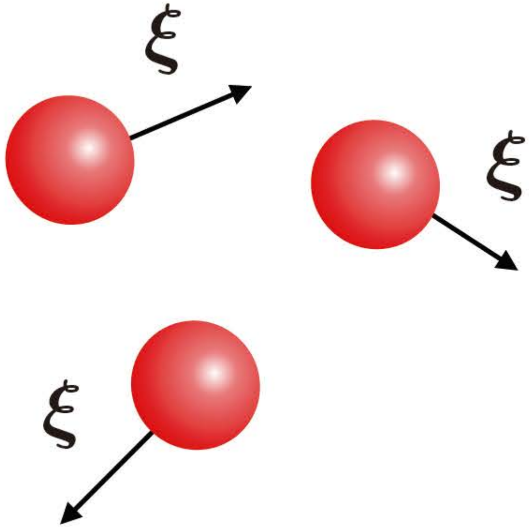
Different scales to describe flow



Macroscopic



Mesoscopic



Microscopic

Navier-Stokes (N-S) equations

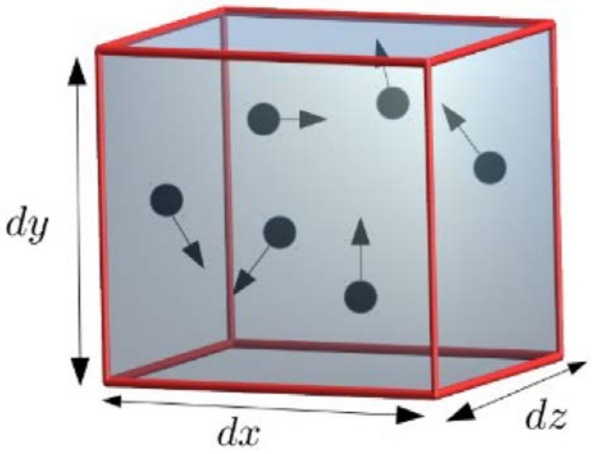
Boltzmann transport equation

Molecular dynamics

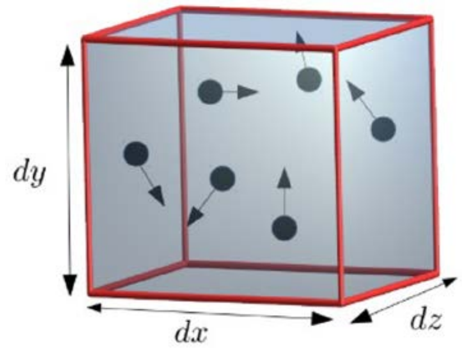
Kinetic formulation

Introducing a mesoscopic description of fluid

- Particle distribution function: $f(\boldsymbol{x}, \boldsymbol{v}, t)$



Kinetic formulation



Introducing a mesoscopic description of fluid

- Particle distribution function: $f(\boldsymbol{x}, \boldsymbol{v}, t)$
- Macroscopic quantities (**moments**)

Density $$\rho = \int f d\boldsymbol{v}$$ Zero order moment

Velocity $$\rho \boldsymbol{u} = \int \boldsymbol{v} f d\boldsymbol{v}$$ First order moment

Pressure $$p = \frac{1}{3} \int \|\boldsymbol{v} - \boldsymbol{u}\|_2^2 f d\boldsymbol{v}$$

Kinetic formulation

Introducing a mesoscopic description of fluid

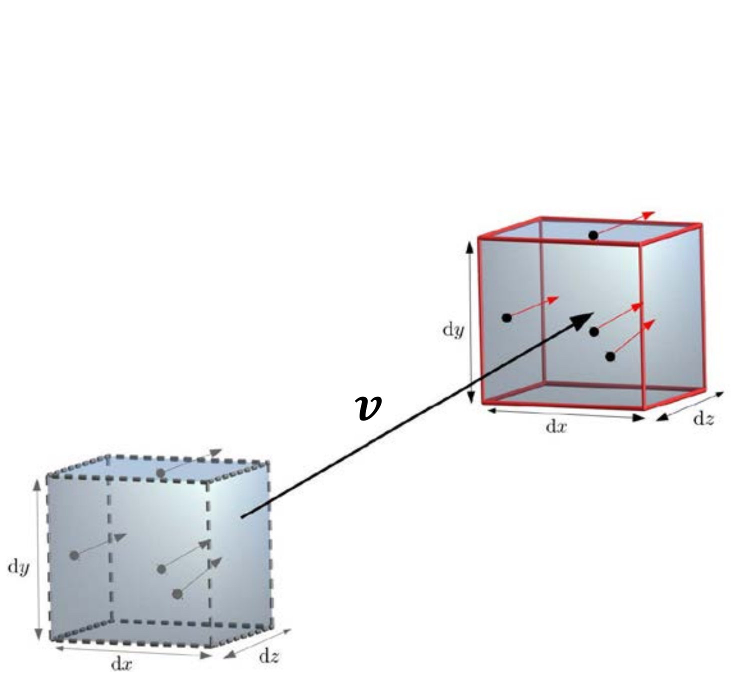
- Particle distribution function: $f(\boldsymbol{x}, \boldsymbol{v}, t)$
- Boltzmann transport equation:

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f = \Omega(f) + \boldsymbol{F} \cdot \nabla_{\boldsymbol{v}} f$$

Kinetic formulation

Introducing a mesoscopic description of fluid

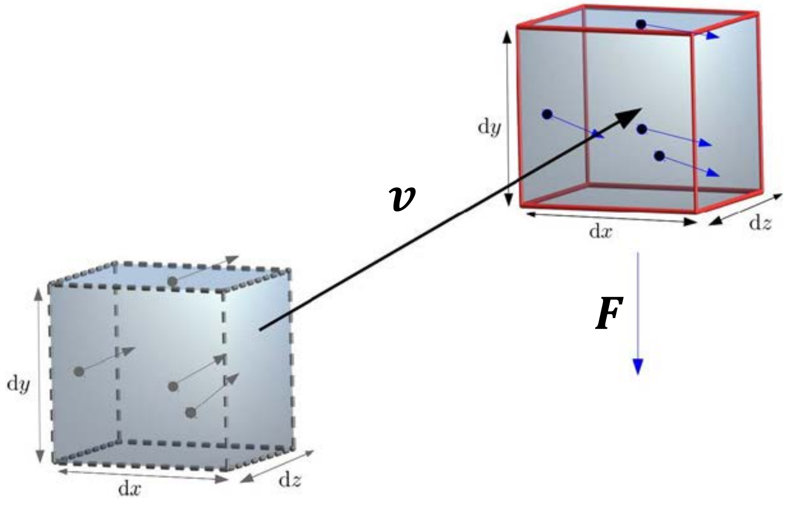
- Boltzmann transport equation: $$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f = \Omega(f) + \boldsymbol{F} \cdot \nabla_{\boldsymbol{v}} f$$

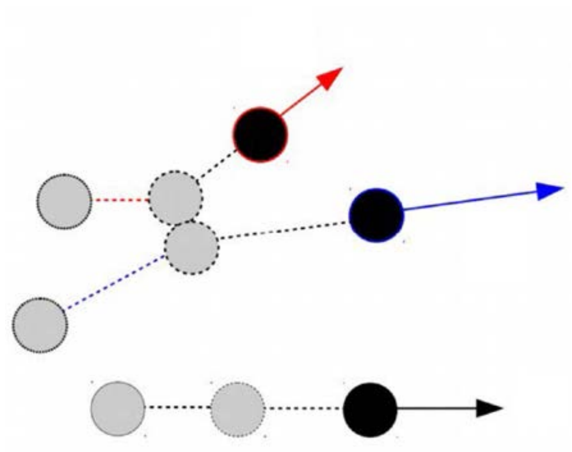


Kinetic formulation

Introducing a <u>mesoscopic</u> description of fluid

- Boltzmann transport equation: $\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f = \Omega(f) + \boldsymbol{F} \cdot \nabla_{\boldsymbol{v}} f$





Kinetic formulation

- **Collision modeling**

$$\int \Omega \, d\boldsymbol{v} = 0$$

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f = \Omega(f) + \boldsymbol{F} \cdot \nabla_{\boldsymbol{v}} f$$

$$\int \boldsymbol{v}\Omega \, d\boldsymbol{v} = 0$$

BGK model

$$\boldsymbol{\Omega} = -\frac{1}{\tau}(\boldsymbol{f} - \bar{\boldsymbol{f}})$$

Equilibrium distribution

$$\bar{f}(\rho, \boldsymbol{u}) = \frac{\rho}{(2\pi)^{D/2}} \exp\left(-\frac{\|\boldsymbol{v} - \boldsymbol{u}\|_2^2}{2}\right)$$

Kinetic formulation

- **Relation between Boltzmann equation and N-S equation**

$$\int \frac{\partial \boldsymbol{f}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{f} \; d\boldsymbol{v} = \int \boldsymbol{\Omega}(\boldsymbol{f}) + \boldsymbol{F} \cdot \nabla_{\boldsymbol{v}} \boldsymbol{f} \; d\boldsymbol{v}$$

$$\int \Omega \, d\boldsymbol{v} = 0$$

$$\rho = \int f d\boldsymbol{v}$$

$$\rho \boldsymbol{u} = \int \boldsymbol{v} f d\boldsymbol{v}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

Kinetic formulation

- **Relation between Boltzmann equation and N-S equation**

$$\int \frac{\partial \boldsymbol{f}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{f} \;\; \boldsymbol{v} d\boldsymbol{v} \;=\; \int \boldsymbol{\Omega}(\boldsymbol{f}) + \boldsymbol{F} \cdot \nabla_{\boldsymbol{v}} \boldsymbol{f} \;\; \boldsymbol{v} d\boldsymbol{v}$$

$$\int \boldsymbol{v} \Omega \, d\boldsymbol{v} = 0$$

$$\rho \boldsymbol{u} = \int \boldsymbol{v} f \, d\boldsymbol{v}$$

$$\boldsymbol{\Pi} = \rho \boldsymbol{u} \otimes \boldsymbol{u} + p\boldsymbol{I} - \boldsymbol{\sigma} = \int \boldsymbol{v}\boldsymbol{v}^T f \, d\boldsymbol{v}$$

$$\downarrow$$

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) = -\nabla p + \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{F}$$

Kinetic formulation

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- **Solve Boltzmann transport equation**

$f(\boldsymbol{x}, \boldsymbol{v}, t)$

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f = -\frac{1}{\tau}(\boldsymbol{f} - \bar{\boldsymbol{f}}) + \boldsymbol{F} \cdot \nabla_{\boldsymbol{v}} f$$

Hard to solve

$$\rho = \int f d\boldsymbol{v}$$

$$\rho \boldsymbol{u} = \int \boldsymbol{v} f d\boldsymbol{v}$$

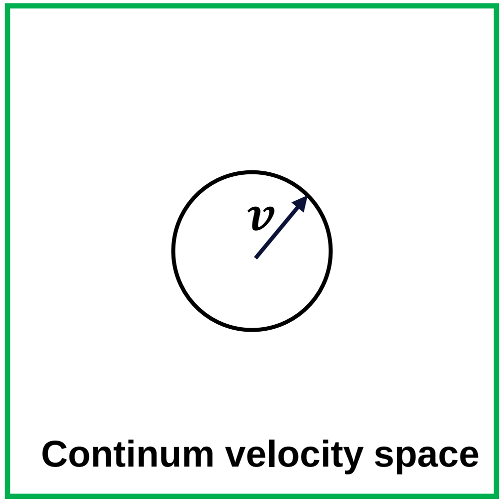
$$p = \frac{1}{3} \int \|\boldsymbol{v} - \boldsymbol{u}\|_2^2 f d\boldsymbol{v}$$

Discretization

- **How calculate the moment: the integrals ?**

$$\rho = \int f d\boldsymbol{v} \qquad \qquad \rho\boldsymbol{u} = \int \boldsymbol{v} f d\boldsymbol{v}$$

- **Gaussian quadrature:** approximate integrals by sums

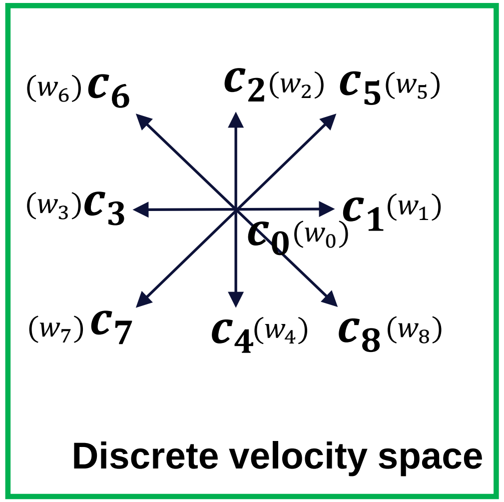


f

$\int d\boldsymbol{v}$

Gaussian quadrature

$$\int w(\boldsymbol{v})H(\boldsymbol{v})d\boldsymbol{v} \approx \sum_{i=0}^{n} w_i H(v_i)$$



$f_i(\boldsymbol{x}, t)$

$\sum_{i=0}^{n-1}$

Discretization

- **Hermite series expansions**

$$f(\boldsymbol{v}, \boldsymbol{x}, t) = \omega(\boldsymbol{v}) \sum_{n=0}^{\infty} \frac{1}{n!} \mathbf{a}^{(n)}(\boldsymbol{x}, t) : \mathbf{H}^{(n)}(\boldsymbol{v}) \ .$$

$$\mathbf{a}^{(n)}(\boldsymbol{x}, t) = \int \frac{f(\boldsymbol{v}, \boldsymbol{x}, t)}{\omega(\boldsymbol{v})} \mathbf{H}^{(n)}(\boldsymbol{v}) \mathrm{d}\boldsymbol{v}$$

$$\mathbf{a}^{(0)} = \rho \ , \quad \mathbf{a}^{(1)} = \rho \boldsymbol{u} \ , \quad \text{and} \quad \mathbf{a}^{(2)} = \Pi - \rho \boldsymbol{I} \ .$$

Discretization

- **Equilibrium distribution**

$$\bar{f}(\rho, \boldsymbol{u}) = \frac{\rho}{(2\pi)^{D/2}} \exp\left(-\frac{\|\boldsymbol{v} - \boldsymbol{u}\|_2^2}{2}\right)$$

$$\Downarrow$$

$$\bar{f}_i(\rho, \boldsymbol{u}) \approx w_i \rho \left(1 + \frac{\boldsymbol{c}_i \cdot \boldsymbol{u}}{c_s^2} + \frac{(\boldsymbol{c}_i \cdot \boldsymbol{u})^2}{2c_s^4} - \frac{\boldsymbol{u} \cdot \boldsymbol{u}}{2c_s^2}\right)$$

Discretization

- **Discretization in mesoscopic velocity space**

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f = -\frac{1}{\tau}(\boldsymbol{f} - \bar{\boldsymbol{f}}) + \boldsymbol{F} \cdot \nabla_{\boldsymbol{v}} f$$

$$\Downarrow$$

$$\frac{\partial f_i}{\partial t} + \boldsymbol{v_i} \cdot \nabla f_i = -\frac{1}{\tau}(f_i - \bar{f_i}) + \boldsymbol{F} \cdot \nabla_{\boldsymbol{v}} f_i$$

Discretization

- **Second-order accurate and explicit numerical scheme**

$$\frac{\partial f_i}{\partial t} + \boldsymbol{v_i} \cdot \nabla f_i = -\frac{1}{\tau}(f_i - \bar{f}_i) + \boldsymbol{F} \cdot \nabla_{\boldsymbol{v}} f_i \qquad \Delta \boldsymbol{x} = \boldsymbol{c_i} \Delta \boldsymbol{t}$$

Trapezoidal rule
Crank–Nicolson method
→ **implicit scheme**

Change of variables → **explicit scheme**

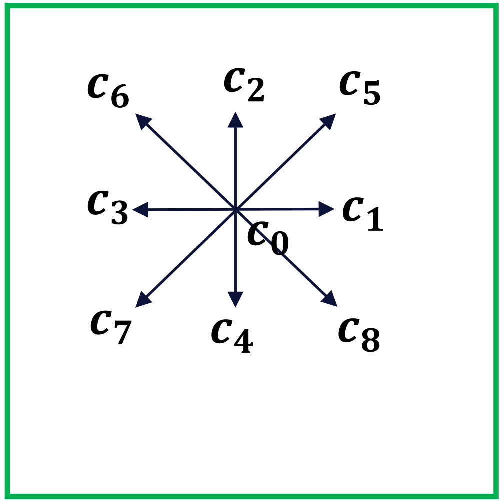
$$f_i(\boldsymbol{x} + \boldsymbol{c_i}, t + 1) - f_i(\boldsymbol{x}, t) = -\frac{1}{\tau}(f_i(\boldsymbol{x}, t) - \bar{f}_i(\boldsymbol{x}, t)) + (1 - \frac{1}{2\tau})F_i$$

$$\Delta \boldsymbol{x} = \Delta \boldsymbol{t} = \boldsymbol{1}$$

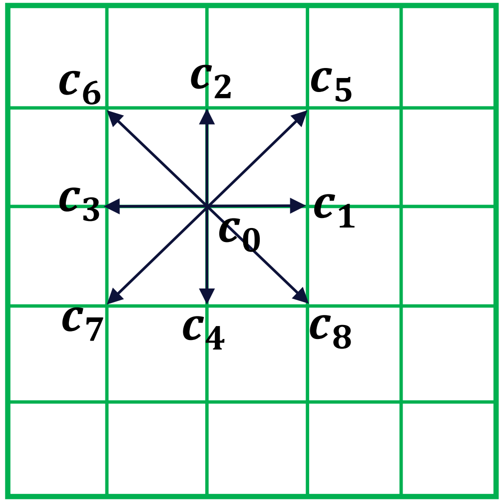
Discretization

- **Second-order accurate and explicit numerical scheme**

$$\frac{\partial f_i}{\partial t} + \boldsymbol{v_i} \cdot \nabla f_i = \Omega(f_i) \longrightarrow f_i(\boldsymbol{x} + \boldsymbol{c_i}, t + 1) - f_i(\boldsymbol{x}, t) = -\frac{1}{\tau}(f_i(\boldsymbol{x}, t) - \overline{f_i}(\boldsymbol{x}, t)) + (1 - \frac{1}{2\tau})F_i$$



Discrete velocity space

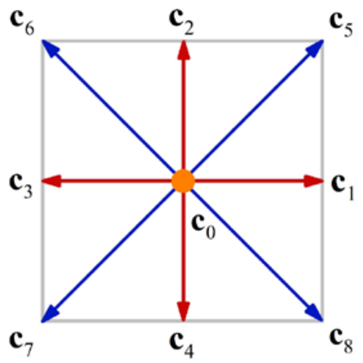


Discrete velocity and position space

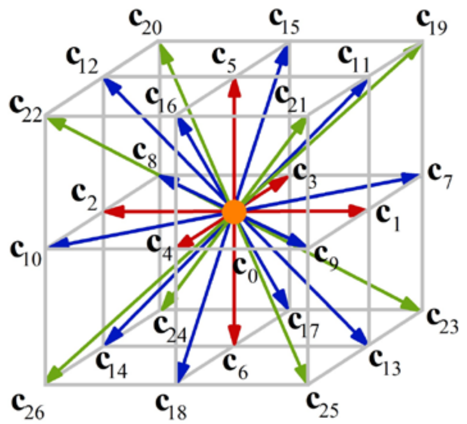
Lattice Boltzmann method

Lattice Boltzmann method (LBM)

- Discretization and time update of distributions:



$$f_i(\boldsymbol{x}+\boldsymbol{c_i}, t+1) - f_i(\boldsymbol{x}, t) = -\frac{1}{\tau}(f_i(\boldsymbol{x}, t) - \overline{f}_i(\boldsymbol{x}, t)) + (1 - \frac{1}{2\tau})F_i$$



Lattice Boltzmann method

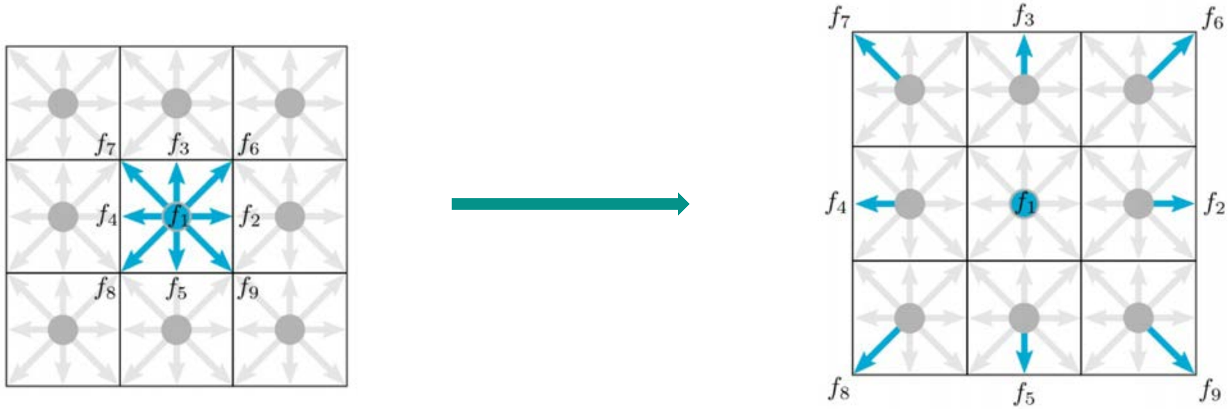
Lattice Boltzmann method (LBM)

- Discretization and time update of distributions:

$$f_i(\boldsymbol{x} + \boldsymbol{c_i}, t + 1) - f_i(\boldsymbol{x}, t) = \Omega_i + F_i$$

- Streaming step:

$$f_i(\boldsymbol{x} + \boldsymbol{c_i}, t + 1) = f_i(\boldsymbol{x}, t)$$



t $\quad$ $t + 1$

Lattice Boltzmann method

Lattice Boltzmann method (LBM)

- Discretization and time update of distributions:

$$f_i(\boldsymbol{x} + \boldsymbol{c_i}, t + 1) - f_i(\boldsymbol{x}, t) = \Omega_i + F_i$$

- Streaming step:

$$f_i(\boldsymbol{x} + \boldsymbol{c_i}, t + 1) = f_i(\boldsymbol{x}, t)$$

- Collision step (e.g., lattice BGK model):

$$\Omega_i = -\frac{1}{\tau}(f_i(\boldsymbol{x}, t) - \overline{f}_i(\boldsymbol{x}, t))$$

Conservative

Parallelizable

Lattice Boltzmann method

Lattice Boltzmann method (LBM)

- Discretization and time update of distributions:

$$f_i(\boldsymbol{x} + \boldsymbol{c_i}, t + 1) - f_i(\boldsymbol{x}, t) = \Omega_i + F_i$$

- Streaming step:

$$f_i(\boldsymbol{x} + \boldsymbol{c_i}, t + 1) = f_i(\boldsymbol{x}, t)$$

- Collision step (e.g., lattice BGK model):

$$\Omega_i = -\frac{1}{\tau}\left(f_i(\boldsymbol{x}, t) - \overline{f_i}(\boldsymbol{x}, t)\right)$$

- Macroscopic quantities still from moments!

$$\rho(\boldsymbol{x}, t) \equiv \sum_{i=0}^{q-1} f_i(\boldsymbol{x}, t) \qquad \boldsymbol{u}(\boldsymbol{x}, t) \equiv \frac{1}{\rho(\boldsymbol{x}, t)}\left(\sum_{i=0}^{q-1} \boldsymbol{c}_i f_i(\boldsymbol{x}, t) + \frac{1}{2}\boldsymbol{F}\right)$$

Lattice Boltzmann method

Lattice Boltzmann method (LBM)

- Discretization and time update of distributions:

$$f_i(\boldsymbol{x} + \boldsymbol{c}_i, t + 1) - f_i(\boldsymbol{x}, t) = \Omega_i + F_i$$

- Streaming step:

Support small Δt naturally!

$$f_i(\boldsymbol{x} + \boldsymbol{c}_i, t + 1) = f_i(\boldsymbol{x}, t)$$

- Collision step (e.g., lattice BGK model):

$$\Omega_i = -\frac{1}{\tau}\left(f_i(\boldsymbol{x}, t) - \overline{f}_i(\boldsymbol{x}, t)\right)$$

- Macroscopic quantities still from moments!

$$\rho(\boldsymbol{x}, t) \equiv \sum_{i=0}^{q-1} f_i(\boldsymbol{x}, t) \qquad \boldsymbol{u}(\boldsymbol{x}, t) \equiv \frac{1}{\rho(\boldsymbol{x}, t)}\left(\sum_{i=0}^{q-1} \boldsymbol{c}_i f_i(\boldsymbol{x}, t) + \frac{1}{2}\boldsymbol{F}\right)$$

Collision modeling

- **Distribution space single-relaxation time model**
  - Lattice BGK model

$$\boldsymbol{\Omega} = -\frac{1}{\tau}(\boldsymbol{f} - \bar{\boldsymbol{f}})$$

$$\boldsymbol{\Omega} = \{\Omega_1, \dots, \Omega_q\}$$

$$\boldsymbol{f} = \{f_1, \dots, f_q\}$$

$$\bar{\boldsymbol{f}} = \{\bar{f}_1, \dots, \bar{f}_q\}$$

$$\frac{1}{\tau} = 3\nu + \frac{1}{2}$$

Collision modeling

- **Moment space multi-relaxation time model**
  - Moment: measure a distribution in statistics

$$k_0 = \sum_{i=0}^{q-1} f_i$$

$$k_1 = \sum_{i=0}^{q-1} \boldsymbol{c}_i f_i$$

$$k_2 = \sum_{i=0}^{q-1} \boldsymbol{c}_i \boldsymbol{c}_i f_i$$

$$\mathbf{K} = \{k_0, \dots, k_q\} = \mathbf{M}\boldsymbol{f}$$

Raw-moment MRT

$$\boldsymbol{\Omega} = -\mathbf{M}^{-1}\mathbf{R}\mathbf{M}(\boldsymbol{f} - \bar{\boldsymbol{f}})$$

Violates Galilean invariance

linear combination of $\boldsymbol{c}_{i,a}{}^{\alpha} \boldsymbol{c}_{i,b}{}^{\beta} \boldsymbol{c}_{i,c}{}^{\gamma}$

$a, b, c \in \{x, y, z\}$ $\alpha, \beta, \gamma \in \{0, 1, 2\}$

Collision modeling

- **Moment space multi-relaxation time model**
  - Central-moment multi-relaxation time model **(CM-MRT)**

$$\mathbf{K} = \{k_0, \dots, k_q\} = \mathbf{M}\boldsymbol{f} \longrightarrow \boldsymbol{\Omega} = -\mathbf{M}^{-1}\mathbf{R}\mathbf{M}(\boldsymbol{f} - \bar{\boldsymbol{f}})$$

linear combination of $(\boldsymbol{c}_{i,a} - \boldsymbol{u})^{\alpha}(\boldsymbol{c}_{i,b} - \boldsymbol{u})^{\beta}(\boldsymbol{c}_{i,c} - \boldsymbol{u})^{\gamma}$ **Ensure Galilean invariance**

$a, b, c \in \{x, y, z\}$

$\alpha, \beta, \gamma \in \{0, 1, 2\}$

Collision modeling

- **Central-moment multi-relaxation time model (CM-MRT)**

$$\boldsymbol{\Omega} = -\mathbf{M}^{-1}\mathbf{R}\mathbf{M}\left(\boldsymbol{f} - \bar{\boldsymbol{f}}\right) = \mathbf{M}^{-1}\mathbf{R}(\mathbf{m} - \bar{\boldsymbol{m}})$$

$$\boldsymbol{diag}(\mathbf{R}) = \{1, 1, 1, 1, \frac{1}{\tau^P}, \frac{1}{\tau^P}, \frac{1}{\tau^P}, \frac{1}{\tau^P}, \frac{1}{\tau^*}, \dots, \frac{1}{\tau^*}\} \longrightarrow \boldsymbol{\Omega}^* = \boldsymbol{\Omega}(\tau^*)$$

Kinematic viscosity ν

$\tau^P = (3\nu + 0.5)$

High order relaxation times to be determined

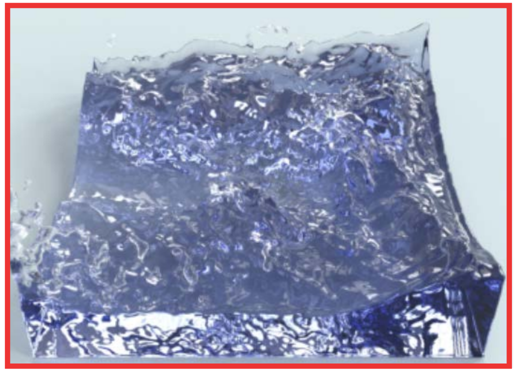
Related work

Kinetic methods

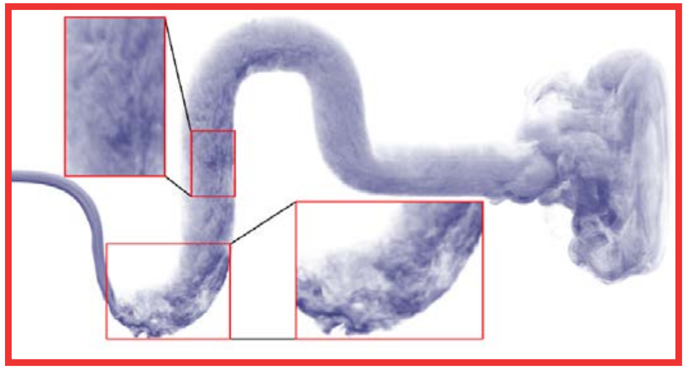
- Lattice BGK model
- Raw-moment MRT
- Central-moment MRT

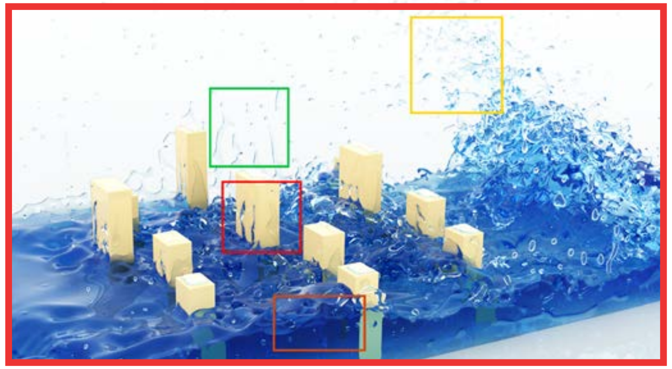

Thuerey et al. VMV’04


Thuerey et al. VMV’06


Guo et al. TVCG ’17


Liu et al. TVCG ’14


Li et al. TVCG ’19

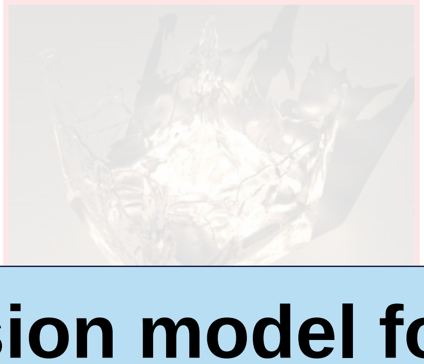

Li et al. TVCG ’20

Related work

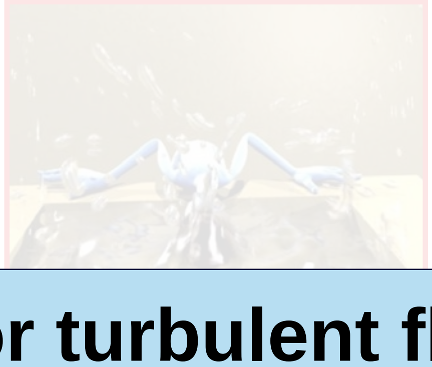
Kinetic methods

- Lattice BGK model
- Raw-moment MRT
- Central-moment MRT

Inaccurate collision model for turbulent flows!



Thuerey et al. VMV'04



Thuerey et al. VMV'06

Guo et al. TVCG '17

Liu et al. TVCG '14

Li et al. TVCG '19

Li et al. TVCG '20

Contributions

- **Low-dissipation and low-dispersion fluid solver**
  - High-order model
  - Evaluate high-order relaxation times
  - A linear regression to estimate relaxation times

- **Turbulent fluid with two-way coupling**
  - Immersed boundary method
  - Calibration between physical and LBM units

Low-dissipation & low-dispersion fluid solver

- **High-order hermit expansion**

$$\bar{f}_i(\rho, \boldsymbol{u}) \approx w_i \rho \left(1 + \frac{\boldsymbol{c}_i \cdot \boldsymbol{u}}{c_s^2} + \frac{(\boldsymbol{c}_i \cdot \boldsymbol{u})^2}{2c_s^4} - \frac{\boldsymbol{u} \cdot \boldsymbol{u}}{2c_s^2}\right) \longrightarrow$$

$$\begin{aligned}
\bar{f}_i \approx w_i \rho \Bigg[& 1 + \frac{\boldsymbol{c}_i \cdot \boldsymbol{u}}{c_s^2} + \frac{1}{2c_s^4} \mathbf{H}^{(2)}(\boldsymbol{c}_i) : \boldsymbol{u} \otimes \boldsymbol{u} \\
& + \frac{1}{2c_s^6} \Big(\mathbf{H}^{(3)}_{ixxy} u_x^2 u_y + \mathbf{H}^{(3)}_{ixxz} u_x^2 u_z + \mathbf{H}^{(3)}_{ixyy} u_x u_y^2 \\
& + \mathbf{H}^{(3)}_{ixzz} u_x u_z^2 + \mathbf{H}^{(3)}_{iyzz} u_y u_z^2 + \mathbf{H}^{(3)}_{iyzz} u_y^2 u_z + \mathbf{H}^{(3)}_{ixyz} u_x u_y u_z \Big) \\
& + \frac{1}{4c_s^8} \Big[\mathbf{H}^{(4)}_{ixxyy} u_x^2 u_y^2 + \mathbf{H}^{(4)}_{ixxzz} u_x^2 u_z^2 + \mathbf{H}^{(4)}_{iyyzz} u_y^2 u_z^2 \\
& + 2 \Big(\mathbf{H}^{(4)}_{ixyzz} u_x u_y u_z^2 + \mathbf{H}^{(4)}_{ixyyz} u_x u_y^2 u_z + \mathbf{H}^{(4)}_{ixxyz} u_x^2 u_y u_z \Big) \Big] \\
& + \frac{1}{4c_s^{10}} \Big(\mathbf{H}^{(5)}_{ixxyzz} u_x^2 u_y u_z^2 + \mathbf{H}^{(5)}_{ixxyyz} u_x^2 u_y^2 u_z \\
& + \mathbf{H}^{(5)}_{ixyyzz} u_x u_y^2 u_z^2 \Big) + \frac{1}{8c_s^{12}} \mathbf{H}^{(6)}_{ixxyyzz} u_x^2 u_y^2 u_z^2 \Bigg].
\end{aligned}$$

Low-dissipation & low-dispersion fluid solver

- **High-order hermit expansion**

M

$$\bar{f}_i \approx w_i \rho \Bigg[1 + \frac{\boldsymbol{c}_i \cdot \boldsymbol{u}}{c_s^2} + \frac{1}{2c_s^4} \mathrm{H}^{(2)}(\boldsymbol{c}_i) : \boldsymbol{u} \otimes \boldsymbol{u}$$
$$+ \frac{1}{2c_s^6} \Big(\mathrm{H}^{(3)}_{ixxy} u_x^2 u_y + \mathrm{H}^{(3)}_{ixxz} u_x^2 u_z + \mathrm{H}^{(3)}_{ixyy} u_x u_y^2$$
$$+ \mathrm{H}^{(3)}_{ixzz} u_x u_z^2 + \mathrm{H}^{(3)}_{iyzz} u_y u_z^2 + \mathrm{H}^{(3)}_{iyzz} u_y^2 u_z + \mathrm{H}^{(3)}_{ixyz} u_x u_y u_z \Big)$$
$$+ \frac{1}{4c_s^8} \Big[\mathrm{H}^{(4)}_{ixxyy} u_x^2 u_y^2 + \mathrm{H}^{(4)}_{ixxzz} u_x^2 u_z^2 + \mathrm{H}^{(4)}_{iyyzz} u_y^2 u_z^2$$
$$+ 2 \Big(\mathrm{H}^{(4)}_{ixyzz} u_x u_y u_z^2 + \mathrm{H}^{(4)}_{ixyyz} u_x u_y^2 u_z + \mathrm{H}^{(4)}_{ixxyz} u_x^2 u_y u_z \Big) \Big]$$
$$+ \frac{1}{4c_s^{10}} \Big(\mathrm{H}^{(5)}_{ixxyzz} u_x^2 u_y u_z^2 + \mathrm{H}^{(5)}_{ixxyyz} u_x^2 u_y^2 u_z$$
$$+ \mathrm{H}^{(5)}_{ixyyzz} u_x u_y^2 u_z^2 \Big) + \frac{1}{8c_s^{12}} \mathrm{H}^{(6)}_{ixxyyzz} u_x^2 u_y^2 u_z^2 \Bigg].$$

→

$$\boldsymbol{q}_0 = \boldsymbol{q}_9 = \rho, \quad \boldsymbol{q}_{17} = \rho c_s^2,$$
$$\boldsymbol{q}_{18} = \rho c_s^4, \quad \text{and } \boldsymbol{q}_{26} = \rho c_s^6$$

Low-dissipation & low-dispersion fluid solver

- **High-order hermit expansion**

$$F_i = \boldsymbol{F}\cdot\nabla_{\boldsymbol{c}_i} f_i \approx \boldsymbol{F}\cdot\nabla_{\boldsymbol{c}_i}\bar{f}_i = (1-\frac{1}{2\tau})w_i\left(\frac{\boldsymbol{c}_i-\boldsymbol{u}}{c_s^2}+\frac{\boldsymbol{c}_i\cdot\boldsymbol{u}}{c_s^4}\boldsymbol{c}_i\right)\cdot\boldsymbol{F}$$

→

$$\begin{aligned}
F_i = w_i\rho(&\frac{\boldsymbol{g}\cdot\boldsymbol{c}_i}{c_s^2} + \frac{1}{2c_s^4}[\mathrm{H}^{(2)}_{ixx}(2u_x g_x) + \mathrm{H}^{(2)}_{iyy}(2u_y g_y) + \mathrm{H}^{(2)}_{izz}(2u_z g_z)\\
&+ 2\mathrm{H}^{(2)}_{ixy}(u_x g_y + u_y g_x) + 2\mathrm{H}^{(2)}_{ixz}(u_x g_z + u_z g_x)\\
&+ 2\mathrm{H}^{(2)}_{iyz}(u_y g_z + u_z g_y)] + \frac{1}{2c_s^6}[\mathrm{H}^{(3)}_{ixxy}(2u_x u_y g_x + u_x^2 g_y)\\
&+ \mathrm{H}^{(3)}_{ixxz}(2u_x u_z g_x + u_x^2 g_z) + \mathrm{H}^{(3)}_{ixyy}(2u_x u_y g_y + u_y^2 g_x)\\
&+ \mathrm{H}^{(3)}_{ixzz}(2u_x u_z g_z + u_z^2 g_x) + \mathrm{H}^{(3)}_{iyzz}(2u_y u_z g_z + u_z^2 g_y)\\
&+ \mathrm{H}^{(3)}_{iyyz}(2u_y u_z g_y + u_y^2 g_z) + 2\mathrm{H}^{(3)}_{ixyz}(u_x u_y g_z + u_x g_y u_z + g_x u_y u_z)]\\
&+ \frac{1}{4c_s^8}[\mathrm{H}^{(4)}_{ixxyy}(2u_x^2 u_y g_y + 2u_x u_y^2 g_x) + \mathrm{H}^{(4)}_{ixxzz}(2u_x^2 u_z g_z +\\
&2u_x u_z^2 g_x) + \mathrm{H}^{(4)}_{iyyzz}(2u_y^2 u_z g_z + 2u_y u_z^2 g_y) + 2\mathrm{H}^{(4)}_{ixyzz}(2u_x u_y u_z g_z\\
&+ u_x g_y u_z^2 + g_x u_y u_z^2) + 2\mathrm{H}^{(4)}_{ixyyz}(2u_x u_y u_z g_y + u_x u_y^2 g_z + g_x u_y^2 u_z)\\
&+ 2\mathrm{H}^{(4)}_{ixxyz}(2u_x u_y u_z g_x + u_y u_x^2 g_z + u_x^2 u_z g_y)]\\
&+ \frac{1}{4c_s^{10}}[\mathrm{H}^{(5)}_{ixxyyz}(2u_x u_y^2 u_z g_x + 2u_x^2 u_y u_z g_y + u_x^2 u_y^2 g_z)\\
&+ \mathrm{H}^{(5)}_{ixxyzz}(2u_x u_y u_z^2 g_x + u_x^2 u_z^2 g_y + 2u_x^2 u_y u_z g_z)\\
&+ \mathrm{H}^{(5)}_{ixyyzz}(u_y^2 u_z^2 g_x + 2u_x u_y u_z^2 g_y + 2u_x u_y^2 u_z g_z)]\\
&+ \frac{1}{8c_s^{12}}\mathrm{H}^{(6)}_{ixxyyzz}(2u_x u_y^2 u_z^2 g_x + 2u_x^2 u_y u_z^2 g_y + 2u_x^2 u_y^2 u_z F_z)).
\end{aligned}$$

Low-dissipation & low-dispersion fluid solver

- **High-order hermit expansion**

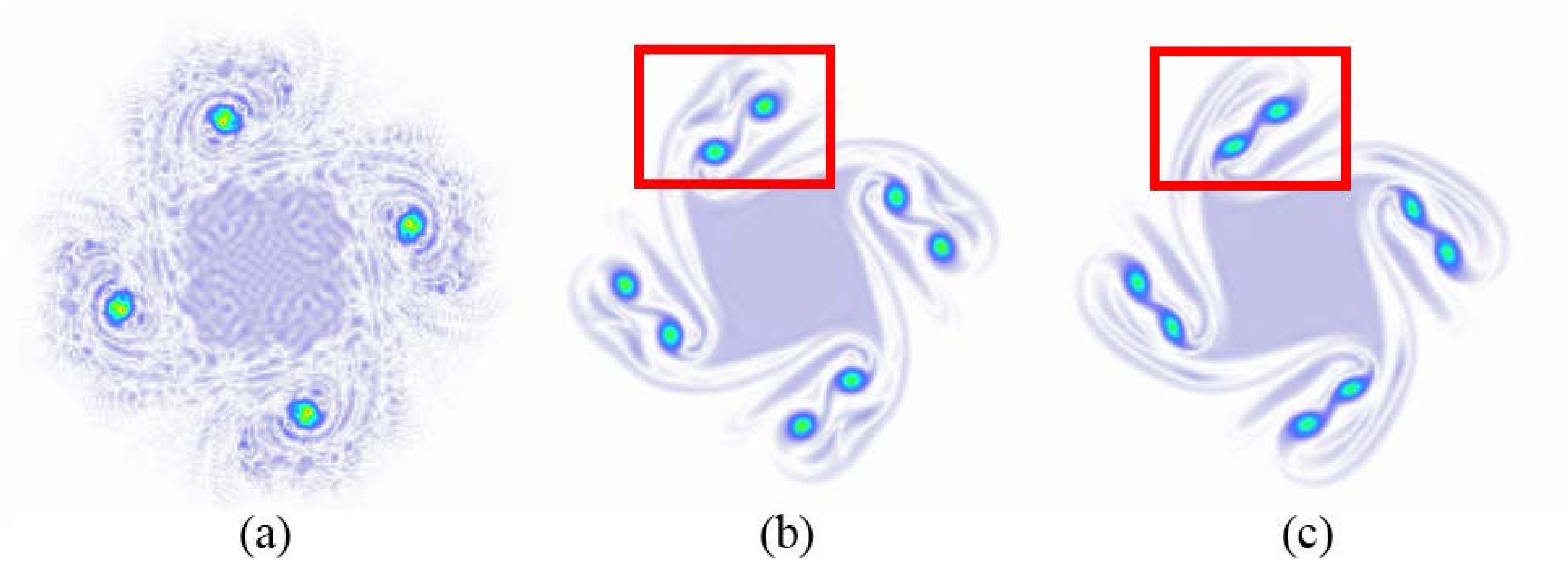
$$
\begin{aligned}
F_i = w_i\rho\Big(&\frac{\boldsymbol{g}\cdot\boldsymbol{c}_i}{c_s^2} + \frac{1}{2c_s^4}[\mathrm{H}^{(2)}_{ixx}(2u_x g_x) + \mathrm{H}^{(2)}_{iyy}(2u_y g_y) + \mathrm{H}^{(2)}_{izz}(2u_z g_z) \\
&+ 2\mathrm{H}^{(2)}_{ixy}(u_x g_y + u_y g_x) + 2\mathrm{H}^{(2)}_{ixz}(u_x g_z + u_z g_x) \\
&+ 2\mathrm{H}^{(2)}_{iyz}(u_y g_z + u_z g_y)] + \frac{1}{2c_s^6}[\mathrm{H}^{(3)}_{ixxy}(2u_x u_y g_x + u_x^2 g_y) \\
&+ \mathrm{H}^{(3)}_{ixxz}(2u_x u_z g_x + u_x^2 g_z) + \mathrm{H}^{(3)}_{ixyy}(2u_x u_y g_y + u_y^2 g_x) \\
&+ \mathrm{H}^{(3)}_{ixzz}(2u_x u_z g_z + u_z^2 g_x) + \mathrm{H}^{(3)}_{iyzz}(2u_y u_z g_z + u_z^2 g_y) \\
&+ \mathrm{H}^{(3)}_{iyyz}(2u_y u_z g_y + u_y^2 g_z) + 2\mathrm{H}^{(3)}_{ixyz}(u_x u_y g_z + u_x g_y u_z + g_x u_y u_z)] \\
&+ \frac{1}{4c_s^8}[\mathrm{H}^{(4)}_{ixxyy}(2u_x^2 u_y g_y + 2u_x u_y^2 g_x) + \mathrm{H}^{(4)}_{ixxzz}(2u_x^2 u_z g_z + \\
&2u_x u_z^2 g_x) + \mathrm{H}^{(4)}_{iyyzz}(2u_y^2 u_z g_z + 2u_y u_z^2 g_y) + 2\mathrm{H}^{(4)}_{ixyzz}(2u_x u_y u_z g_z \\
&+ u_x g_y u_z^2 + g_x u_y u_z^2) + 2\mathrm{H}^{(4)}_{ixyyz}(2u_x u_y u_z g_y + u_x u_y^2 g_z + g_x u_y^2 u_z) \\
&+ 2\mathrm{H}^{(4)}_{ixxyz}(2u_x u_y u_z g_x + u_y u_x^2 g_z + u_x^2 u_z g_y)] \\
&+ \frac{1}{4c_s^{10}}[\mathrm{H}^{(5)}_{ixxyyz}(2u_x u_y^2 u_z g_x + 2u_x^2 u_y u_z g_y + u_x^2 u_y^2 g_z) \\
&+ \mathrm{H}^{(5)}_{ixxyzz}(2u_x u_y u_z^2 g_x + u_x^2 u_z^2 g_y + 2u_x^2 u_y u_z g_z) \\
&+ \mathrm{H}^{(5)}_{ixyyzz}(u_y^2 u_z^2 g_x + 2u_x u_y u_z^2 g_y + 2u_x u_y^2 u_z g_z)] \\
&+ \frac{1}{8c_s^{12}}\mathrm{H}^{(6)}_{ixxyyzz}(2u_x u_y^2 u_z^2 g_x + 2u_x^2 u_y u_z^2 g_y + 2u_x^2 u_y^2 u_z F_z)\Big).
\end{aligned}
$$

M →

$$
\begin{aligned}
\tilde{\mathbb{F}}_1 &= F_x, & \tilde{\mathbb{F}}_2 &= F_y, & \tilde{\mathbb{F}}_3 &= F_z, \\
\tilde{\mathbb{F}}_{10} &= 2c_s^2 F_x, & \tilde{\mathbb{F}}_{11} &= 2c_s^2 F_y, & \tilde{\mathbb{F}}_{12} &= 2c_s^2 F_z, \\
\tilde{\mathbb{F}}_{23} &= c_s^4 F_x, & \tilde{\mathbb{F}}_{24} &= c_s^4 F_y, & \tilde{\mathbb{F}}_{25} &= c_s^4 F_z\ .
\end{aligned}
$$

Low-dissipation & low-dispersion fluid solver

- **How to determine high-order relaxation time τ^* ?** $\Omega^* = \Omega(\tau^*)$
  - Still an unsolved but very important problem



Effects of high-order relaxation rate

Different high-order relaxation times

Small relaxation time

Large relaxation time

“Optimal” relaxation time

Low-dissipation & low-dispersion fluid solver

- **Measurement functional**
  - Dissipation or dispersion error results in larger fluid low-order moment variation
  - Measure fluid variation in **a time-step**

$$\epsilon(\mathbf{x}_k, t) = \frac{||\delta^t(\rho)_k||}{\overline{\rho}} + \frac{||\delta^t(\rho\mathbf{u})_k||}{||\overline{\rho\mathbf{u}}||} + \frac{||\delta^t(\mathbf{\Pi})_k||}{||\overline{\mathbf{\Pi}}||}$$

Low-dissipation & low-dispersion fluid solver

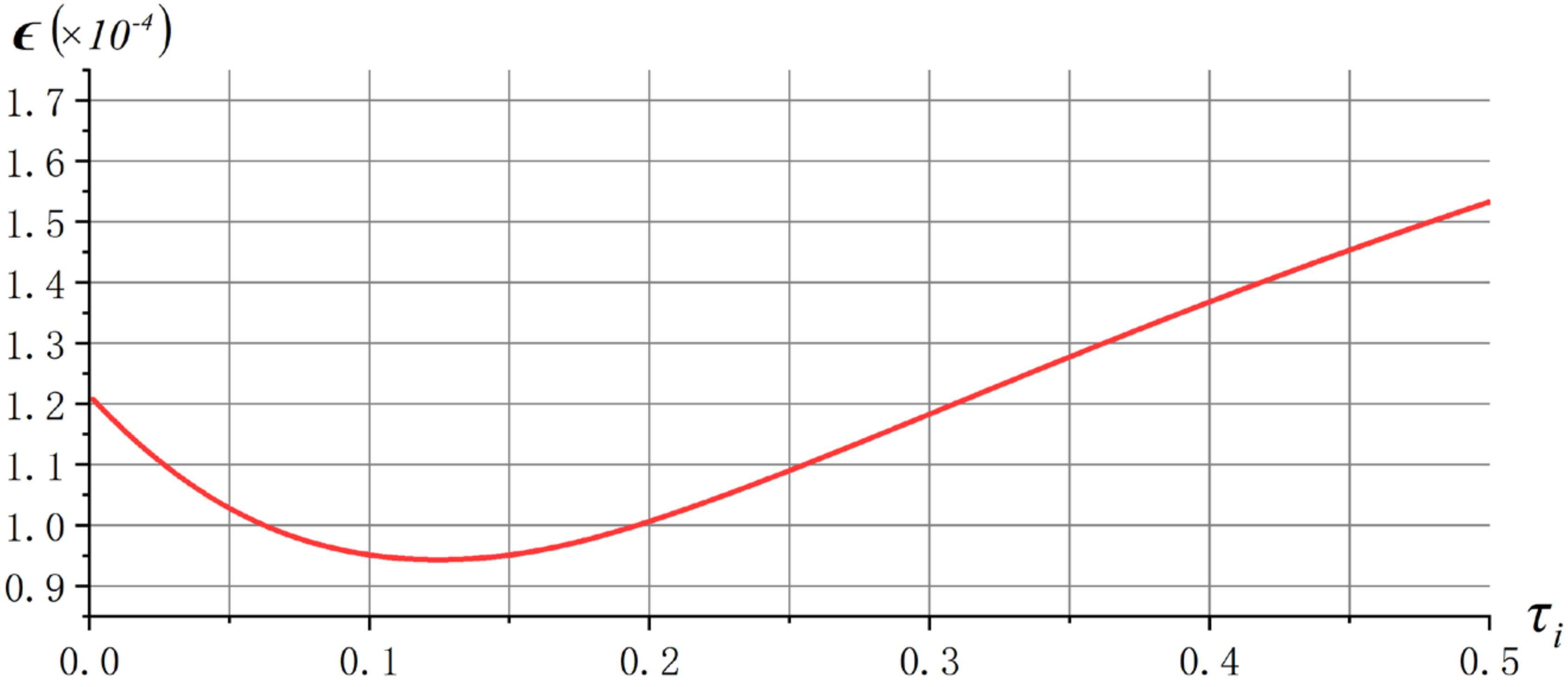
- **Measurement functional**

$$\epsilon(\mathbf{x}_k, t) = \frac{||\delta^t(\rho)_k||}{\overline{\rho}} + \frac{||\delta^t(\rho\mathbf{u})_k||}{||\overline{\rho\mathbf{u}}||} + \frac{||\delta^t(\mathbf{\Pi})_k||}{||\overline{\mathbf{\Pi}}||}$$

Measure the functional with different relaxation time τ^* in one time step

Low-dissipation & low-dispersion fluid solver

- **Measurement functional**



Low-dissipation & low-dispersion fluid solver

- **Numerical optimization of τ^***
  - Brute-force search to find optimal value
  - Gradient decent optimization

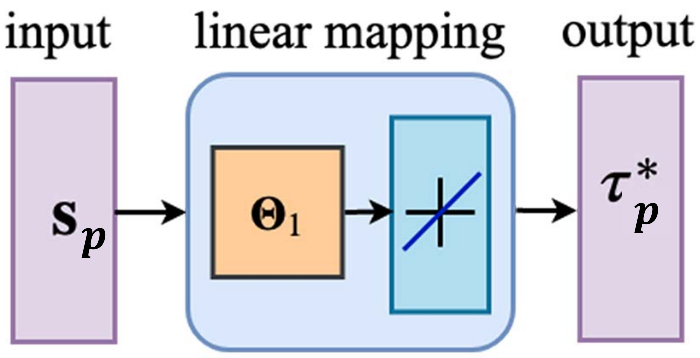
One optimization step ⟷ **One simulation time step**

Many time steps

Very inefficient in practice !!!

Low-dissipation & low-dispersion fluid solver

- **Regression-based evaluation of local $\boldsymbol{\tau}^*$**
  - Linear regression offers a simple and accurate estimate
    - Linear regression
      - Input state: $\mathbf{s}_p = (\rho_p/\overline{\|\rho\|}, \|\rho_p \mathbf{u}_p\|/\overline{\|\rho \boldsymbol{u}\|}, \|\boldsymbol{\Pi}_p\|/\overline{\|\boldsymbol{\Pi}\|}, 1)$
    - Offline pre-computation: collect data $(\boldsymbol{S}_p$ and $\boldsymbol{\tau}^*)$



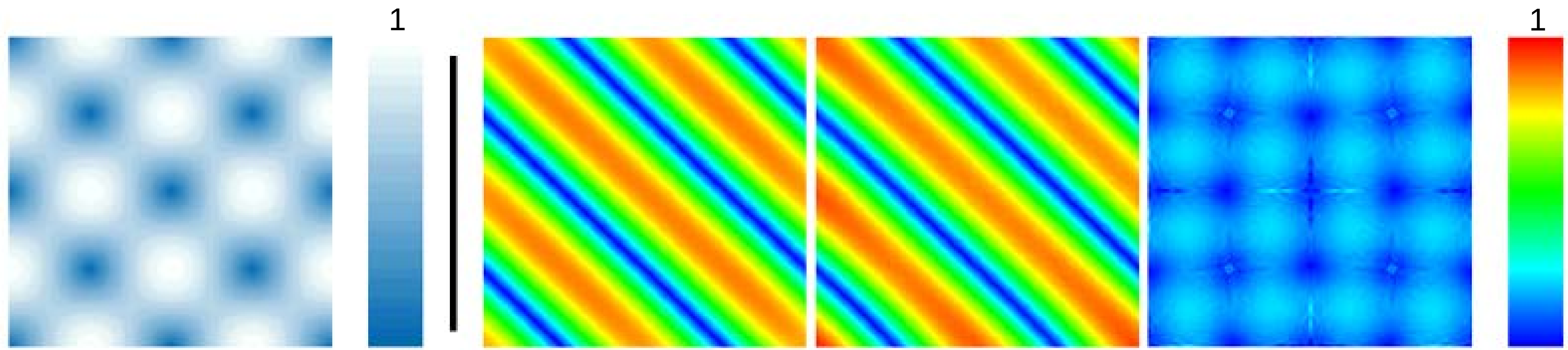
$$\theta^* = \mathrm{argmin}_{\theta} \sum_p \left(\hat{\tau}_p^*(\theta) - \tau_p^*\right)^2$$

$$\hat{\tau}_p^* \approx \boldsymbol{\theta}^T \mathbf{s}_p$$

Evaluating the resulting LBM solver

2D Taylor-Green vortex

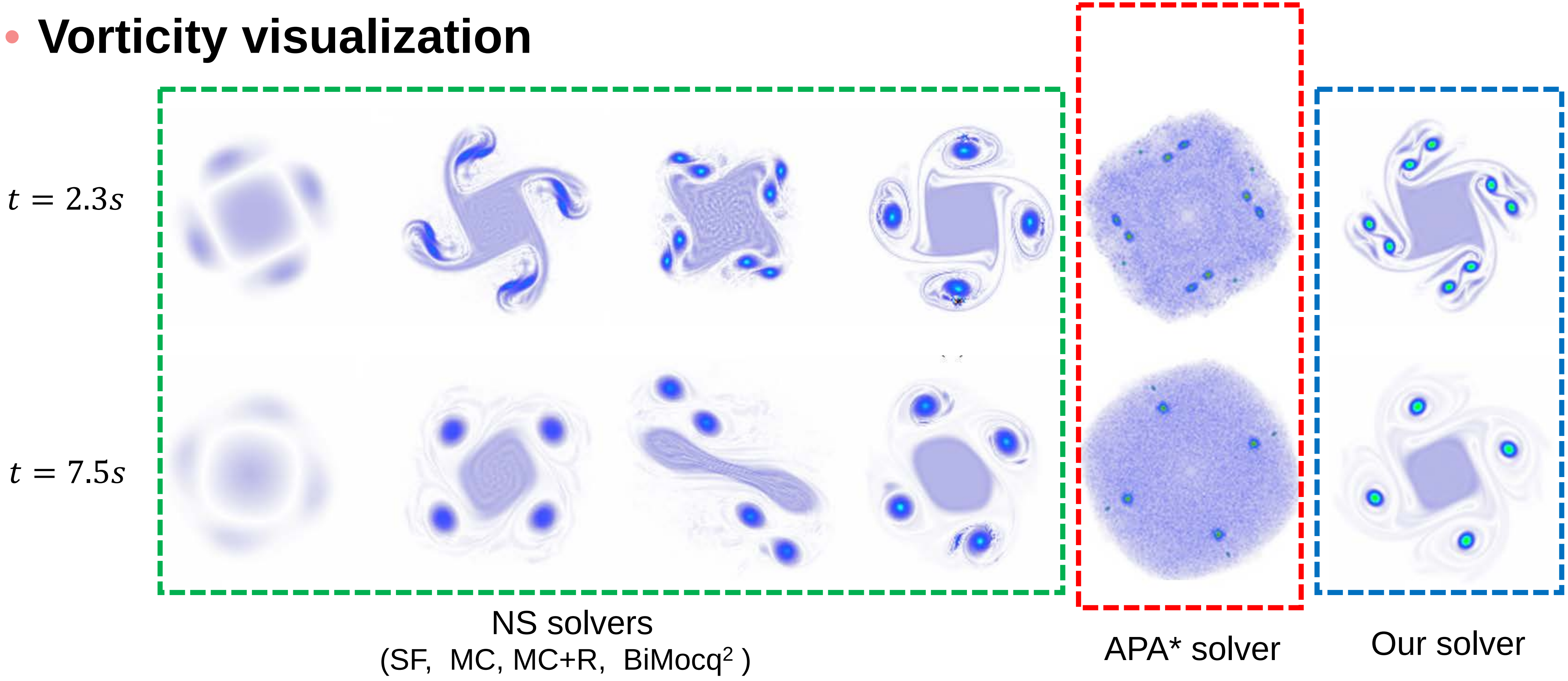
- **Visualization of velocity and error magnitudes**



(a) Velocity (b) MC+R NS solver (c) BiMocq² NS solver (d) Our kinetic solver

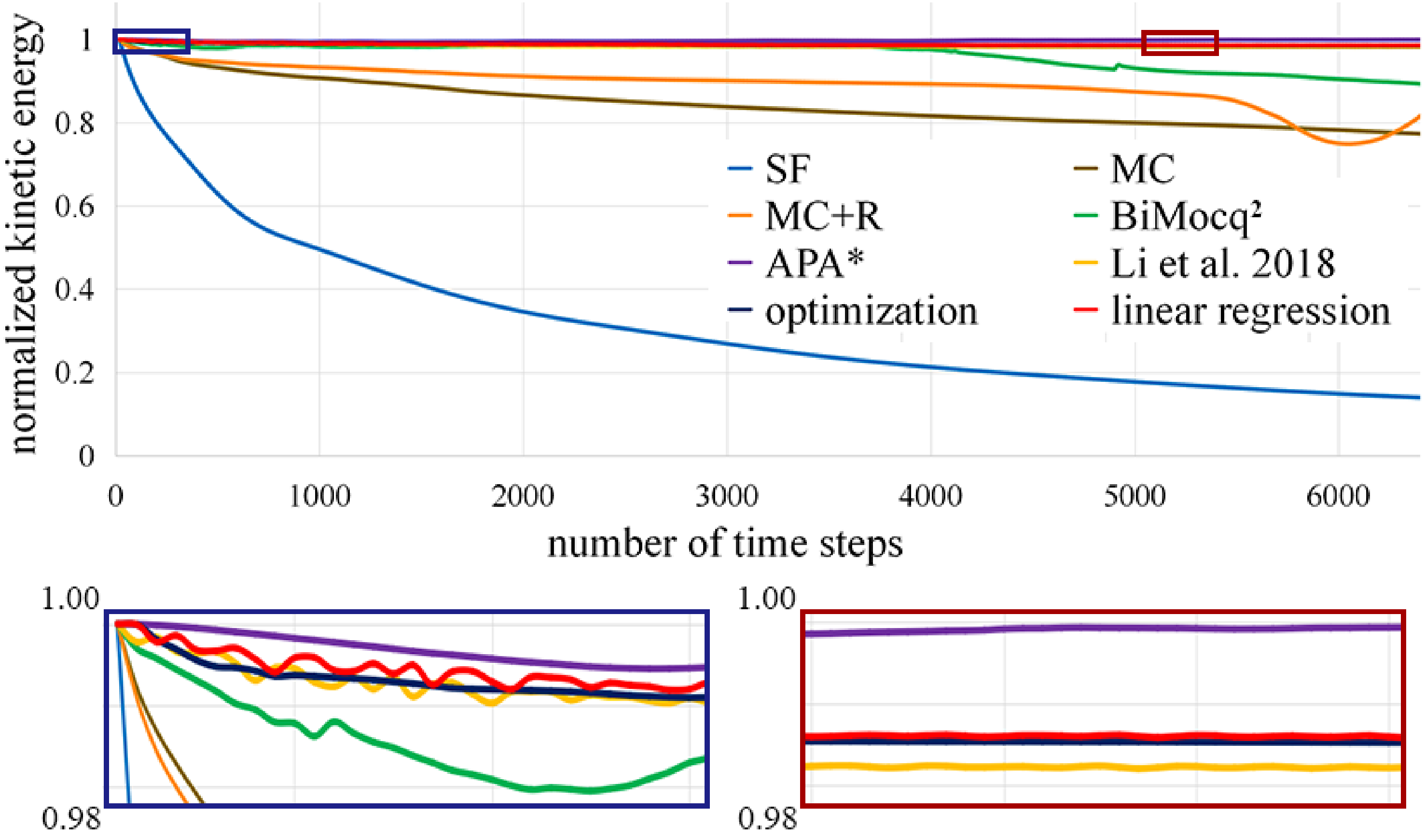
2D vortex sheet simulation

- **Vorticity visualization**



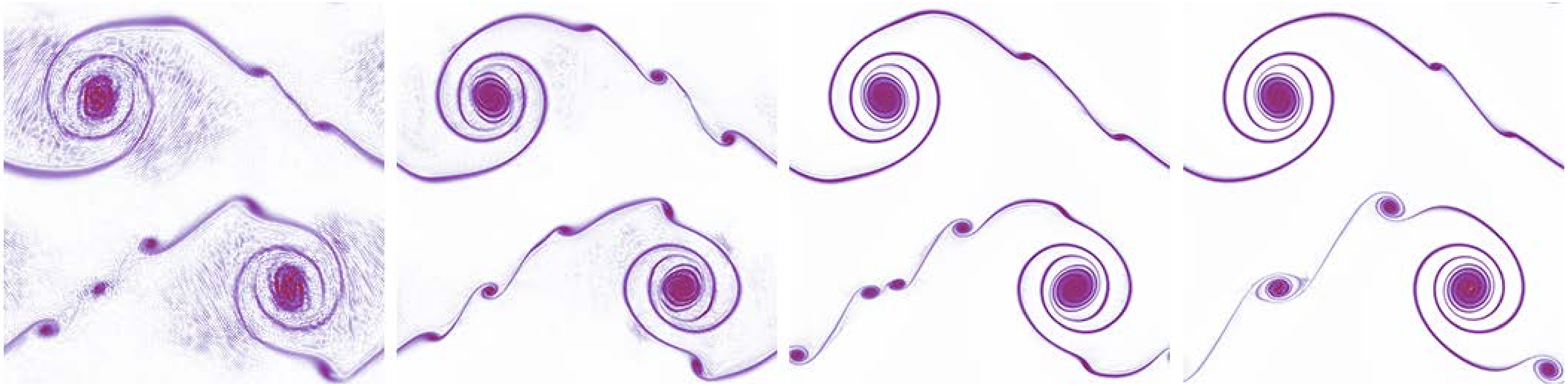
2D vortex sheet simulation

- **Energy preservation**



2D double layer vortex

- **Vorticity visualization**
  - MC+R NS solver with small Δt

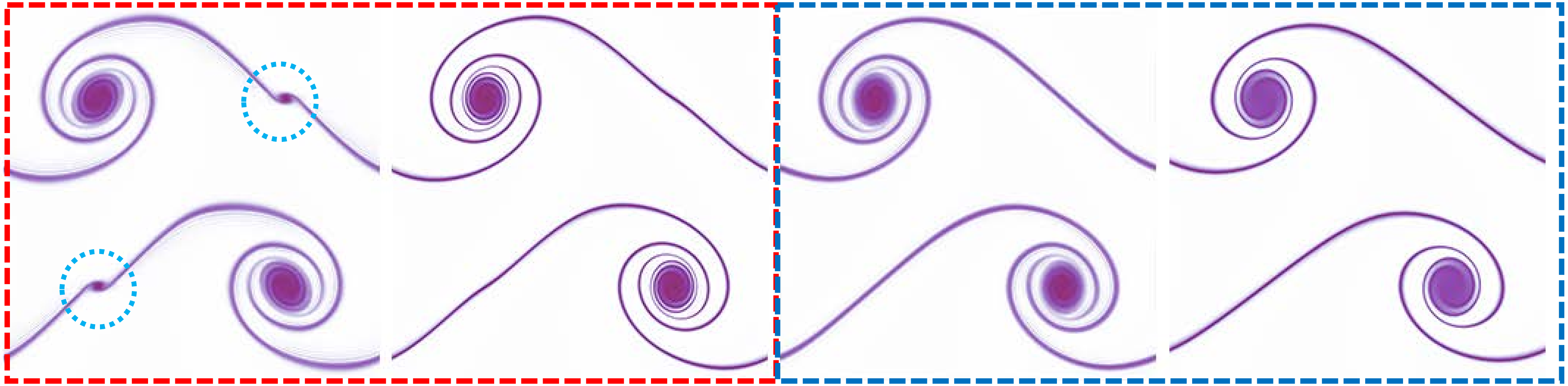


256×256 512×512 1024×1024 2048×2048

2D double layer vortex

- **Vorticity visualization**

MC+R NS solver (large Δt) | Our solver (small Δt)



256×256 | 1024×1024 | 256×256 | 1024×1024

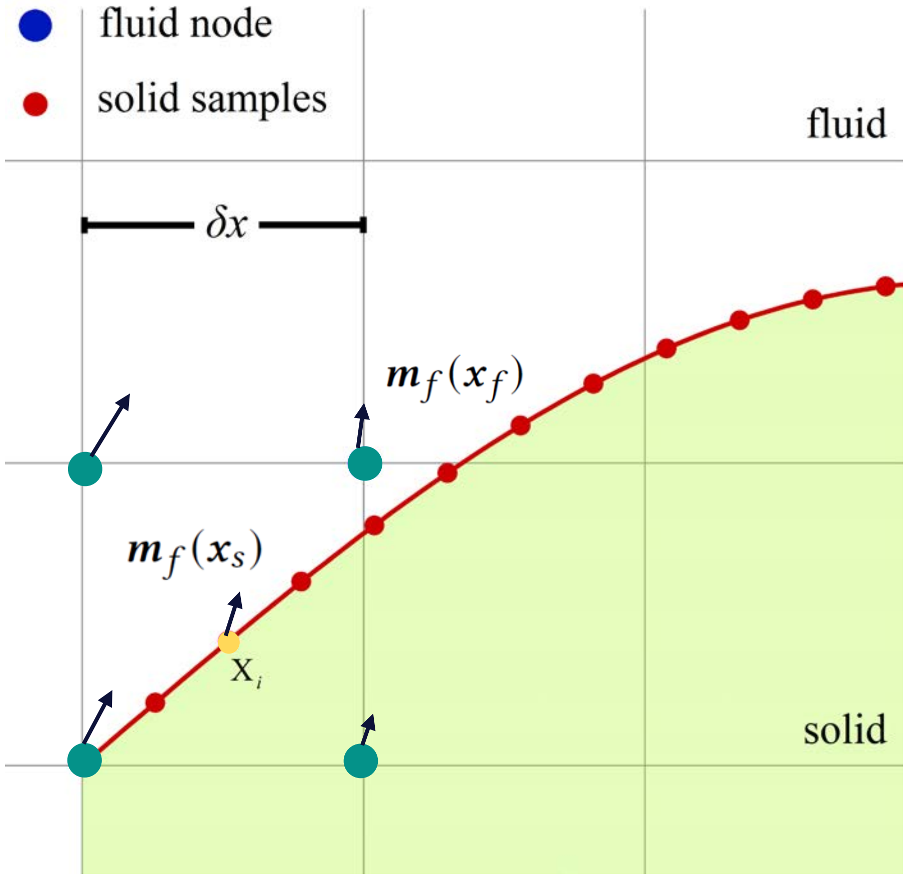
Turbulent flows with two-way coupling

Turbulent fluid with two-way coupling

- **Immersed boundary method**
  - Interpolation process

$$m_f(x_s) = \int m_f(x)\delta(x - x_s)dx$$
$$\approx \sum_{x_f \in \mathcal{D}_s} m_f(x_f)\bar{\delta}(x_f - x_s)\Delta v$$

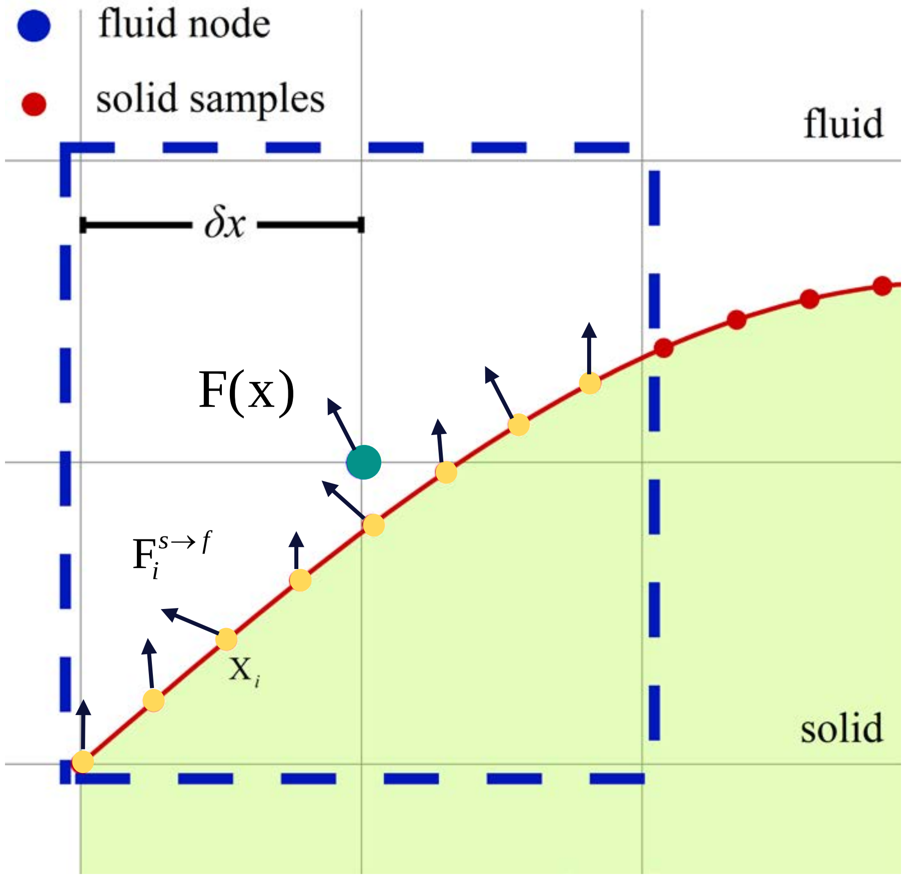
$$F_{f\to s}(x_s) = \left(m_f(x_s) - \rho v_s(x_s)\right) / \Delta t$$



Turbulent fluid with two-way coupling

- **Immersed boundary method**
  - Spreading process

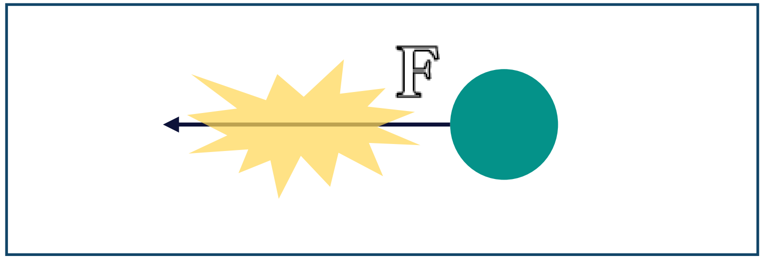
$$F_{s\to f}(x_f) = -\int F_{f\to s}(x)\delta(x - x_f)dS$$
$$= -\sum_{x_s \in D_f} F_{f\to s}(x_s)\bar{\delta}(x_s - x_f)\Delta s$$



Turbulent fluid with two-way coupling

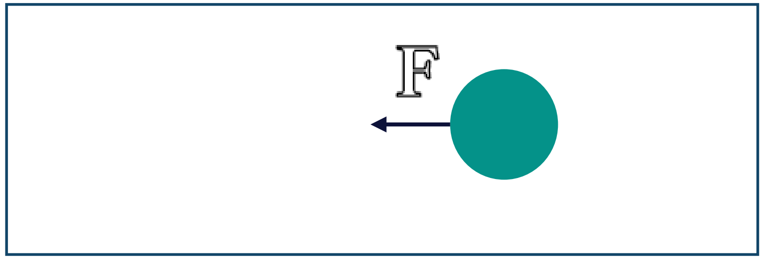
- **Coupling force in immersed boundary method**

Fast moving solid



Not stable

Increase iteration numbers



Smaller time step integration

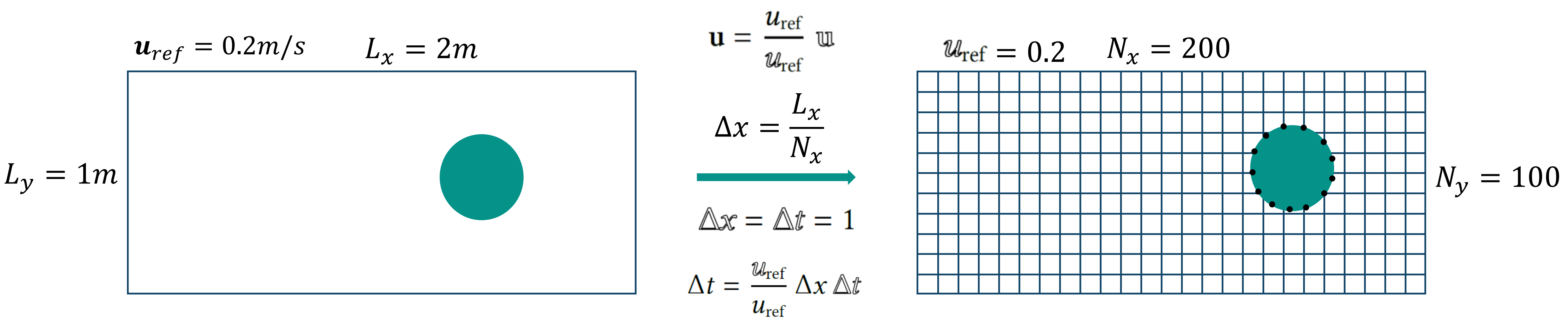
Stable

Turbulent fluid with two-way coupling

- **Dimensionless scaling in LBM space**
  - Stability velocity range: [0, 0.2]
  - Set reference LBM velocity $\mathbb{u}_{ref}$ = 0.2: maximize efficiency
  - Reference Physical velocity $\boldsymbol{u}_{ref}$

Outlined letter in LBM space

Solid letter in physical space

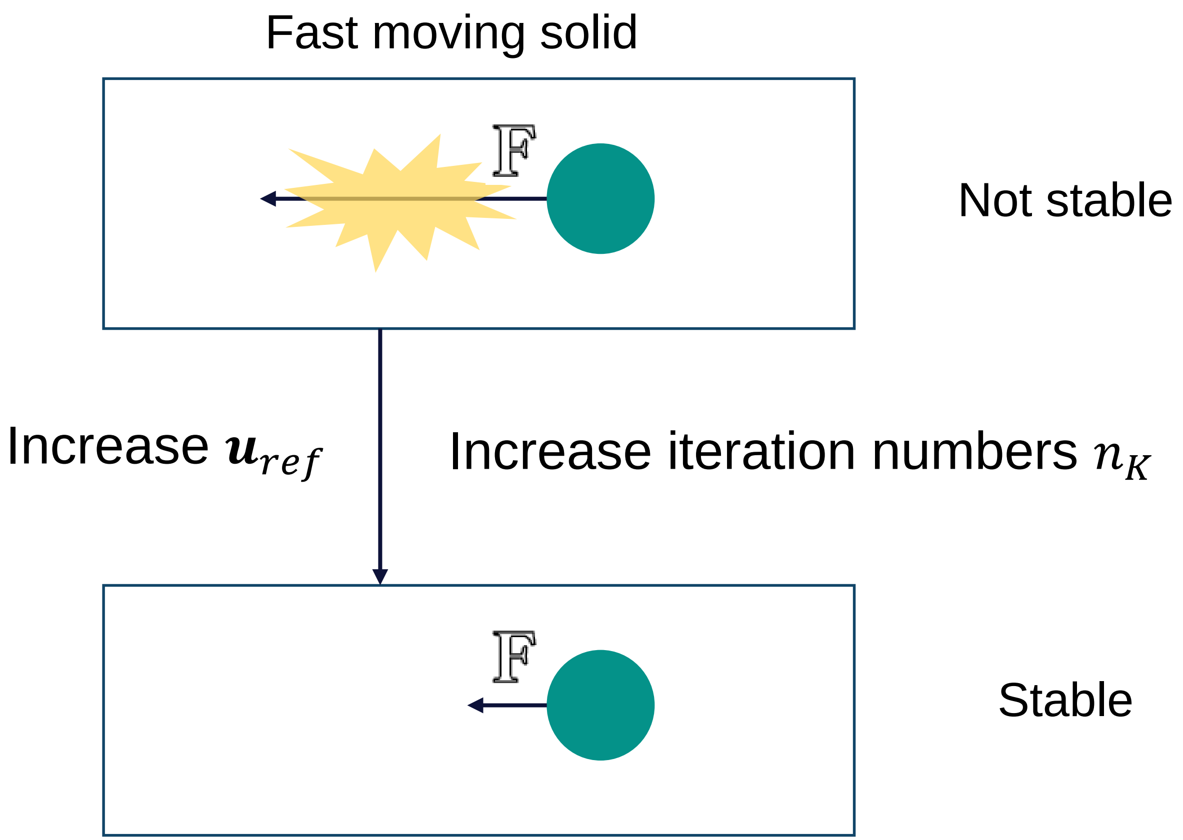


Turbulent fluid with two-way coupling

- **Coupling force in immersed boundary method**
  - Physical space to LBM space:

$$\mathbb{F} = \frac{u_{\text{ref}}^2}{\rho_{\text{ref}}\, u_{\text{ref}}^2} \boldsymbol{F}$$

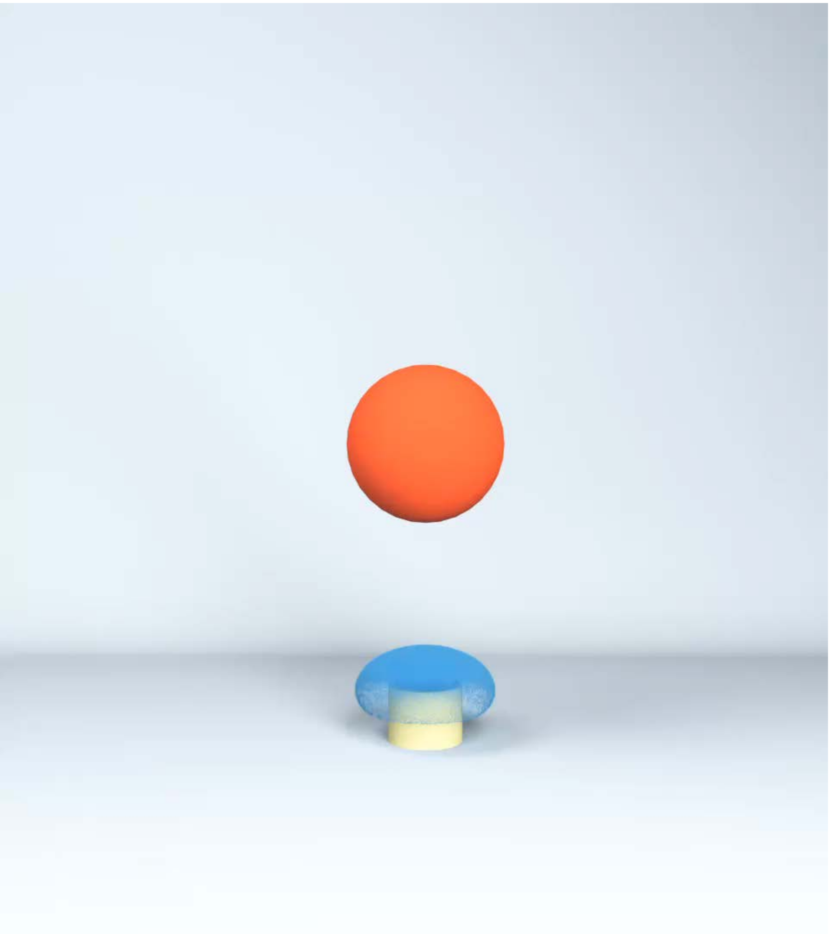
$$n_K = \frac{u_{\text{ref}}}{K u_{\text{ref}}\, \Delta x}$$



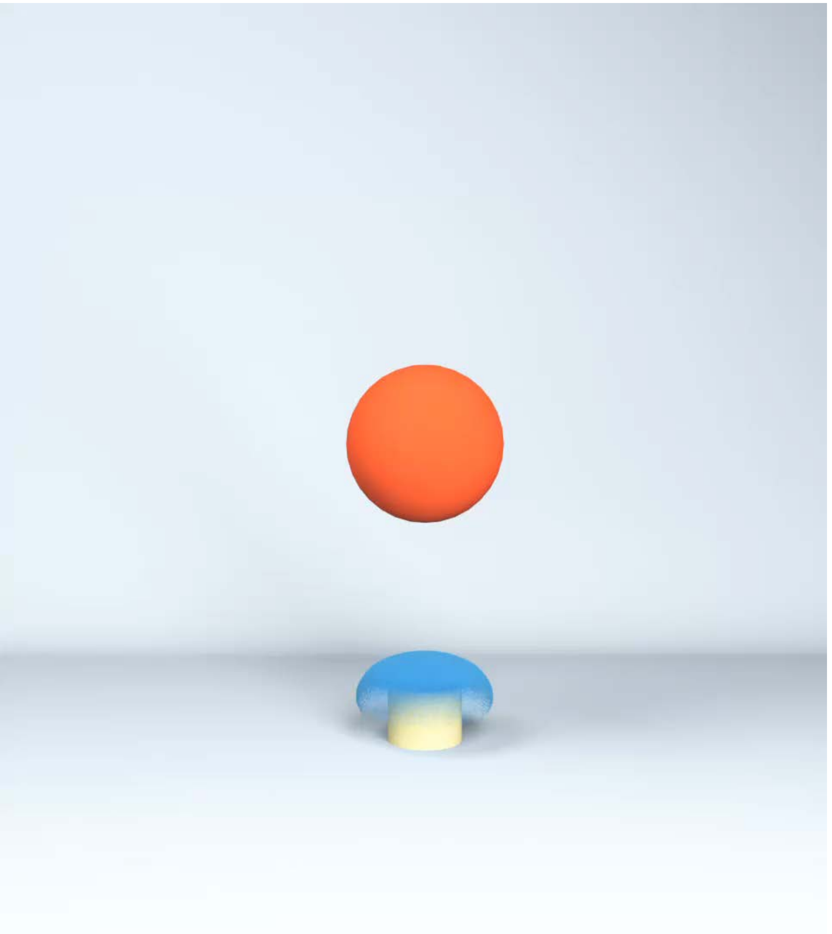
Results

Comparisons

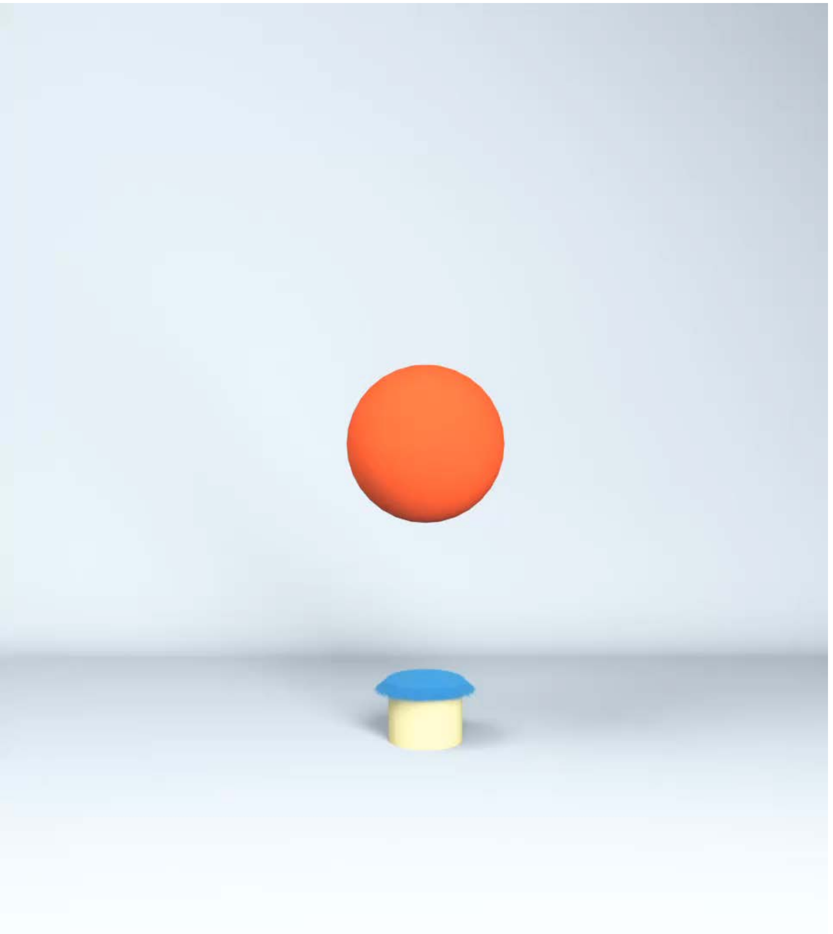
Efficiency vs. Accuracy (I)



MC+R Δt

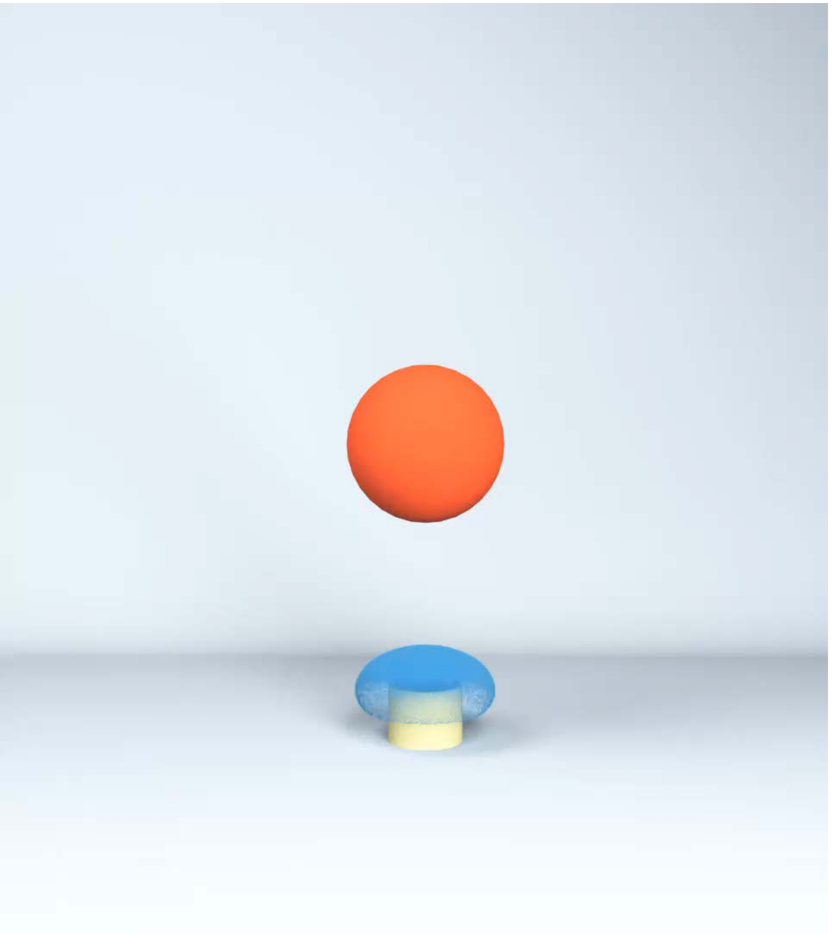


MC+R $7\Delta t$

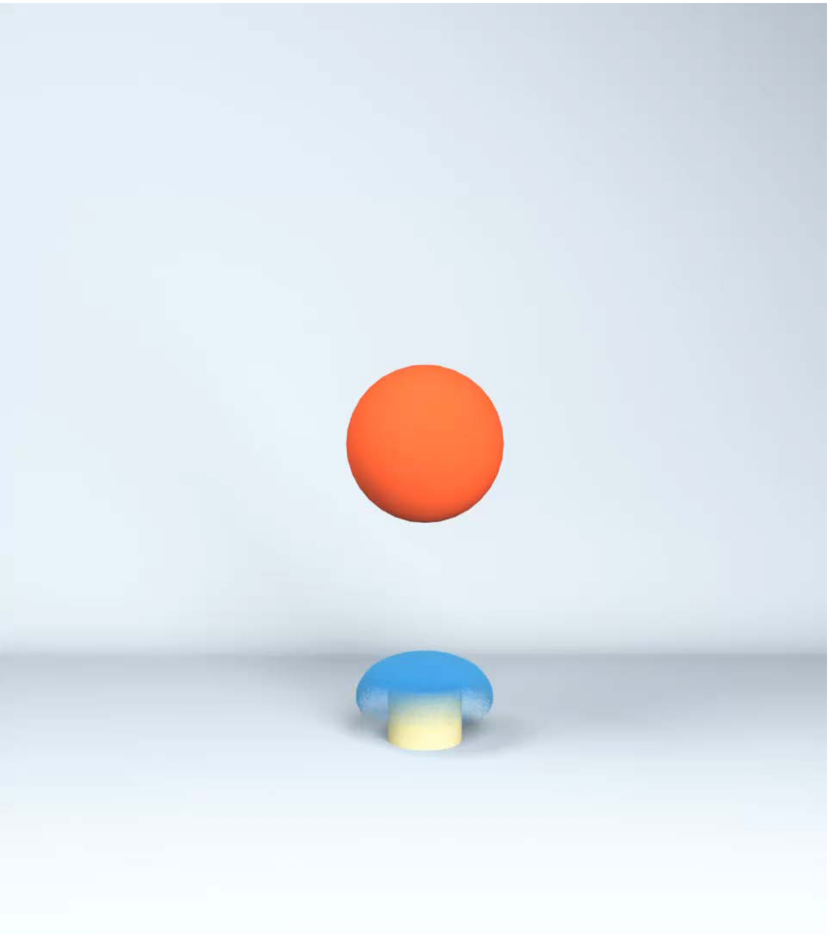


MC+R $42\Delta t$

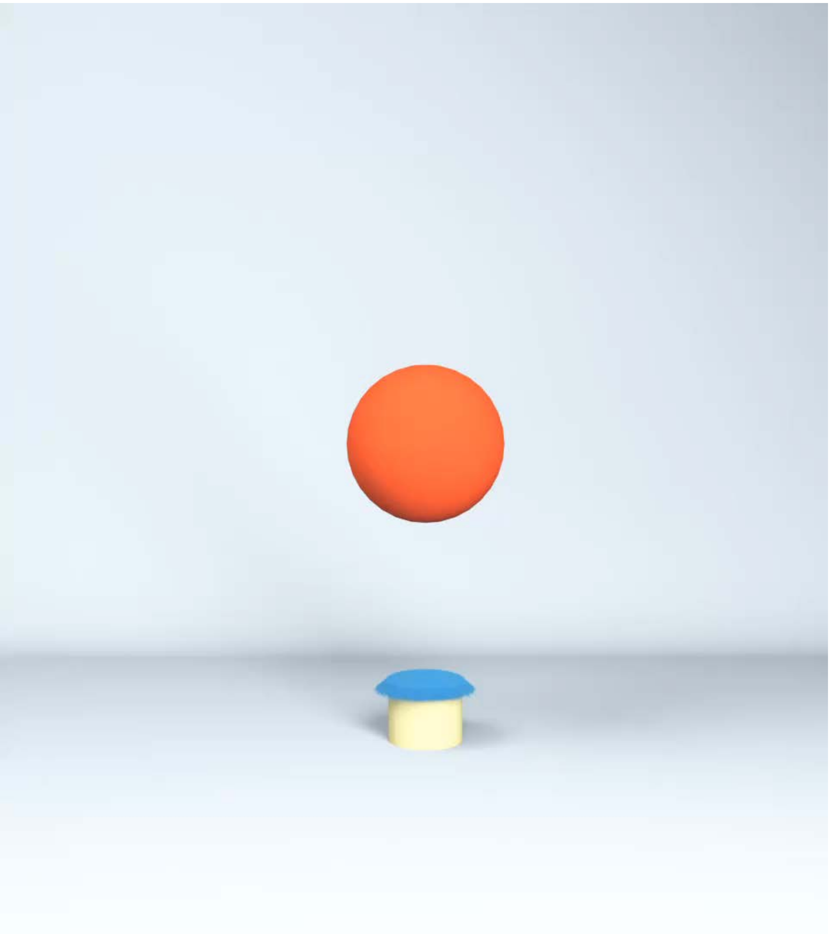
Efficiency vs. Accuracy (I)



MC+R Δt

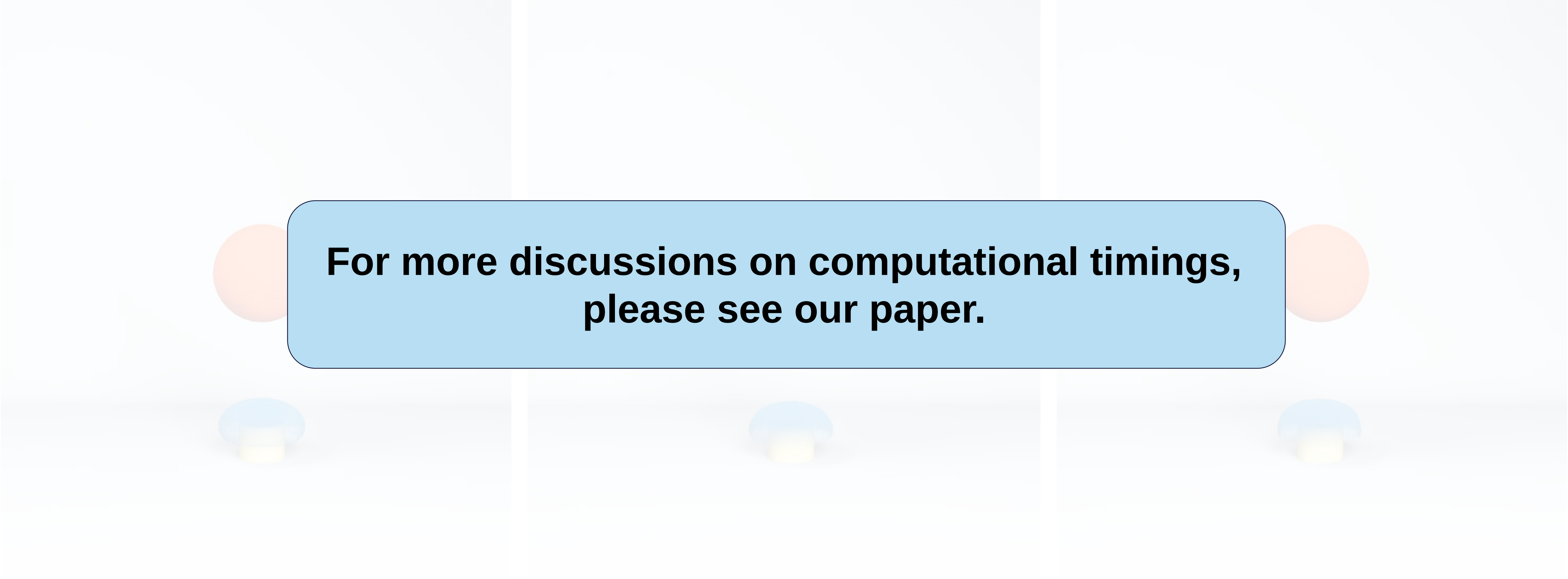


MC+R $7\Delta t$



MC+R $42\Delta t$

Efficiency vs. Accuracy (II)



For more discussions on computational timings, please see our paper.

MC+R Δt | Ours Δt | Ours (8× coarser resolution)

Limitations

- **Inaccurate coupling with thin solid structures**
  - Leakage problems in immersed boundary method
- **Memory usage is relatively larger than traditional N-S solver**
  - Three times larger than current high order solver
  - Try to do spatial adaptive simulation in the future

Conclusions

☑ **Large-scale simulation**
- Scalable

☑ **Turbulent flow**
- Low-dissipation
- Low-dispersion

☑ **Two-way coupling**
- Force evaluation

Thank you!

Q&A