SIGGRAPH THINK BEYOND

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IQ-MPM: AN INTERFACE QUADRATURE MATERIAL POINT METHOD FOR NON-STICKY STRONGLY TWO-WAY COUPLED NONLINEAR SOLIDS AND FLUID

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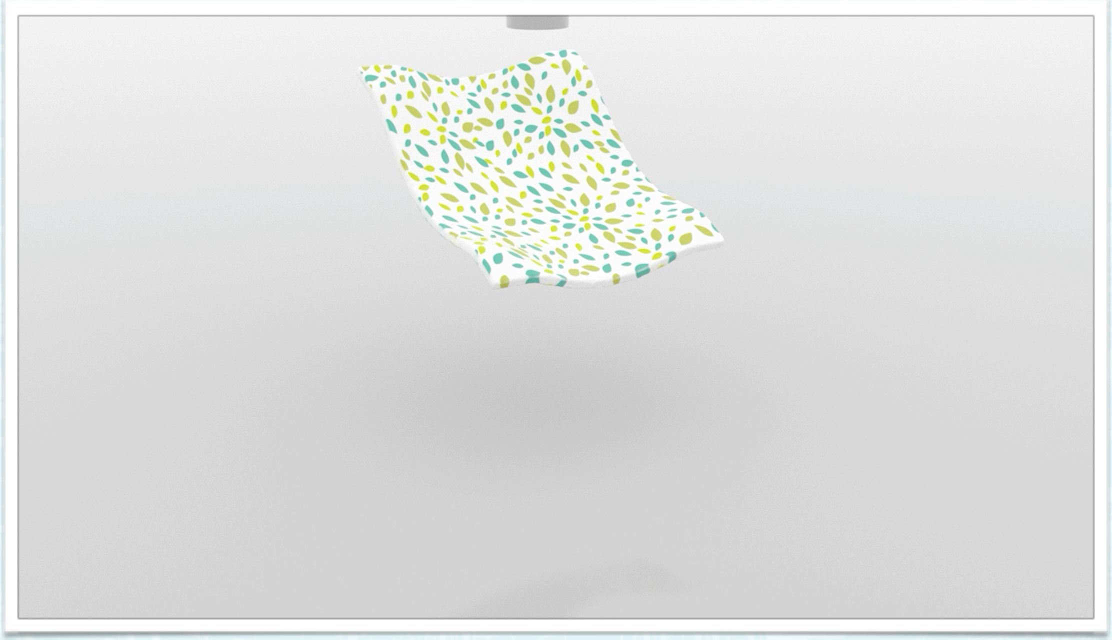
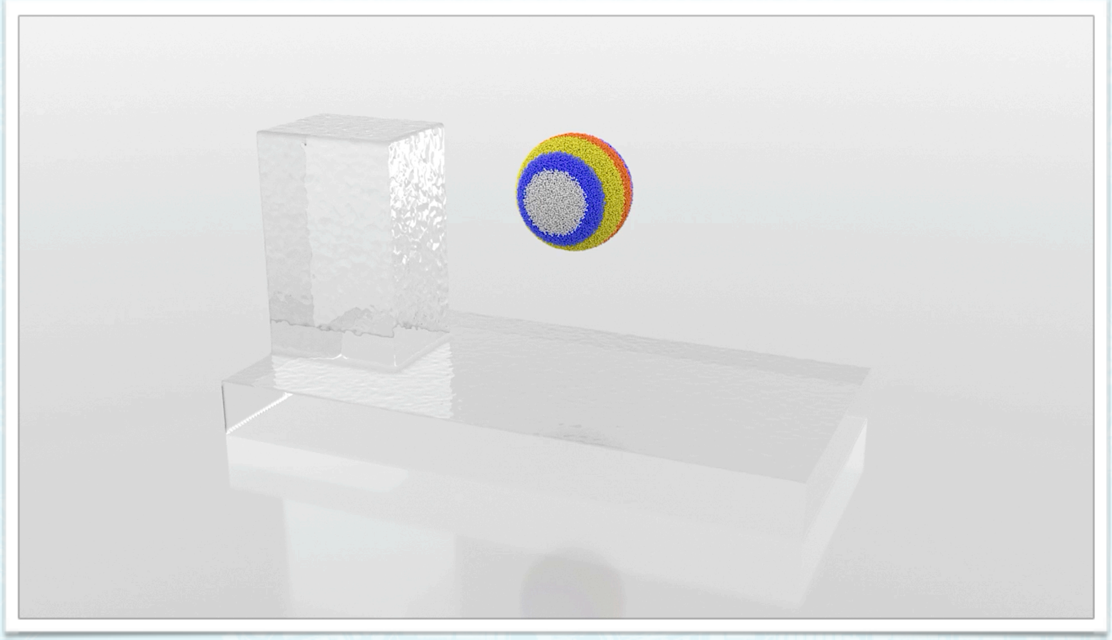
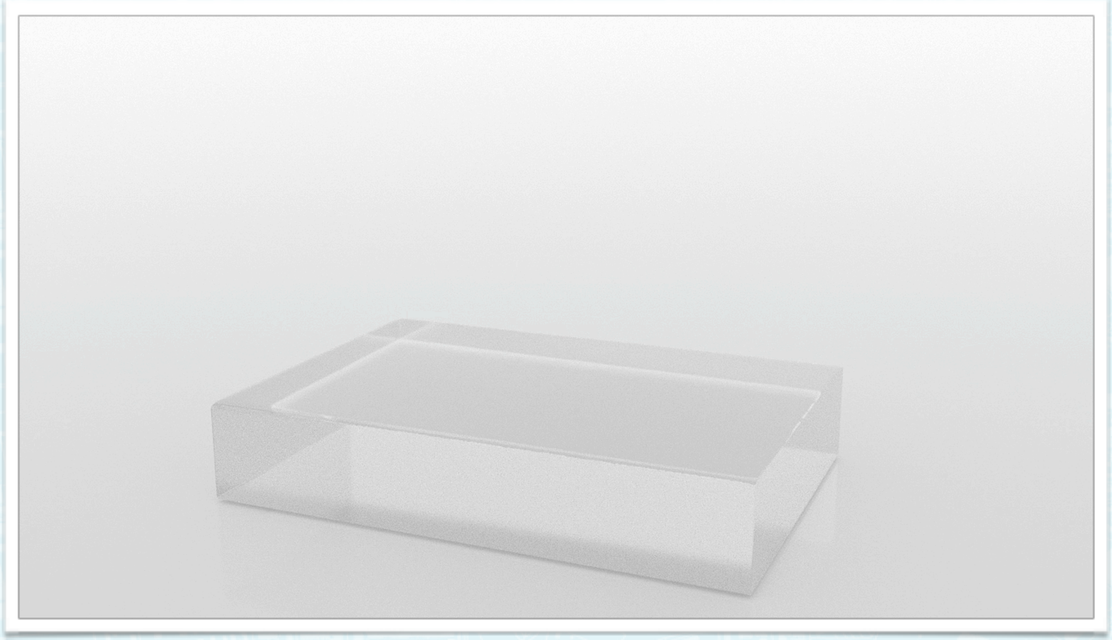






MOTIVATION

Challenge: free slip interface

Traditional MPM
[Stomakhin et al. 2013]

Augmented MPM
[Stomakhin et al. 2014]

Ours

PREVIOUS WORK

- **Lagrangian Solids and Eulerian Fluids** [Chentanez et al. 2006] [Batty et al. 2007]
- **Lagrangian Solids and Lagrangian Fluids** [Solenthaler et al. 2007] [Muller et al. 2015]
- **Eulerian Solids and Eulerian Fluids** [Levin et al. 2011] [Teng et al. 2016]
- **Hybrid Particle-Grid Methods** [Hu et al. 2018] [Han et al. 2019]

Most of these works do not address strong two-way coupling of incompressible fluids with non-linear elastic solids without requiring multiple monolithic solves.

CORE IDEAS

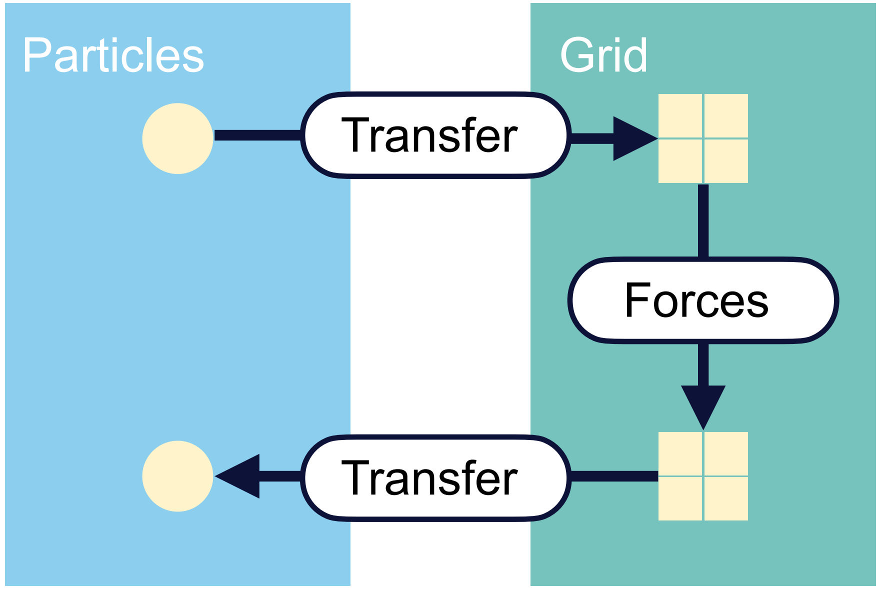
Governing Equation → Weak Form → Discretization

$$\frac{\rho^{s,n}\mathbf{v}^{s,n+1}}{\Delta t} + \nabla p^{g,n+1} - h^{n+1}\mathbf{n}^{s,n} = \frac{\rho^{s,n}\mathbf{v}^{s,*}}{\Delta t}.$$

$$\int_{\Omega^s} \frac{\rho^{s,n} q^s \cdot v^{s,n+1}}{\Delta t} dx + \int_{\Omega^s} q^s \cdot \nabla p^{g,n+1} dx$$
$$- \int_{\Gamma} q^s \cdot n^{s,n} h^{n+1} ds = \int_{\Omega^s} \frac{\rho^{s,n} q^s \cdot v^{s,*}}{\Delta t} dx$$

$$\mathbf{v}^{s,\{n,n+1\}}(\mathbf{x}) = \sum_i \mathbf{v}_i^{s,\{n,n+1\}} N_i^{s,2}(\mathbf{x}),$$
$$\mathbf{q}^s(\mathbf{x}) = \sum_j \mathbf{q}_j^s N_j^{s,2}(\mathbf{x}).$$

Material Point Method (MPM)



MPM Coupling Issues:

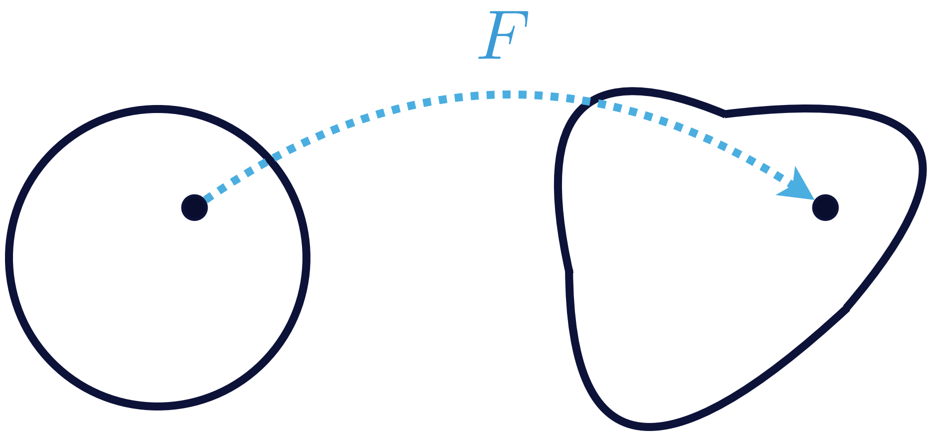
- Single grid: self-collision suffers from numerical stickiness
- Multiple grid: tangentially discontinuous velocity by using different grids is limited to explicit time integration

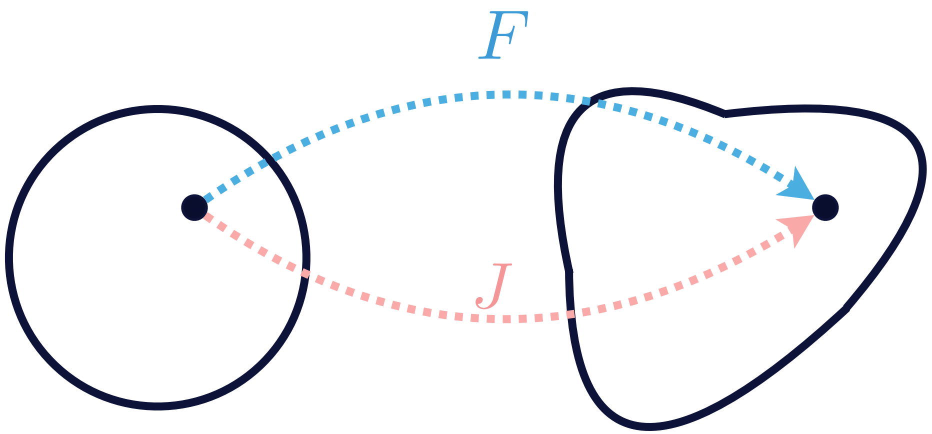
The Material Point Method for Simulating Continuum Materials, SIGGRAPH course notes, 2016.

On hybrid lagrangian-eulerian simulation methods: practical notes and high-performance aspects, SIGGRAPH course notes, 2019.

IQ-MPM METHOD

SPLITTING





- **Traditional Solid:**

$$\Psi(F^s) = \Psi^s(F)$$

Arbitrary nonlinear hyper-elasticity supported:
Fixed Corotated model [Stomakhin et al. 2012]
Neo-Hookean model
Saint Vernant-Kirchhoff model

- **Solid with ghost matrix:**

Similar to compressible air

$$\Psi(F^s, J^g) = \Psi^s(F^s) + \Psi^g(J^g)$$

$$\Psi^g(J^g) = \frac{1}{2}\lambda^J(J^g - 1)^2$$

Half Lame's first parameter

GOVERNING EQUATIONS (GENERAL)

- For any phase k (fluid or solid):

$$\frac{D\rho^k}{Dt} + \rho^k \nabla \cdot \mathbf{v}^k = 0, \qquad \mathbf{x} \in \Omega^k, \qquad \text{Mass conservation}$$

$$\rho^k \frac{D\mathbf{v}^k}{Dt} - \nabla \cdot \sigma^k - \rho^k \mathbf{g} = 0, \qquad \mathbf{x} \in \Omega^k \qquad \text{Momentum conservation}$$

$$\sigma^k \cdot \mathbf{n}^k = \mathbf{b}, \qquad \mathbf{x} \in \partial\Omega_N^k \qquad \text{Free surface}$$

$$\mathbf{v}^k \cdot \mathbf{n}^k = v_S^k, \qquad \mathbf{x} \in \partial\Omega_S^k \qquad \text{Slip boundary}$$

$$\mathbf{v}^k = \mathbf{v}_{\text{NS}}^k, \qquad \mathbf{x} \in \partial\Omega_{\text{NS}}^k \qquad \text{No-slip boundary}$$

g: gravity
n^k: outward normal
Ω^k: solid/fluid domain
ρ^k: density
v^k:velocity
σ^k: Cauchy stress
$\mathbf{b}$: free surface bc
v_S^k: slip bc
v_{NS}^k: no-slip bc

- Automatic MPM coupling is inherently restricted to sticky and no-slip interactions
- To allow free-slip boundary conditions, we add velocity and pressure continuity for solid and fluid interfaces

$$(\mathbf{v}^s - \mathbf{v}^f) \cdot \mathbf{n}^s = 0, \qquad \mathbf{x} \in \Gamma \qquad\qquad p^s - p^f = 0, \qquad \mathbf{x} \in \Gamma.$$

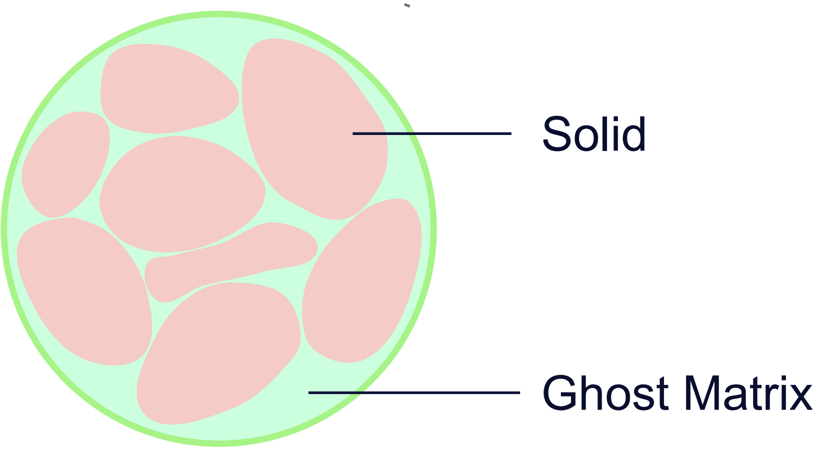
GOVERNING EQUATIONS (SOLID)

Energy based solid-wall collision

Strong coupling force exerted from the fluid to the solid

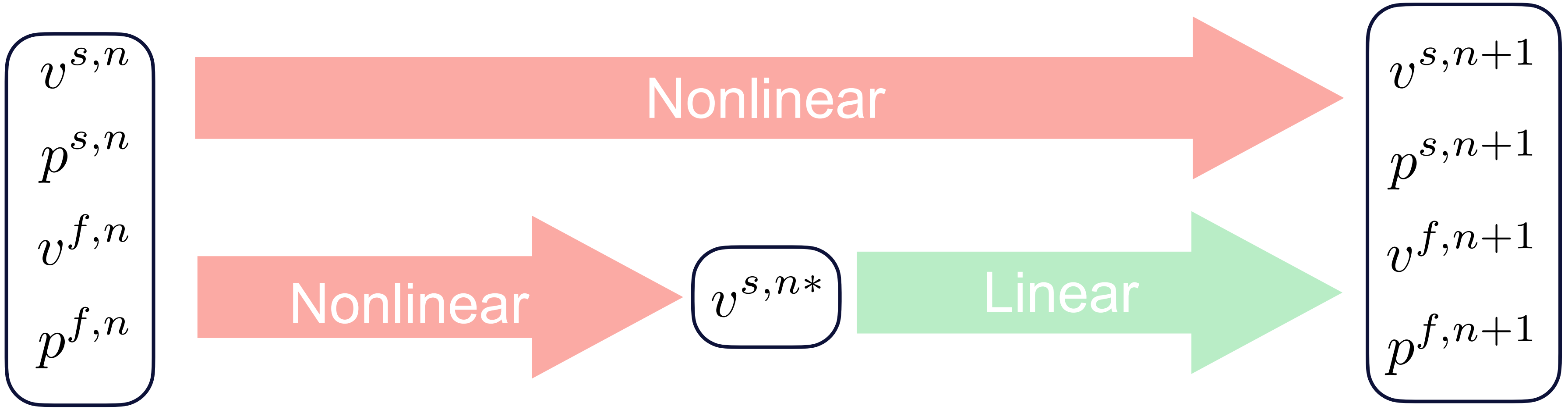
$$\frac{\rho^{s,n} v^{s,n+1}}{\Delta t} - \nabla \cdot \sigma^{s,n+1} + \nabla p^{g,n+1} = \frac{\rho^{s,n} v^{s,n}}{\Delta t} + f^{fs,n+1}$$

$$\nabla \cdot v^{s,n+1} + \frac{p^{g,n+1}}{\lambda^g J^{g,n} \Delta t} = \frac{p^{g,n}}{\lambda^g J^{g,n} \Delta t}$$



- Solid and ghost matrix share velocity
  $v^s = v^g$
- Coupling force is along normal direction
  $f^{fs,n+1} = h^{n+1} n^{s,n}$

SPLITTING SOLVE



$$\frac{\rho^{s,n} v^{s,*}}{\Delta t} - \nabla \cdot \sigma^{s,*} = \frac{\rho^{s,n} v^{s,n}}{\Delta t}$$

- Fully nonlinear Newton solve

$$\frac{\rho^{s,n} v^{s,n+1}}{\Delta t} + \nabla p^{g,n+1} - h^{n+1} n^{s,n} = \frac{\rho^{s,n} v^{s,*}}{\Delta t}$$

- Linear system coupled with fluid
- Solve pressure first and substitute them back

Inspired by [Teng et.al 2016]

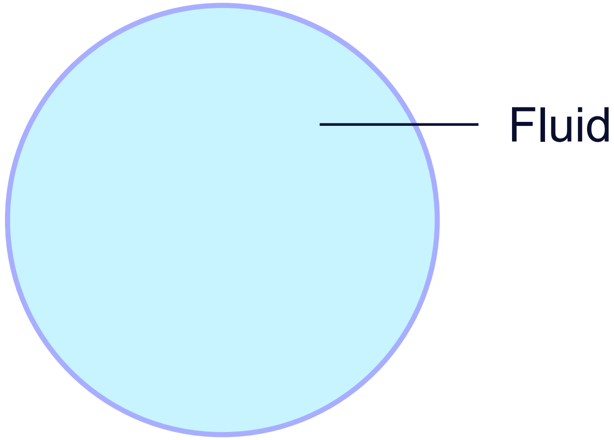
GOVERNING EQUATIONS (FLUID)

Density does not change over time
for incompressible fluid

$$\frac{\rho^f v^{f,n+1}}{\Delta t} + \nabla p^{f,n+1} = \frac{\rho^f v^{f,n}}{\Delta t} + f^{sf,n+1} + f^{w,n+1}$$

$$\nabla \cdot v^{f,n+1} = 0$$

Divergence free condition



- Fluid touches slip boundary walls by constraints
  $$v^{f,n+1}|_{\partial\Omega^f_S} \cdot n^f = v^f_S$$
- Coupling force is also along normal direction
- Incompressibility is assuming that
  $$\lambda = \infty$$

WEAK FORM

Governing equation

$$\frac{\rho^{s,n} v^{s,n+1}}{\Delta t} + \nabla p^{g,n+1} - h^{n+1} n^{s,n} = \frac{\rho^{s,n} v^{s,*}}{\Delta t}$$

Unknown variables

Weak form

$$\int_{\Omega^s} \frac{\rho^{s,n} q^s \cdot v^{s,n+1}}{\Delta t} dx + \int_{\Omega^s} q^s \cdot \nabla p^{g,n+1} dx - \int_{\Gamma} q^s \cdot n^{s,n} h^{n+1} ds = \int_{\Omega^s} \frac{\rho^{s,n} q^s \cdot v^{s,*}}{\Delta t} dx$$

Solid

Solid

Solid-fluid interface

Solid

Test function

Integration domain

DISCRETIZATION I

Weak form

$$\int_{\Omega^s} \frac{\rho^{s,n} q^s \cdot v^{s,n+1}}{\Delta t} dx + \int_{\Omega^s} q^s \cdot \nabla p^{g,n+1} dx - \int_{\Gamma} q^s \cdot n^{s,n} h^{n+1} ds = \int_{\Omega^s} \frac{\rho^{s,n} q^s \cdot v^{s,*}}{\Delta t} dx$$

Continuous variables

Discretization on variables

$$v^{s,\{n,n+1\}}(x) = \sum_i v_i^{s,\{n,n+1\}} N_i^{s,2}(x)$$

$$q^s(x) = \sum_j q_j^s N_j^{s,2}(x)$$

Continuous variables | Discretized variables | Order

DISCRETIZATION I

MPM Limitation

Solid velocity order ≥ 1

- Prevent the cell-crossing instability
- Use **B-spline** to eliminate nonnegative cases

Computational Limitation

Any order ≤ 2

- Reduce kernel size and achieve efficiency

Stability Limitation

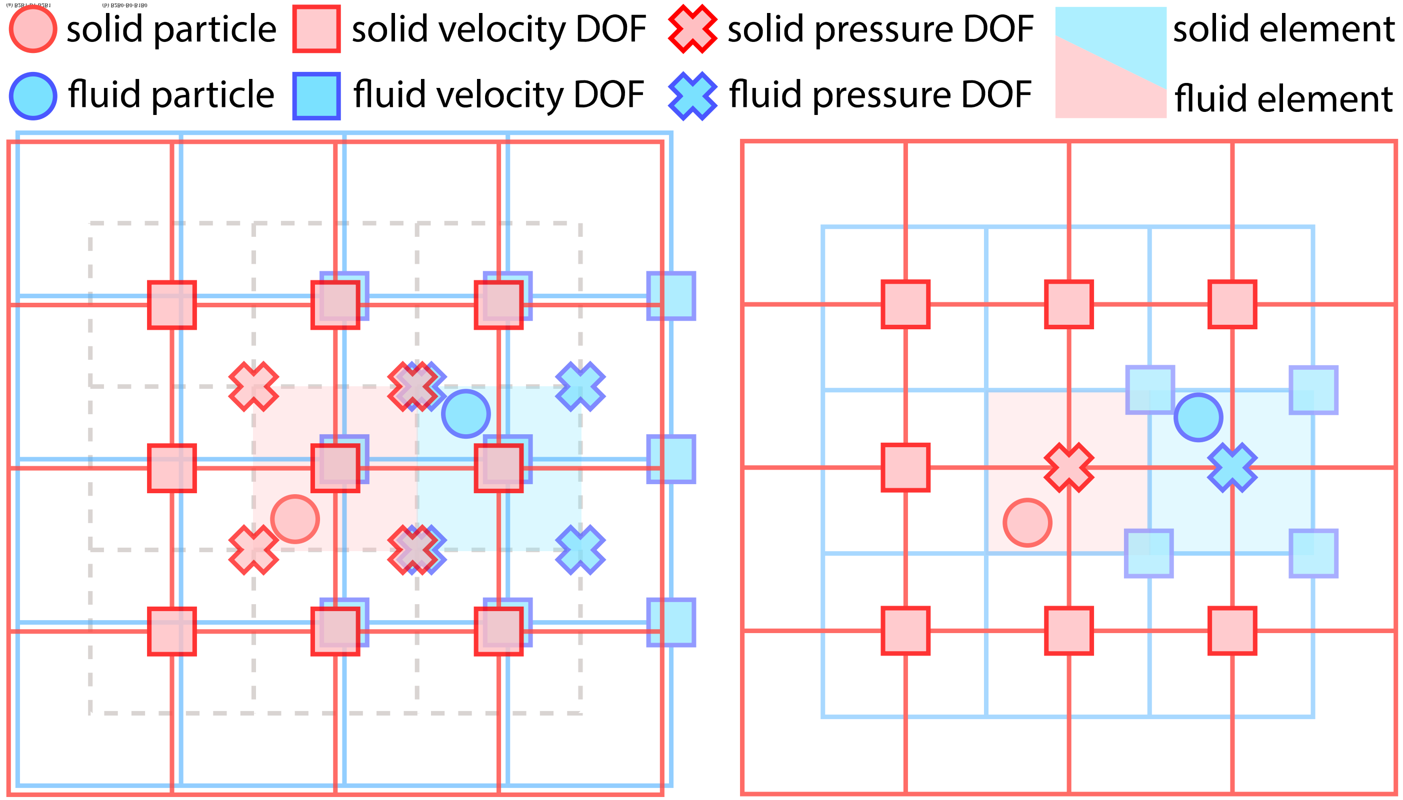
velocity order > pressure order

- Prevent kinetic locking
- Satisfy inf-sup stability
- Avoid null space in velocity-pressure system

DISCRETIZATION I

$$B\{v^s\}B\{p^s\} - B\{h\} - B\{v^f\}B\{p^f\}$$

- Solid's velocity and pressure order
- Solid-fluid coupling pressure order
- Fluid's velocity and pressure order



$B2B1 - B1 - B2B1$ $\quad$ $B2B0 - B0 - B1B0$

DISCRETIZATION II

Weak form

$$\int_{\Omega^s} \frac{\rho^{s,n} q^s \cdot v^{s,n+1}}{\Delta t} dx + \int_{\Omega^s} q^s \cdot \nabla p^{g,n+1} dx - \int_{\Gamma} q^s \cdot n^{s,n} h^{n+1} ds = \int_{\Omega^s} \frac{\rho^{s,n} q^s \cdot v^{s,*}}{\Delta t} dx$$

Continuous integrations

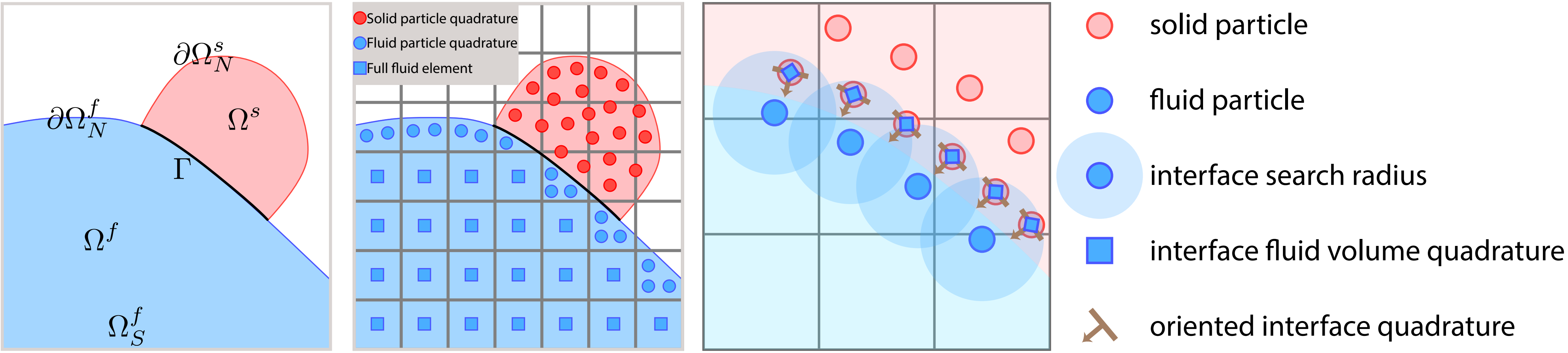
Discretization on integrations

For solid (MPM), $\int_{\Omega^s} \rho^{s,n} N_i^{s,2}(x) dx = \sum_p m_p^s N_i^{s,2}(x_p)$

Particles quadrature

For fluid, ?

DISCRETIZATION II



Methods	Efficiency	Accuracy	DoF consistency
Particles quadrature	Good	Bad	Bad
Cut cell + analytical integration	Bad	Good	Bad
Interface quadrature	Good	Good	Good

GHOST MATRIX UPDATE J^g

B2B1-B1-B2B1

$$p^{s,n+1} = -\lambda(J^{g,n+1} - 1)$$

$$J^{g,n+1} = \det(F^{s,n+1})$$

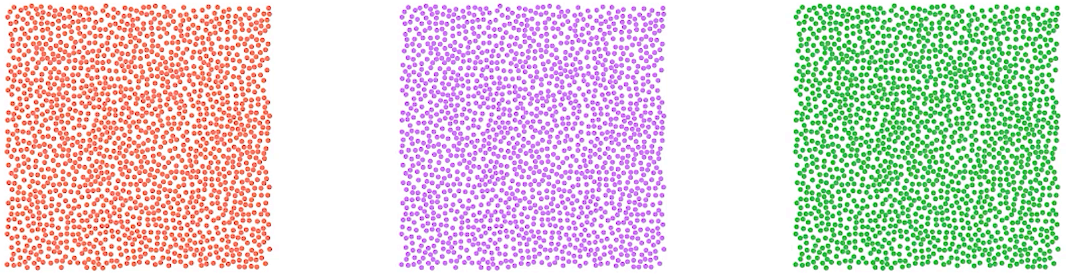
$$J_p^{g,n+1} = (1 + \Delta t(\nabla \cdot v^{n+1})(x_p))J_p^{g,n}$$

B2B0-B0-B1B0

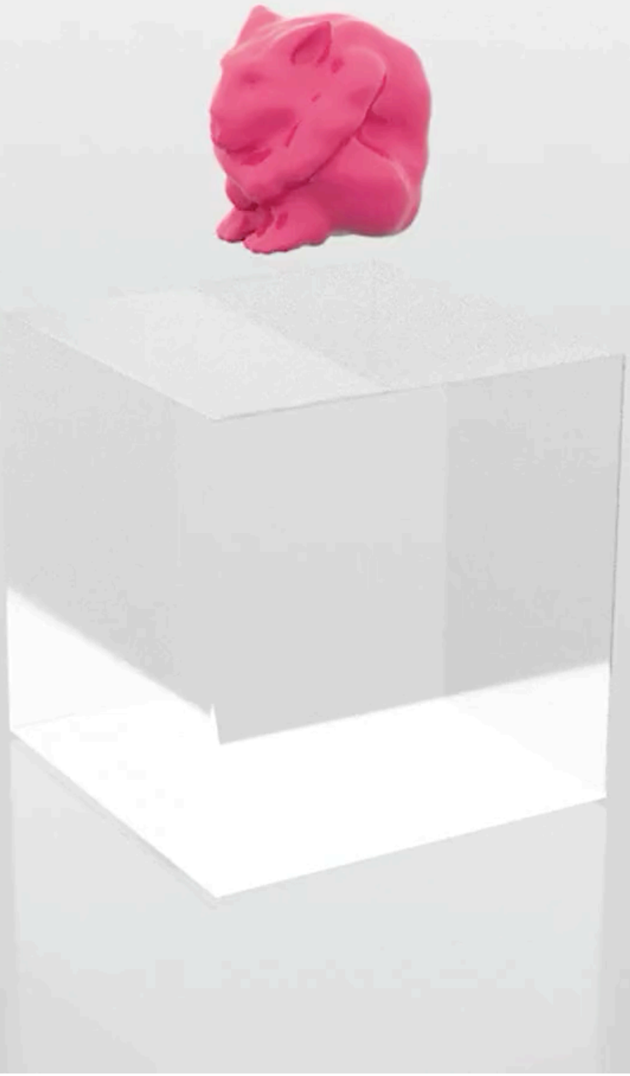
$$p^{s,n+1} = -\lambda(J^{g,n+1} - 1)$$

- Discontinuous pressure and high order velocity in inconsistency
- Derived from weak form

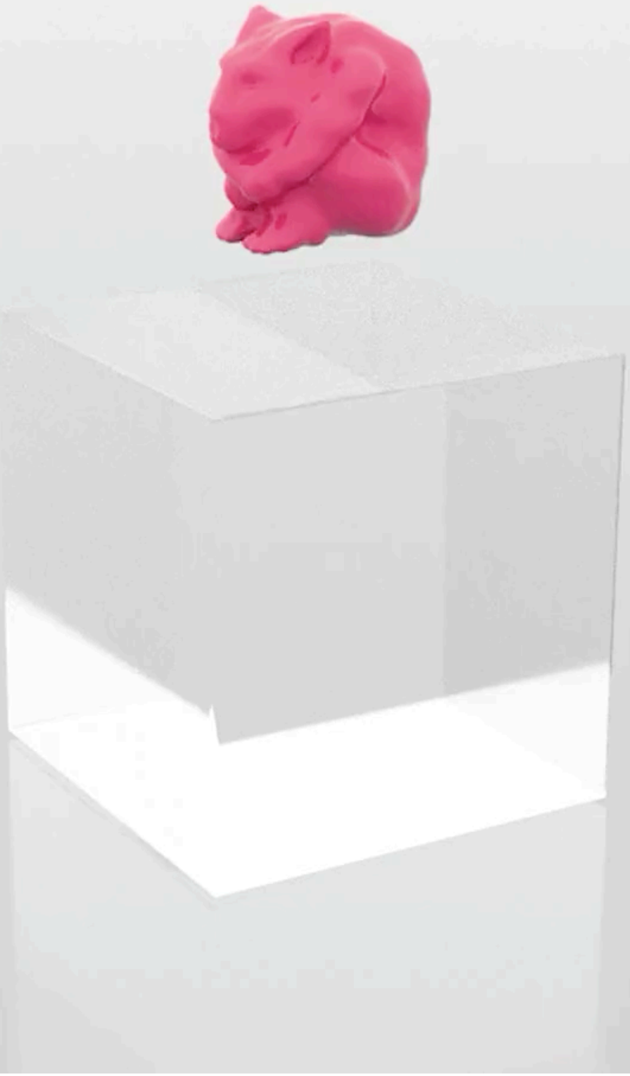
All tests under B2B0-B0-B1B0



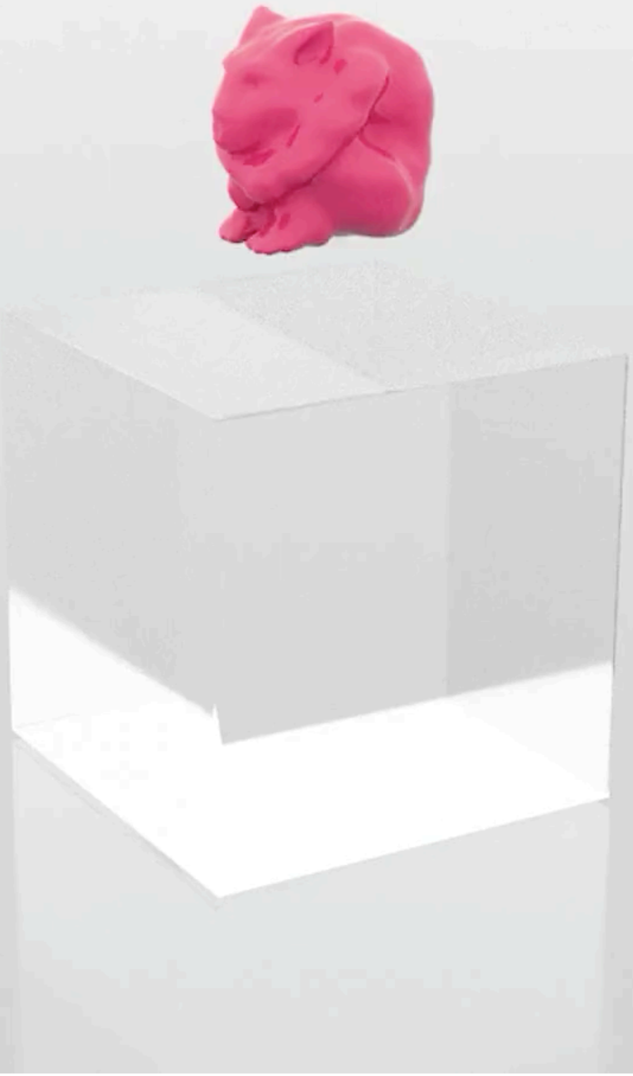
MORE RESULTS



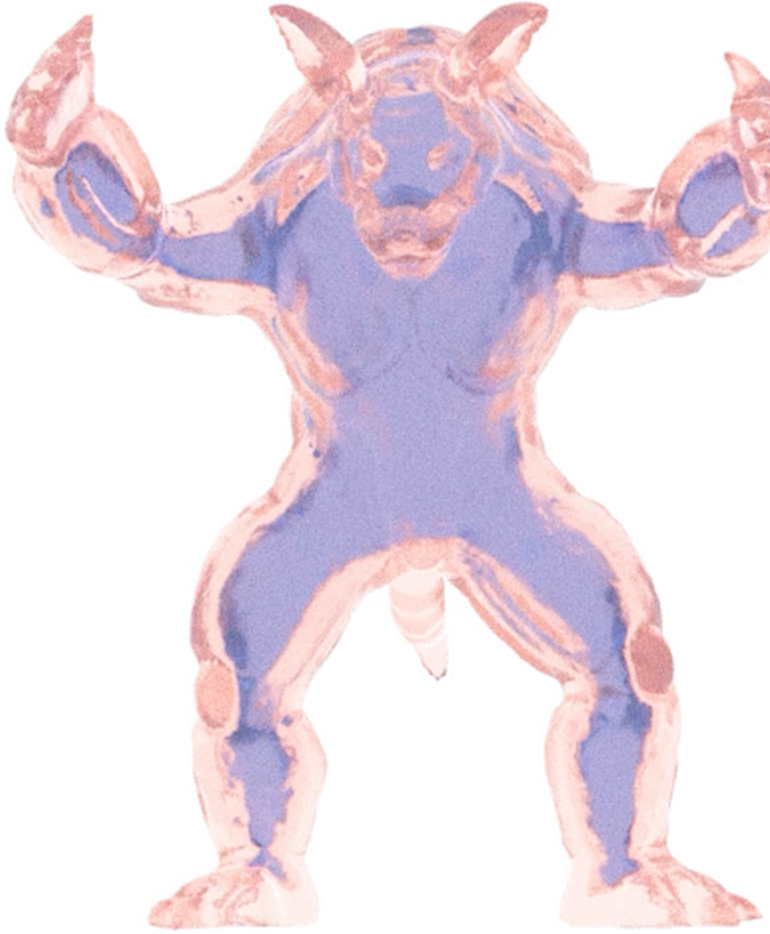
Density = 200 kg/m^3

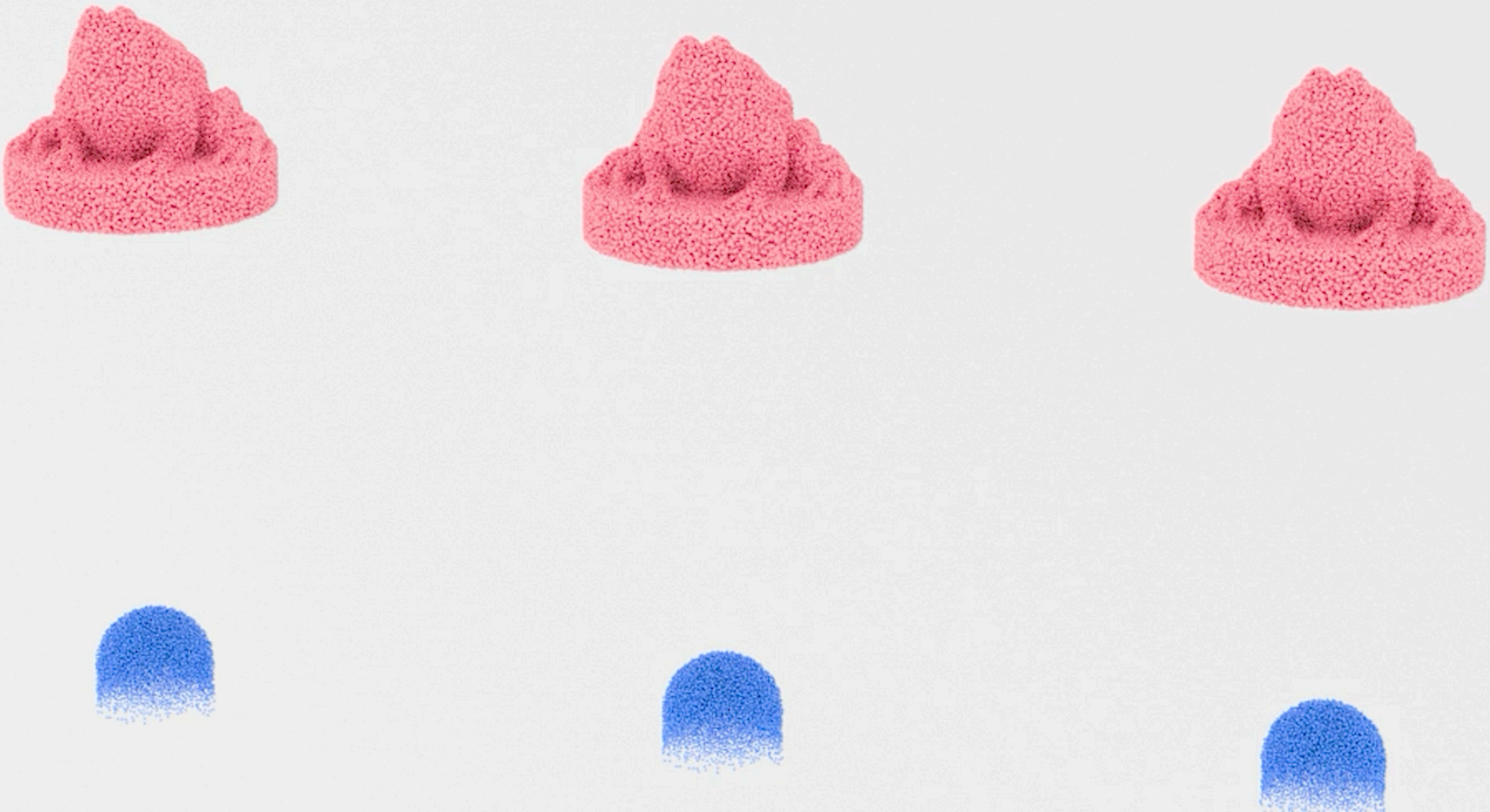


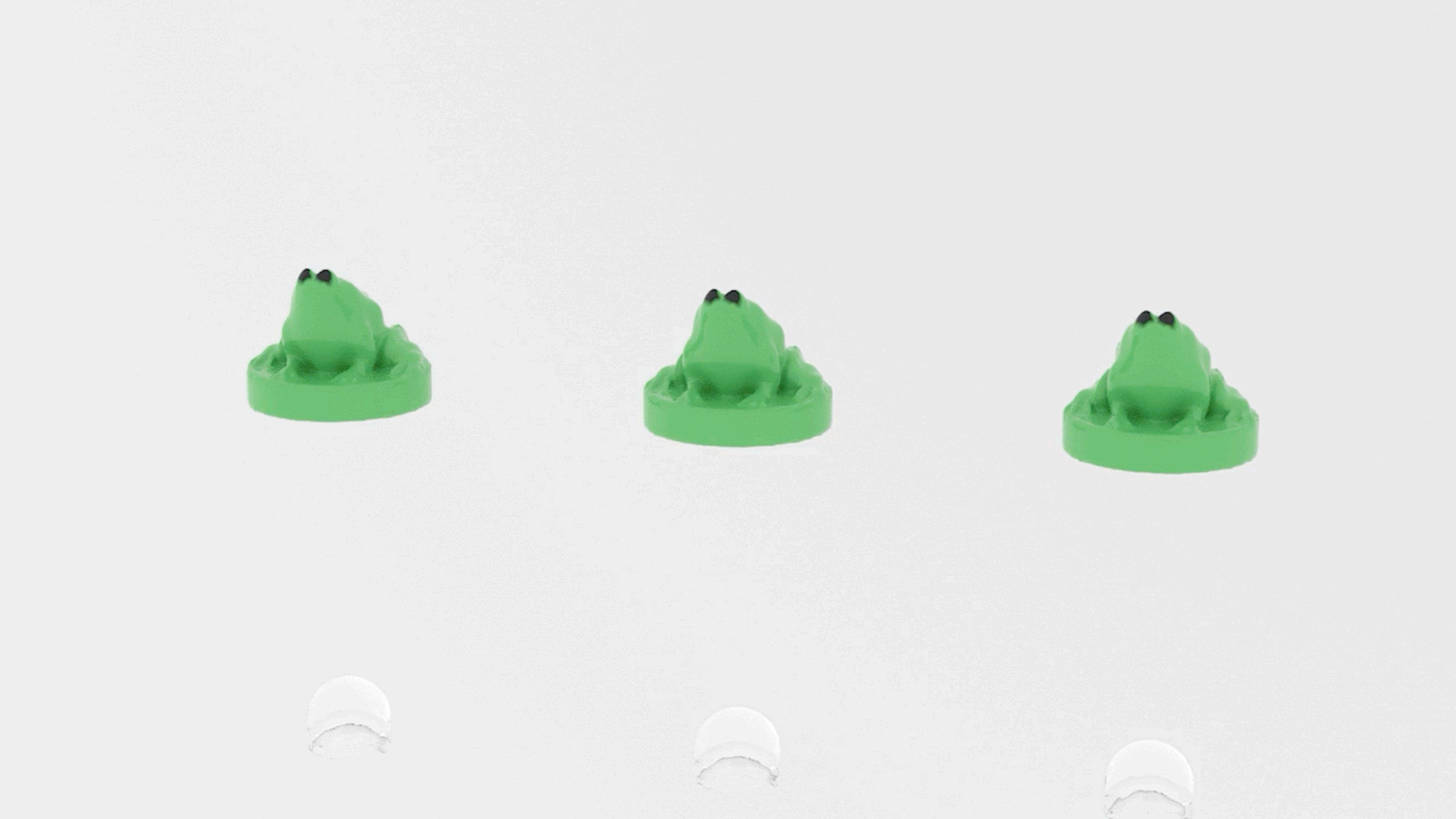
Density = 700 kg/m^3



Density = 1200 kg/m^3







THANK YOU!