Graphics And Mixed Environment Seminar
12/16/2021

Incompressible fluid simulation based on vortex surface field

Shiying Xiong

Dartmouth College

Content

- Background
- Vortex surface field and Clebsch representation
- Construction of knotted Clebsch fields
- Clebsch gauge fluid

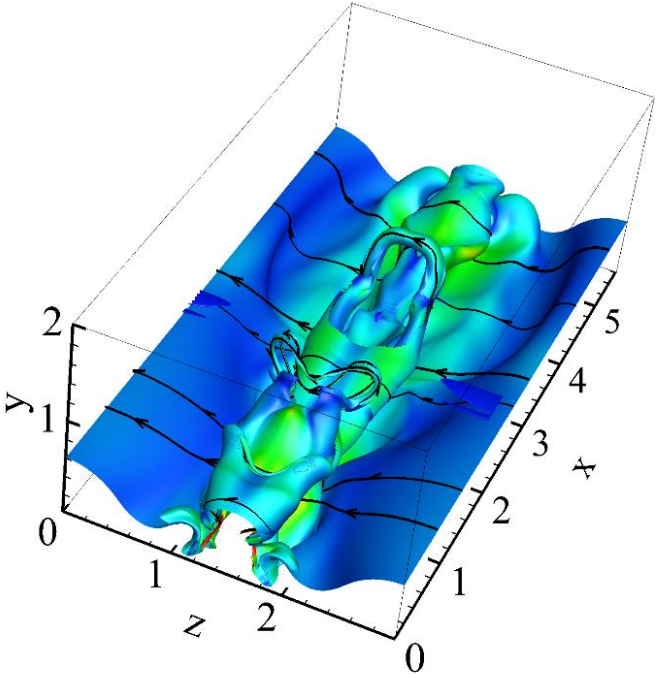
Votical flows

Jellyfish vs bubble ring.

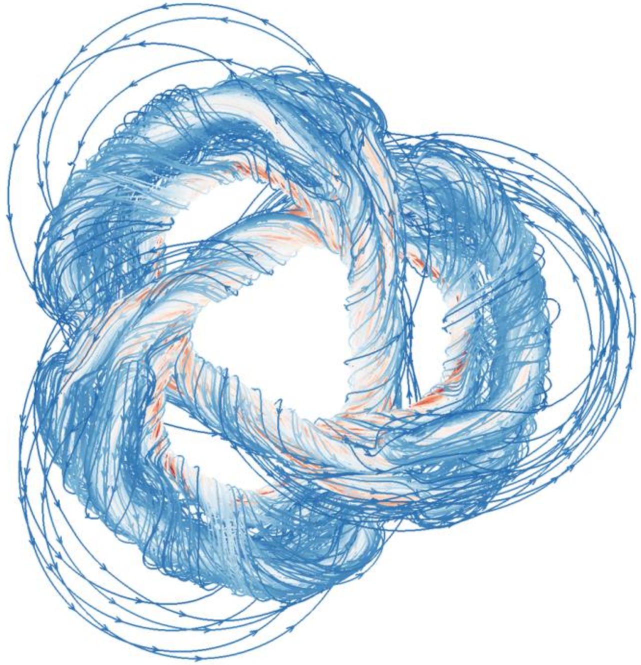
Wingtip vortices from airplanes.

https://www.quora.com/What-is-the-vortex-ring-state-in-a-helicopter

Vortex surfaces and vortex lines in fluids



Vortex surface in the transitional channel flow.
S. Xiong & Y. Yang,
J. Comput. Phys., 2017

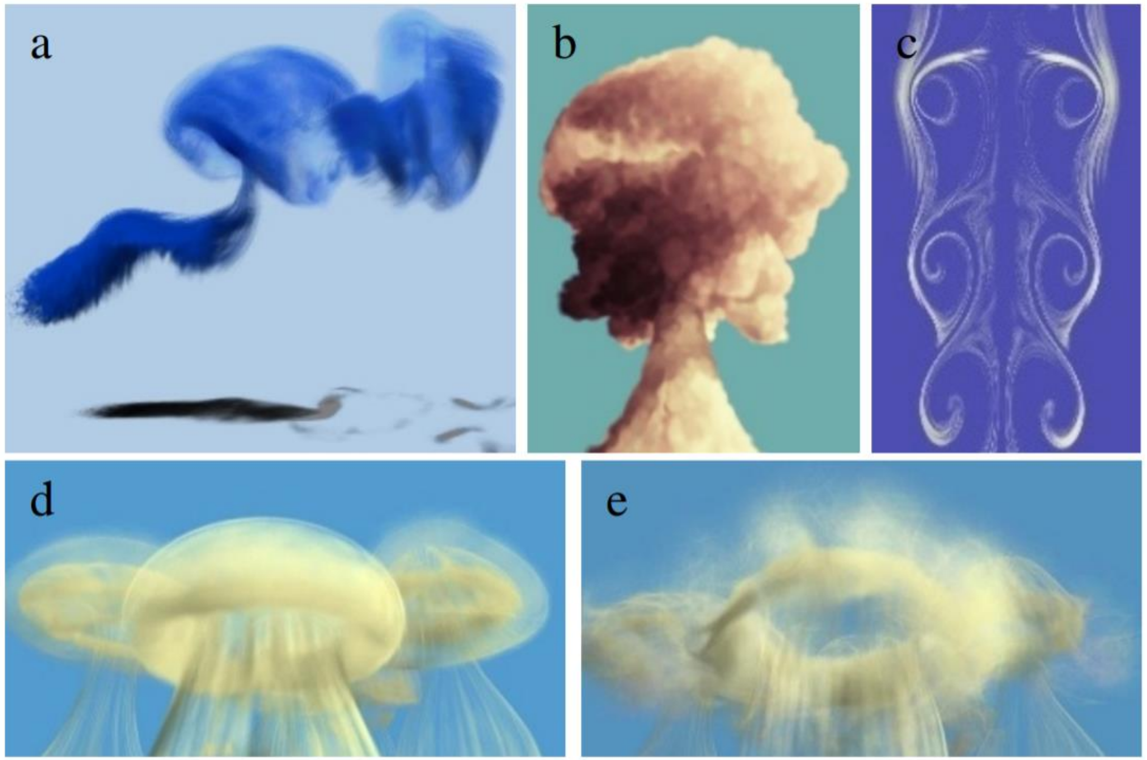


Chaotic flux lines inside the magnetic knot.
S. Xiong & Y. Yang,
J. Fluid Mech., 2020

Vortex surfaces in isotropic turbulence.
S. Xiong & Y. Yang, *J. Fluid Mech.*, 2019

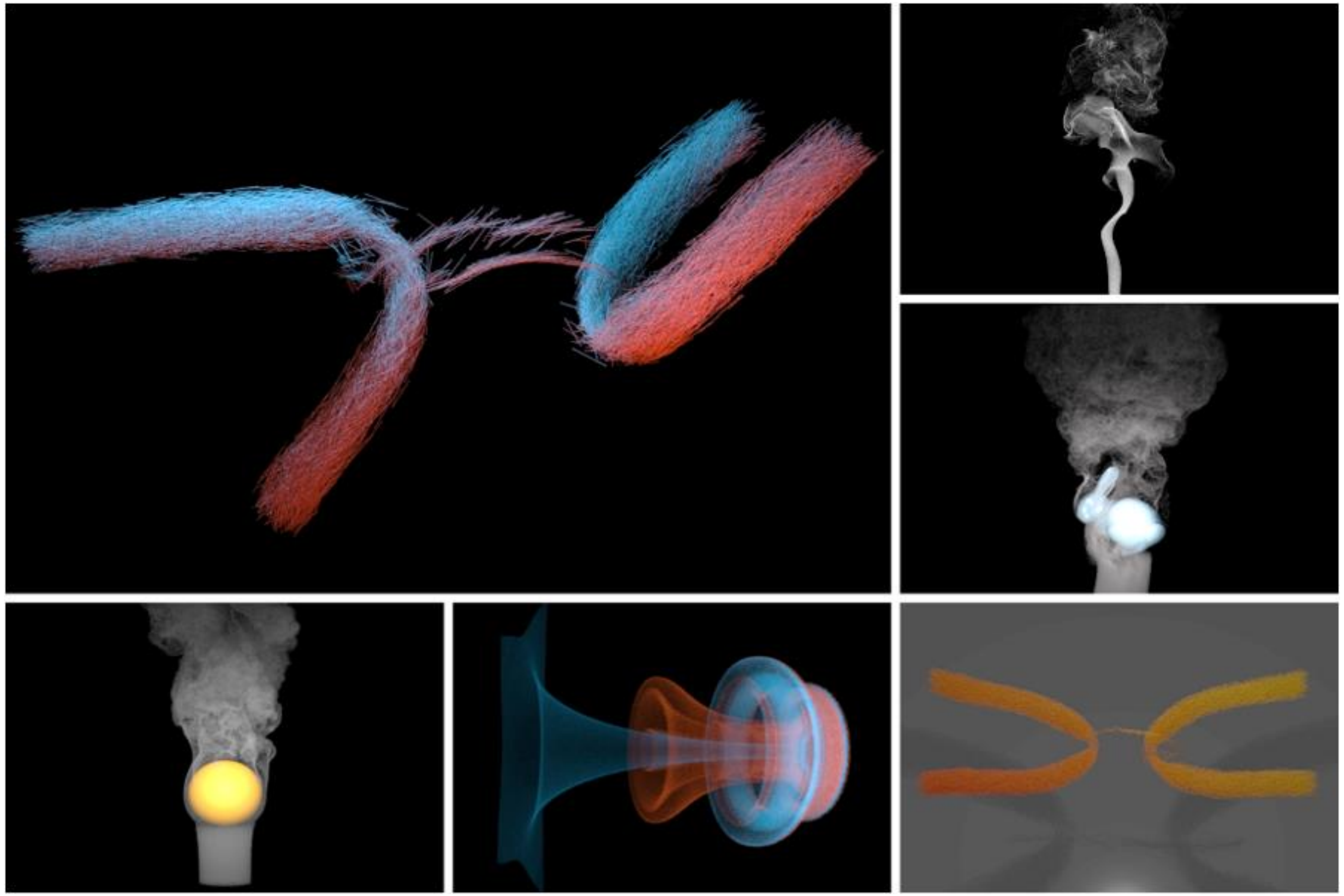
Vortex filament method

Simulation of a jet.
S. Weißmann & U. Pinkall, *SIGGRAPH,* 2010

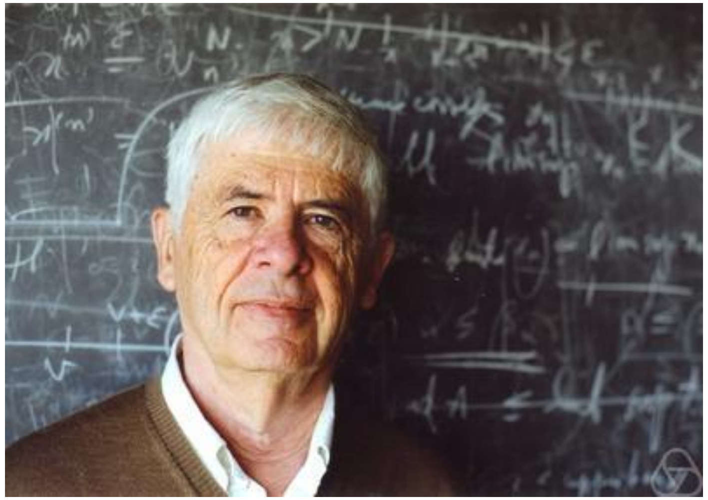


Various examples of animated flows.
A. Angelidis & F. Neyret, *Eurographics,* 2015

Vortex segment method



Various examples of animated flows.
S. Xiong *et al*., *SIGGRAPH*, 2021



“a physical vortex is approximated by a cloud of tubular vortices” (Chorin, 1990)

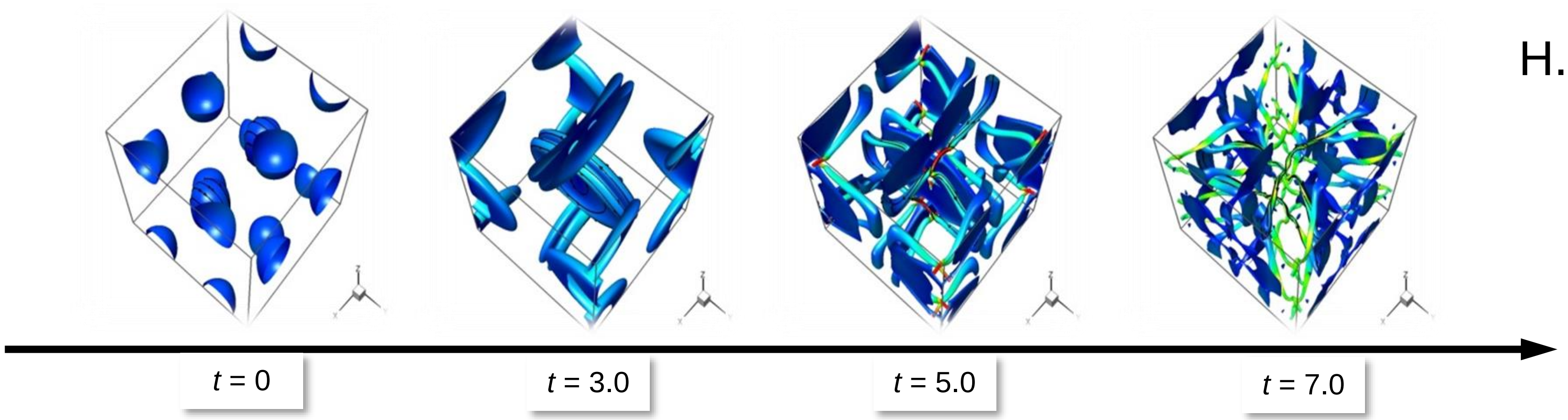
Content

- Background
- **Vortex surface field and Clebsch representation**
- Construction of knotted Clebsch fields
- Clebsch gauge fluid

Vortex surface field (VSF)

- ❑ Vortex surfaces: tangent to vorticity lines on every point
- ❑ Vortex surfaces are material surfaces in inviscid flows
- ❑ Timewise evolution of smooth VSFs in viscous flows elucidates continuous vortex dynamics

H. von Helmholtz
(1821-1894)



Evolution of vortex surface in Taylor-Green flow
Y. Yang and D. I. Pullin, *J. Fluid Mech.*, 2011

Vorticity Clebsch representation

- Vorticity Clebsch map (Alfred Clebsch, 1985)

$$\boldsymbol{\omega} = \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\chi$$

- Advantages
  - Vortex surface field (Y. Yang & D. I. Pullin, 2010) $\boldsymbol{\omega} \cdot \boldsymbol{\nabla}\psi = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\chi = 0$
  - Hamiltonian mechanics

- Limitations
  - Can only represent fields with zero helicity $H = \int \boldsymbol{u} \cdot \boldsymbol{\omega} = 0$
  - Clebsch map may not exist near points of vanishing vorticity (Graham and Henyey, 2000)

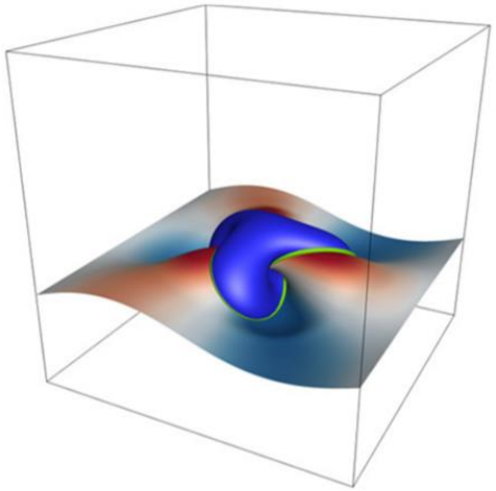
Spherical Clebsch representation

- Spherical Clebsch map (Kuznetsov & Mikhailov 1980, Chern *et al*., 2016)

$$s_1^2 + s_2^2 + s_3^2 = 1$$

$$\boldsymbol{\omega} = \frac{\hbar}{2}(s_1 \nabla s_2 \times \nabla s_3 + s_2 \nabla s_3 \times \nabla s_1 + s_3 \nabla s_1 \times \nabla s_2)$$

- Vortex surface field $\boldsymbol{\omega} \cdot \nabla s_p = 0, \quad p = 1, 2, 3$
- Work well with fields carrying non-zero helicity
- Hamiltonian mechanics

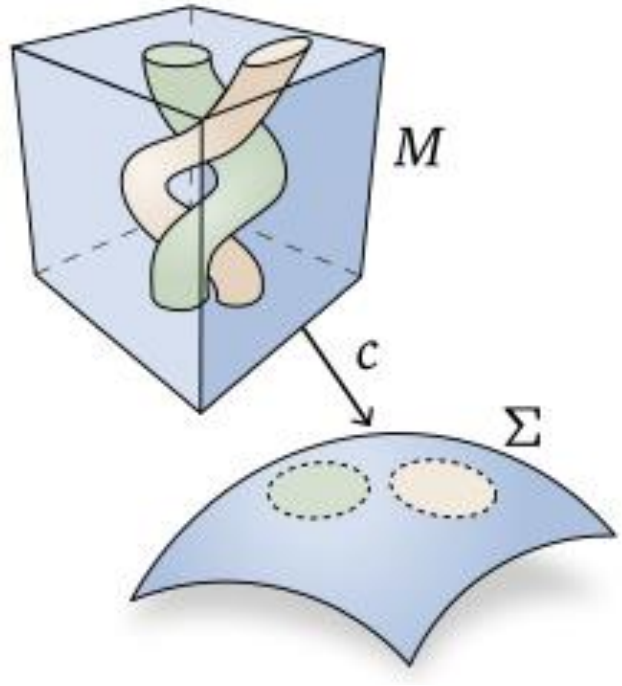


R. Tao, H. Ren, Y. Tong, & S. Xiong*, *Phys. Fluids.*, 2021

Geometry of Clebsch variables

- For a flow described by a vorticity 2-form $\boldsymbol{\omega}$, a function $c : M \to \Sigma$ for some 2-dimensional manifold Σ equipped with an area form $\mathrm{d}A_\Sigma$ is called a Clebsch variable if $\boldsymbol{\omega} = c^\star \mathrm{d}A_\Sigma$ where $c^\star \mathrm{d}A_\Sigma$ denotes the pull back of the area form.

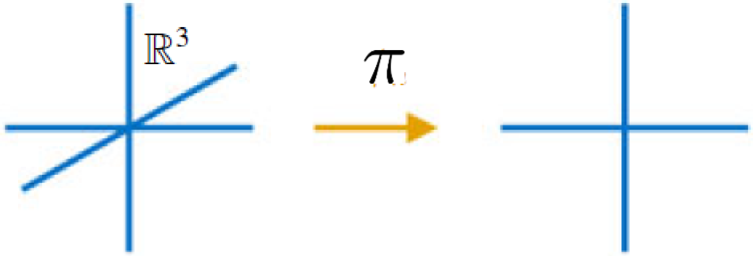
$$\int_\Omega \omega = \int_{c(\Omega)} dA_\Sigma = \mathrm{Area}_\Sigma(c(\Omega))$$



Schrödinger's Smoke
Chern et al., *SIGGRAPH*, 2016

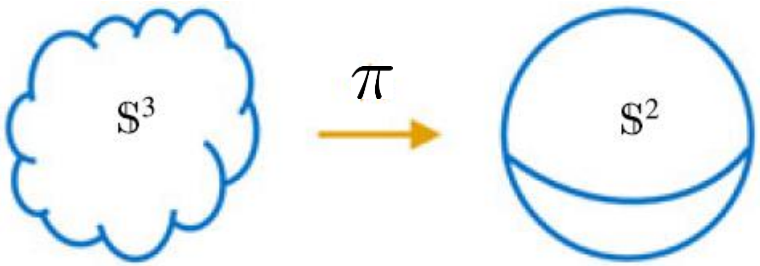
Lift

- Trivial bundle (Clebsch map)



$$\begin{tikzcd} & Q \arrow[d, "\pi"] \\ M \arrow[ur, dotted, "\phi"] \arrow[r, "s"'] & \Sigma \end{tikzcd}$$

- Hopf bundle (Spherical clebsch map)



$$\boldsymbol{\phi} = (\phi_1, \phi_2)^T = (a + bi, c + di)^T, \quad \text{with} \quad \langle \boldsymbol{\phi}, \boldsymbol{\phi} \rangle_{\mathbb{R}} = 1$$

$$s_1 = a^2 + b^2 - c^2 - d^2, \; s_2 = 2(bc - ad), \; s_3 = 2(ac + bd)$$

Component form of spherical Clebsch map

❑ Wave function

$$\boldsymbol{\phi} = (\phi_1, \phi_2)^T = (a + b\mathrm{i}, c + d\mathrm{i})^T, \quad \text{with} \quad \langle \boldsymbol{\phi}, \boldsymbol{\phi} \rangle_{\mathbb{R}} = 1$$

$$\boldsymbol{u} = \hbar \langle \boldsymbol{\nabla} \boldsymbol{\phi}, \mathrm{i}\boldsymbol{\phi} \rangle_{\mathbb{R}} = \hbar(a\boldsymbol{\nabla} b - b\boldsymbol{\nabla} a + c\boldsymbol{\nabla} d - d\boldsymbol{\nabla} c)$$

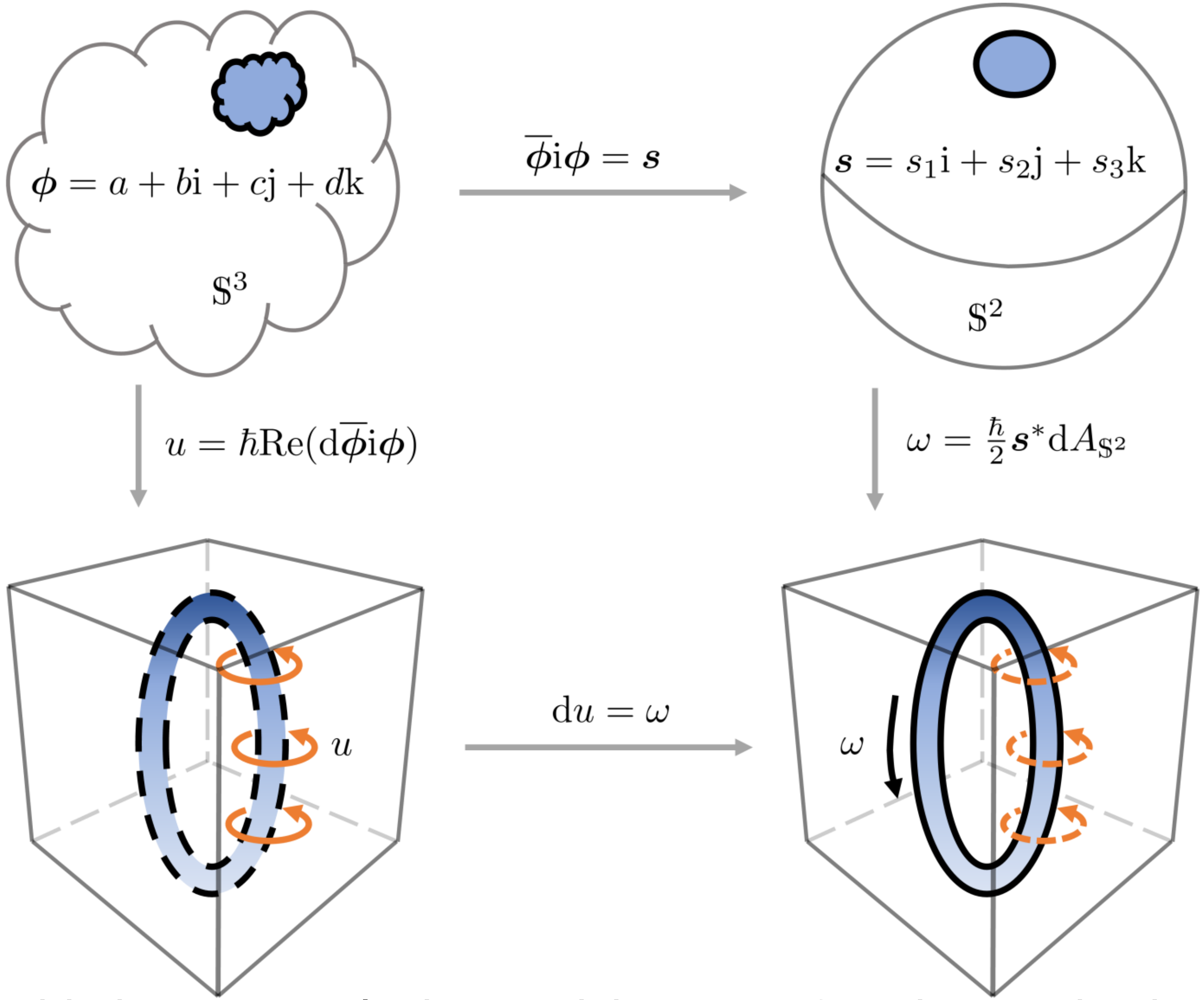
❑ Spin vector

$$\boldsymbol{s} = (s_1, s_2, s_3)$$

$$s_1 = a^2 + b^2 - c^2 - d^2, \;\; s_2 = 2(bc - ad), \;\; s_3 = 2(ac + bd)$$

$$\boldsymbol{\omega} = \frac{\hbar}{2}(s_1 \boldsymbol{\nabla} s_2 \times \boldsymbol{\nabla} s_3 + s_2 \boldsymbol{\nabla} s_3 \times \boldsymbol{\nabla} s_1 + s_3 \boldsymbol{\nabla} s_1 \times \boldsymbol{\nabla} s_2)$$

$$\boldsymbol{\omega} \cdot \boldsymbol{\nabla} s_p = 0, \quad p = 1, 2, 3$$



Relationship between velocity, vorticity, wave function, and spin vectors.
R. Tao, H. Ren, Y. Tong, & S. Xiong*, *Phys. Fluids.*, 2021

Gauge transformation

- Gauge transformation of wave functions

$$\boldsymbol{\phi} \to \boldsymbol{\phi}\exp\left(\mathrm{i}\frac{\varphi}{\hbar}\right) \quad \Longleftrightarrow \quad \boldsymbol{u} \to \boldsymbol{u} + \boldsymbol{\nabla}\varphi$$

- Incompressible flow

$$\langle \mathrm{i}\boldsymbol{\phi}, \Delta\boldsymbol{\phi}\rangle_{\mathbb{R}} = 0 \quad \Longleftrightarrow \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

- Constant background velocity

$$\boldsymbol{\psi} = \boldsymbol{\phi}\exp\left(\mathrm{i}\frac{\rho^i}{\hbar^i}\boldsymbol{U}\cdot\boldsymbol{x}\right) \quad \Longleftrightarrow \quad \boldsymbol{u}_{\psi} = \boldsymbol{u}_{\phi} + \boldsymbol{U}$$

Obejectives

- Propose feasible methods for **constructing spherical Clebsch maps** with finite energy, arbitrary geometry, and tunable helicity

- Propose a **gauge fluid framework based on spherical Clebsch wave functions** to solve incompressible fluids

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- Vortex surface field and Clebsch representation
- **Construction of knotted Clebsch fields**
- Clebsch gauge fluid

Construction of Clebsch map (wave function)

- Transform the Cartesian coordinates

$$\begin{cases} \alpha = \dfrac{2(x+\mathrm{i}y)f(r)}{1+r^2} \\ \beta = \dfrac{2(z-\mathrm{i})f(r)+(1+r^2)\mathrm{i}}{1+r^2} \end{cases}$$

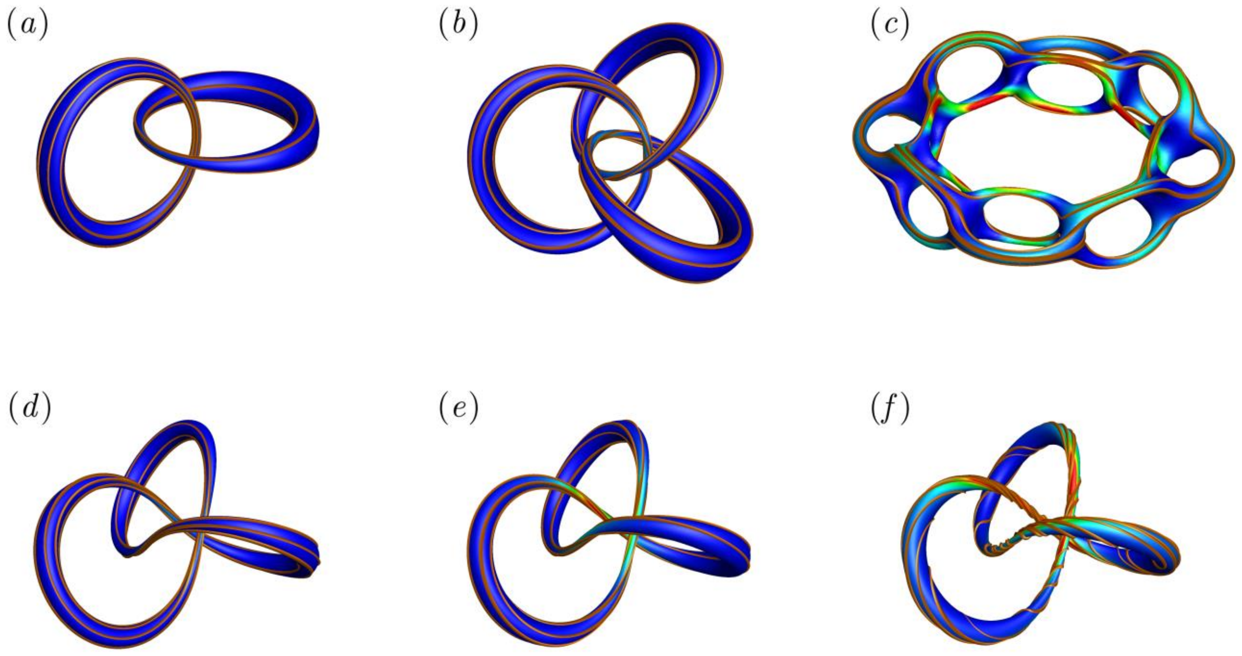
- Construct polynomials to represent the **twist** and the **geometric position** of the flux tube

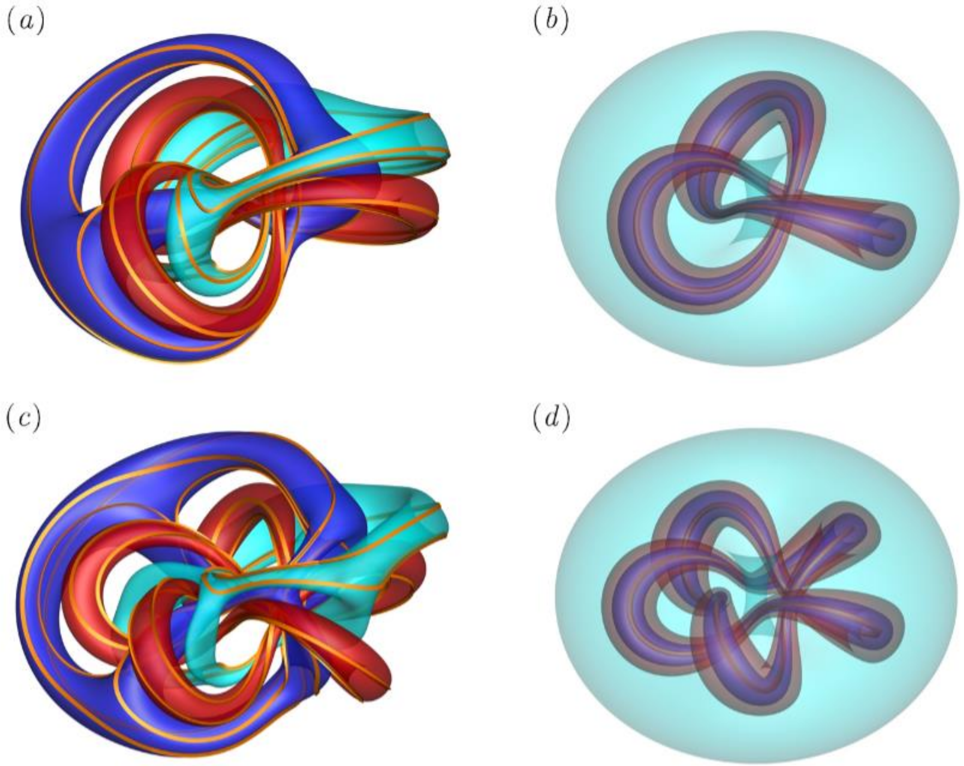
$$P = P(\alpha, \beta, \overline{\alpha}, \overline{\beta}) \qquad Q = Q(\alpha, \beta, \overline{\alpha}, \overline{\beta})$$

- Normalization and solenoidal projection

$$\phi_1 = \frac{Pe^{-\mathrm{i}q/\hbar}}{\sqrt{|P|^2+|Q|^2}}, \qquad \phi_2 = \frac{Qe^{-\mathrm{i}q/\hbar}}{\sqrt{|P|^2+|Q|^2}}, \qquad \Delta q = \hbar\,\langle \Delta\psi, \mathrm{i}\psi\rangle_{\mathbb{R}}$$

R. Tao, H. Ren, Y. Tong, & S. Xiong*, *Phys. Fluids.*, 2021



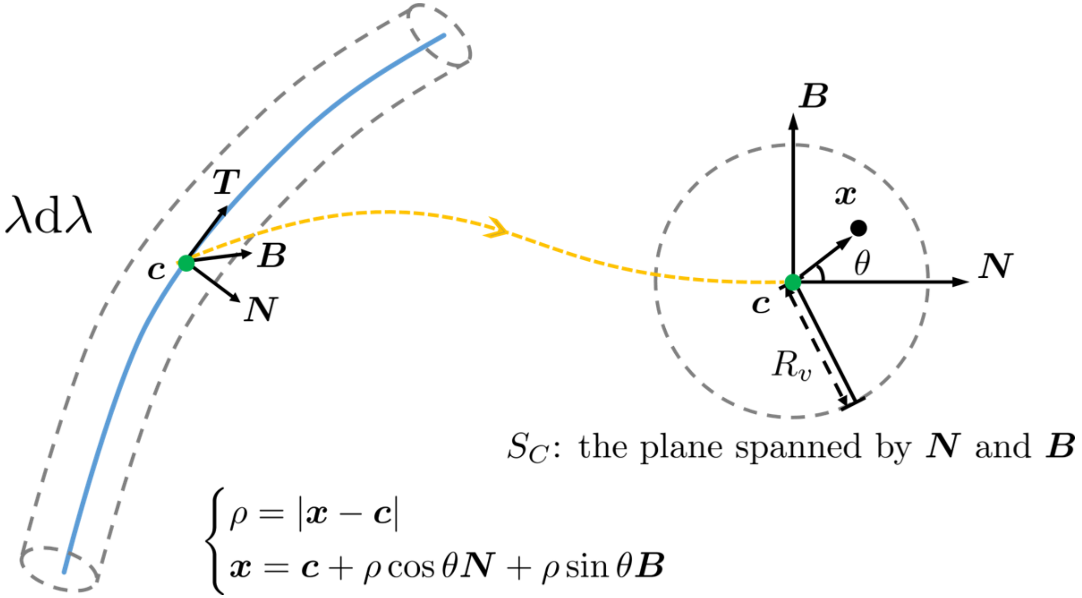


Isosurfaces of spin vector of initial knotted vortex tubes with six different wave functions.
R. Tao, H. Ren, Y. Tong, & S. Xiong*, *Phys. Fluids.,* 2021

Construction of Clebsch map (Spin vector)

- **Assumptions of the field (Gold & Hoyle, 1960)**
  - The axial component only depends on distance from the axis
  - All points of a vector line have the same distance from the axis
  - On vector lines, azimuth changes with the rate only depends on arclength

$$
\begin{cases}
s_1 = 2\sqrt{\dfrac{F}{\Gamma}\left(1-\dfrac{F}{\Gamma}\right)}\cos(2\pi q G), \\
s_2 = 2\sqrt{\dfrac{F}{\Gamma}\left(1-\dfrac{F}{\Gamma}\right)}\sin(2\pi q G), \\
s_3 = 1 - 2\dfrac{F}{\Gamma},
\end{cases}
\qquad
\begin{cases}
F(\rho) = 2\pi \displaystyle\int_0^{\rho} f(\lambda)\lambda \mathrm{d}\lambda \\
G = \dfrac{\theta}{2\pi} - \Xi(s)
\end{cases}
$$

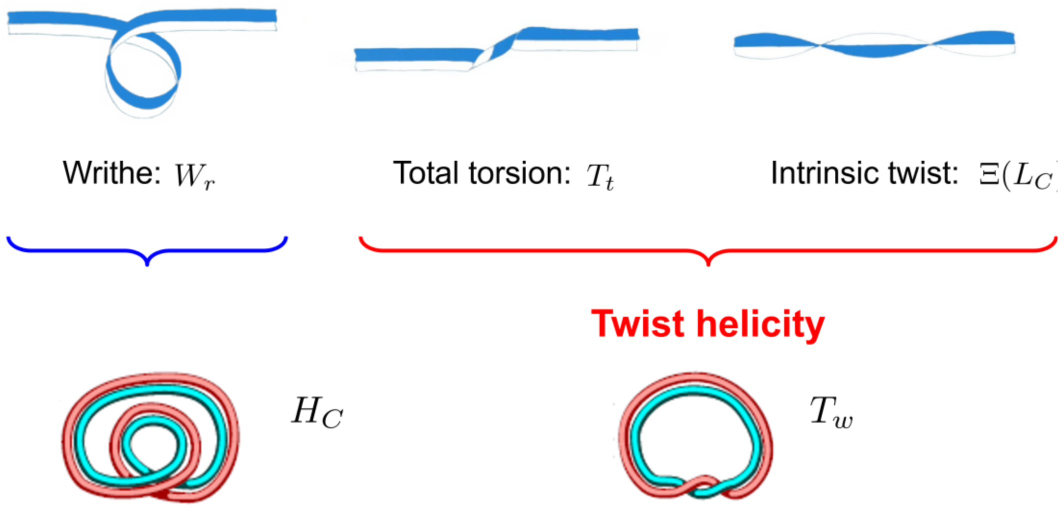


S. Xiong & Y. Yang, *Phys. Fluids.*, 2019, *J. Fluid Mech.*, 2020

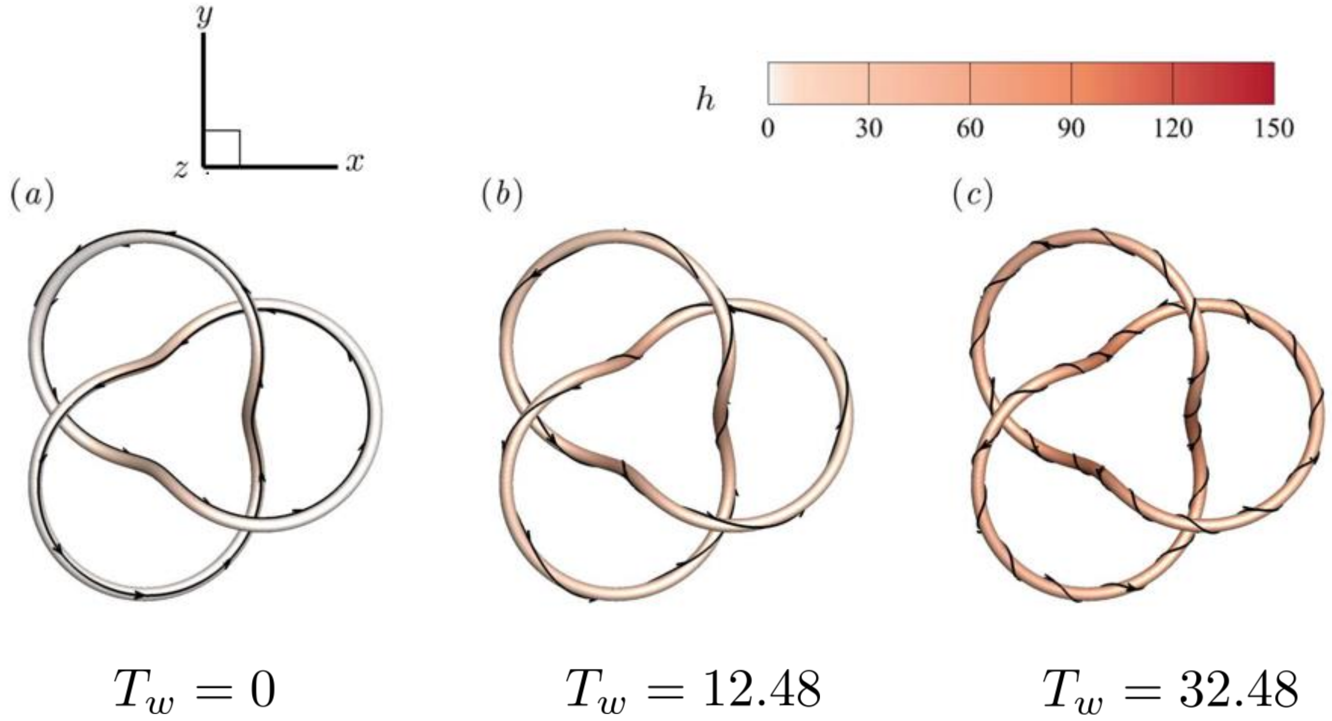
Decomposition of the helicity

- Helicity decomposes into **central-line helicity** and **twist helicity**

$$H = H_C + T_w$$

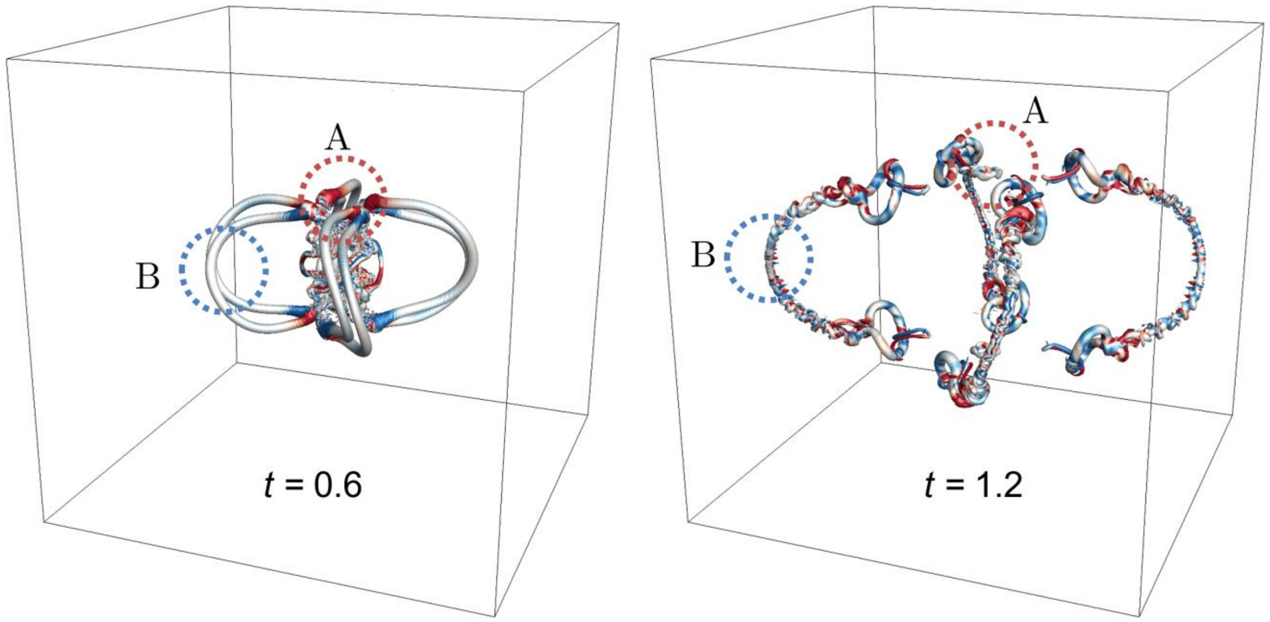


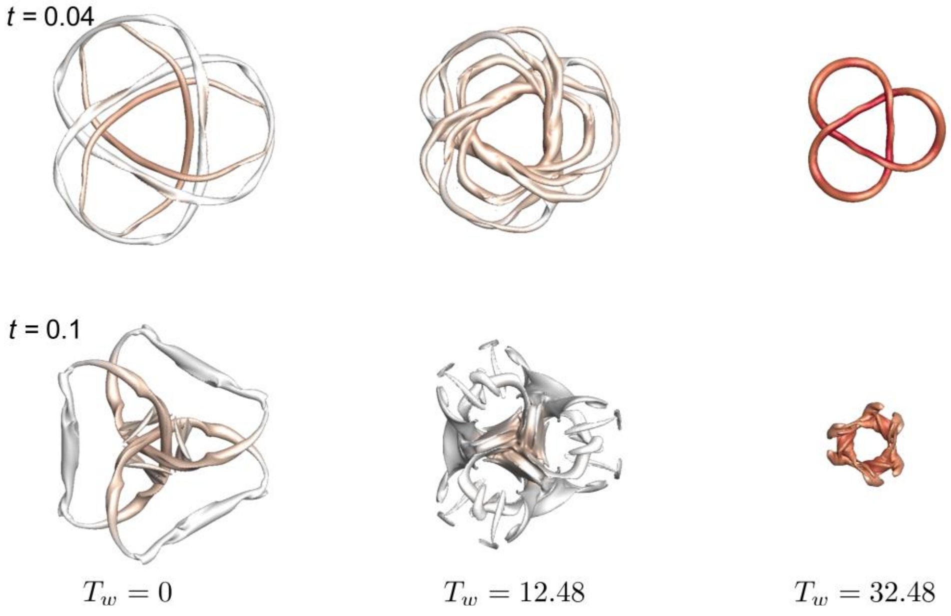
Scheeler *et al.*, *PNAS*, 2014



S. Xiong & Y. Yang, *Phys. Fluids.*, 2019, *J. Fluid Mech*., 2020

Evolution of flux tubes





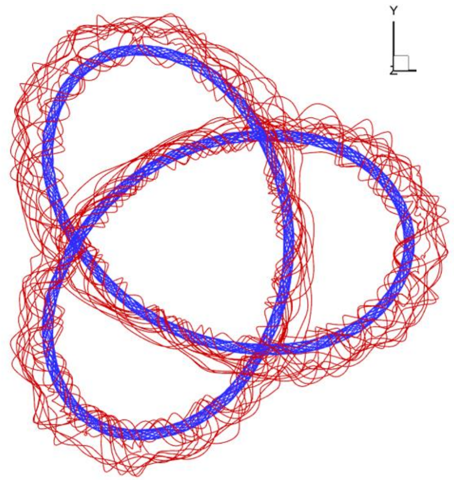
Linked rings with opposite chirality can reconnect after a short time, resulting in rapid scale cascade towards a turbulent-like state

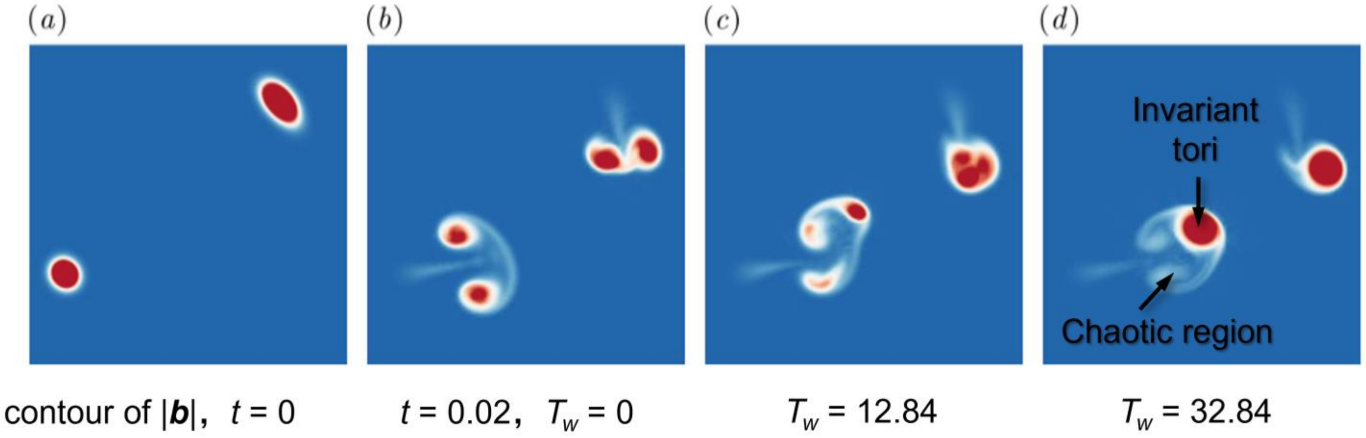
In the **relaxation of knotted magnetic tubes**, magnetic energy is gradually released and converted to kinetic energy

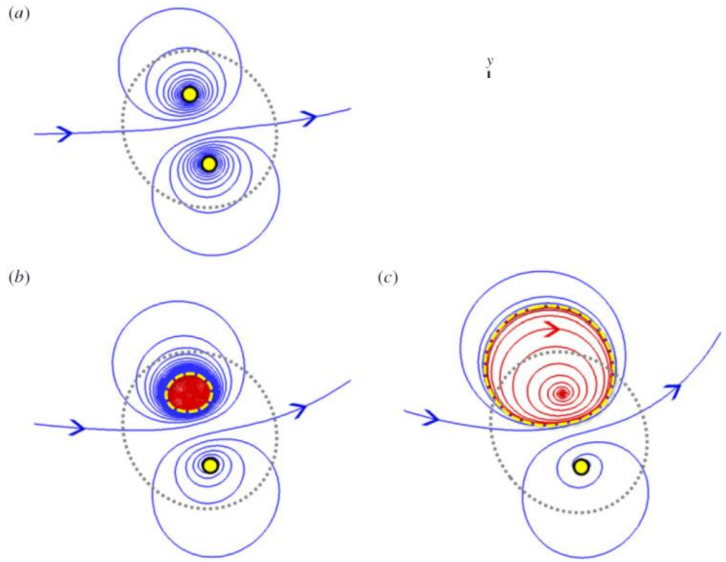
S. Xiong & Y. Yang, *Phys. Fluids.,* 2019, *Sci. Sin-Phys. Mech. Astron*., 2020, *J. Fluid Mech*., 2020

Splitting of magnetic tubes

- Pressure is approximated as a **solenoidal projection of Lorentz force**
- Concentration of magnetic field increases with **twist** increasing







S. Xiong & Y. Yang, *J. Fluid Mech.*, 2020

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- Construction of knotted Clebsch fields
- **Clebsch gauge fluid**

Lagrangian advection

$$\begin{cases} \boldsymbol{\phi} = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k} \\ \boldsymbol{s} = s_1\mathrm{i} + s_2\mathrm{j} + s_3\mathrm{k} \\ \boldsymbol{f}^{\phi} = f_1^{\phi}\mathrm{i} + f_2^{\phi}\mathrm{j} + f_3^{\phi}\mathrm{k} \\ \boldsymbol{f}^{s} = \overline{\boldsymbol{\phi}}\boldsymbol{f}^{\phi}\boldsymbol{\phi} = f_1^{s}\mathrm{i} + f_2^{s}\mathrm{j} + f_3^{s}\mathrm{k} \end{cases}$$

$$\begin{cases} \dfrac{D\boldsymbol{\phi}}{Dt} = \boldsymbol{f}^{\phi}\boldsymbol{\phi} \\ \dfrac{D\boldsymbol{s}}{Dt} = 2\boldsymbol{s} \times \boldsymbol{f}^{s} \\ \dfrac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}\left(\dfrac{|\boldsymbol{u}|^2}{2} - \hbar\left\langle \boldsymbol{s}, \boldsymbol{f}^{s}\right\rangle_{\mathbb{R}}\right) - \hbar\left\langle (\boldsymbol{\nabla}\boldsymbol{s}), \boldsymbol{f}^{s}\right\rangle_{\mathbb{R}} \\ \dfrac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) - \hbar\boldsymbol{\nabla} \times \left\langle (\boldsymbol{\nabla}\boldsymbol{s}), \boldsymbol{f}^{s}\right\rangle \end{cases}$$

Euler flows

$$\boldsymbol{f}^{\phi} = -\mathrm{i}\frac{1}{\hbar}\left(\frac{p}{\rho} - G - \frac{|\boldsymbol{u}|^2}{2}\right)$$

$$\begin{cases} \dfrac{D\boldsymbol{\phi}}{Dt} = -\mathrm{i}\dfrac{1}{\hbar}\left(\dfrac{p}{\rho} - G - \dfrac{|\boldsymbol{u}|^2}{2}\right)\boldsymbol{\phi} \\ \dfrac{D\boldsymbol{s}}{Dt} = \boldsymbol{0} \\ \dfrac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}\left(\dfrac{p}{\rho} - G\right) \\ \dfrac{\partial\boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{\omega}) \end{cases}$$

Schrödinger flows (Chern *et al*., 2016)

$$\boldsymbol{f}^{\phi} = -\frac{\hbar}{4}\boldsymbol{\phi}\boldsymbol{s}(\boldsymbol{s}\times\Delta\boldsymbol{s})\overline{\boldsymbol{\phi}} + \mathrm{i}\frac{1}{\hbar}\left(\frac{|\boldsymbol{u}|^2}{2} - \frac{p}{\rho}\right)$$

$$\Pi = \hbar^2\frac{|\boldsymbol{\nabla}\boldsymbol{\phi}|^2}{2} + \hbar^2\frac{\langle\boldsymbol{s},\Delta\boldsymbol{s}\rangle}{4} + \frac{p}{\rho} - \frac{|\boldsymbol{u}|^2}{2}$$

$$\begin{cases} \dfrac{\partial\boldsymbol{\phi}}{\partial t} - \mathrm{i}\dfrac{\hbar}{2}\Delta\boldsymbol{\phi} + \mathrm{i}\dfrac{1}{\hbar}\Pi\boldsymbol{\phi} = \boldsymbol{0} \\ \dfrac{D\boldsymbol{s}}{Dt} = \dfrac{\hbar}{2}\boldsymbol{s}\times\Delta\boldsymbol{s} \\ \dfrac{D\boldsymbol{u}}{Dt} = -\boldsymbol{\nabla}\dfrac{p}{\rho} - \left(\dfrac{\hbar}{2}\right)^2\langle(\boldsymbol{\nabla}\boldsymbol{s}),\Delta\boldsymbol{s}\rangle_{\mathbb{R}} \\ \dfrac{\partial\boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{\omega}) - \left(\dfrac{\hbar}{2}\right)^2\boldsymbol{\nabla}\times\langle(\boldsymbol{\nabla}\boldsymbol{s}),\Delta\boldsymbol{s}\rangle_{\mathbb{R}} \end{cases}$$

S. Yang, S. Xiong*, Y. Zhang, F. Feng, J. Liu, and B. Zhu, *SIGGRAPH*, 2021
R. Tao, H. Ren, Y. Tong, & S. Xiong*, *Phys. Fluids*., 2021

Gauge transformation for divergence-free projection

$$
\begin{cases}
\dfrac{D\boldsymbol{\phi}}{Dt} = -\mathrm{i}\dfrac{\rho^i}{\hbar^i}\left(\dfrac{p}{\rho^i} - G - \dfrac{|\boldsymbol{u}|^2}{2}\right)\boldsymbol{\phi}, & \boldsymbol{x} \in \Omega^i \\
\langle \mathrm{i}\boldsymbol{\phi}, \Delta\boldsymbol{\phi}\rangle_{\mathbb{R}} = 0, & \boldsymbol{x} \in \Omega^i \\
\dfrac{\hbar^i}{\rho^i}\langle \boldsymbol{\nabla}\boldsymbol{\phi}, \mathrm{i}\boldsymbol{\phi}\rangle_{\mathbb{R}} \cdot \boldsymbol{n} = \boldsymbol{u}_\partial \cdot \boldsymbol{n}, & \boldsymbol{x} \in \partial\Omega_b^i \\
[p]_{\mathrm{jump}} = \gamma\kappa, & \boldsymbol{x} \in \partial\Omega_f
\end{cases}
$$

$$
\Rightarrow
$$

$$
\boxed{\boldsymbol{\psi} = \boldsymbol{\phi}\exp\left(\mathrm{i}\varphi\frac{\rho^i}{\hbar^i}\right)}
$$

$$
\begin{cases}
\dfrac{D\boldsymbol{\psi}}{Dt} = -\mathrm{i}\dfrac{\rho^i}{\hbar^i}\left[\dfrac{p}{\rho^i} - G - \dfrac{D\varphi}{Dt} - \dfrac{|\boldsymbol{u}|^2}{2}\right]\boldsymbol{\psi}, & \boldsymbol{x} \in \Omega^i \\
\boldsymbol{u}_m = \dfrac{\hbar^i}{\rho^i}\langle \boldsymbol{\nabla}\boldsymbol{\psi}, \mathrm{i}\boldsymbol{\psi}\rangle_{\mathbb{R}}, & \boldsymbol{x} \in \Omega^i, \\
\Delta\varphi = \boldsymbol{\nabla} \cdot \boldsymbol{u}_m, & \boldsymbol{x} \in \Omega^i \\
\boldsymbol{u} = \boldsymbol{u}_m - \boldsymbol{\nabla}\varphi, & \boldsymbol{x} \in \Omega^i \\
\dfrac{\hbar^i}{\rho^i}\langle \boldsymbol{\nabla}\boldsymbol{\phi}, \mathrm{i}\boldsymbol{\phi}\rangle_{\mathbb{R}} \cdot \boldsymbol{n} = \boldsymbol{u}_\partial \cdot \boldsymbol{n}, & \boldsymbol{x} \in \partial\Omega_b^i \\
\partial_n\varphi = 0, & \boldsymbol{x} \in \partial\Omega_b^i \\
[p]_{\mathrm{jump}} = \gamma\kappa, & \boldsymbol{x} \in \partial\Omega_f
\end{cases}
$$

S. Yang, S. Xiong*, Y. Zhang, F. Feng, J. Liu, and B. Zhu, *SIGGRAPH*, 2021

Gauge transformation for surface tension

- We introduce **an auxiliary variable *q* to handle interface conditions**

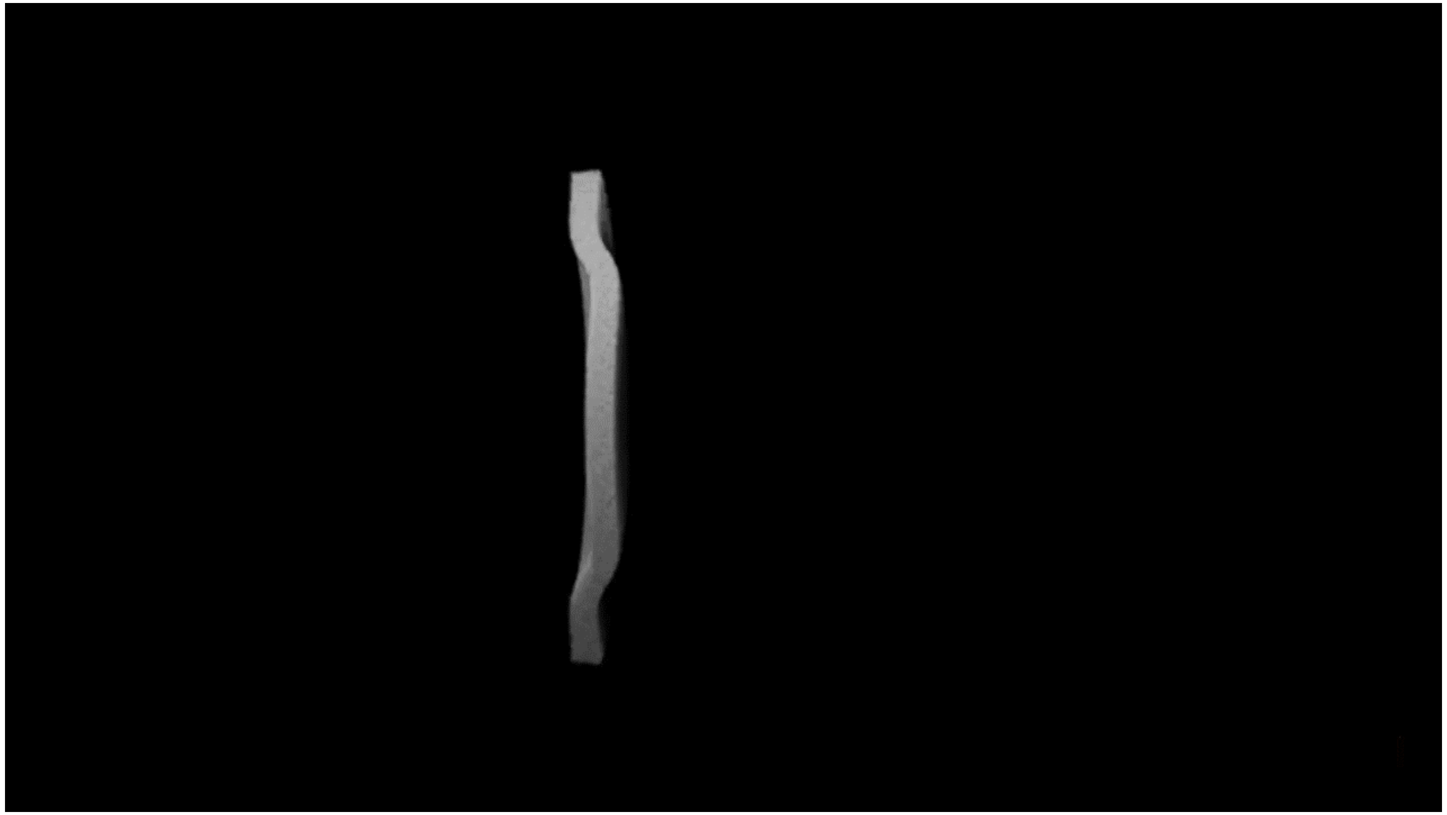
$$\begin{cases} \dfrac{D\boldsymbol{\psi}}{Dt} = -\mathrm{i}\dfrac{\rho^i}{\hbar^i} q\boldsymbol{\psi} \\ \boldsymbol{u}_m = \dfrac{\hbar^i}{\rho^i}\langle \boldsymbol{\nabla}\boldsymbol{\psi}, \mathrm{i}\boldsymbol{\psi}\rangle_{\mathbb{R}} \\ \Delta q = 0 \\ \Delta\varphi = \boldsymbol{\nabla}\cdot\boldsymbol{u}_m \\ \boldsymbol{u} = \boldsymbol{u}_m - \boldsymbol{\nabla}\varphi \end{cases} \quad \boldsymbol{x}\in\Omega^i$$

$$\begin{cases} \boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{n}\cdot\boldsymbol{u}_\partial \\ \partial_n q = 0 \\ \partial_n \varphi = 0 \end{cases} \quad \boldsymbol{x}\in\partial\Omega_b$$

$$\begin{cases} [\varphi]_{\mathrm{jump}} = [\partial_n\varphi]_{\mathrm{jump}} = 0 \\ \left[\rho^i\left(q + G + \dfrac{D\varphi}{Dt} + \dfrac{|\boldsymbol{u}|^2}{2}\right)\right]_{\mathrm{jump}} = \gamma\kappa \end{cases} \quad \boldsymbol{x}\in\partial\Omega_f^i$$

S. Yang, S. Xiong*, Y. Zhang, F. Feng, J. Liu, and B. Zhu, *SIGGRAPH*, 2021

Leapfrogging vortex rings



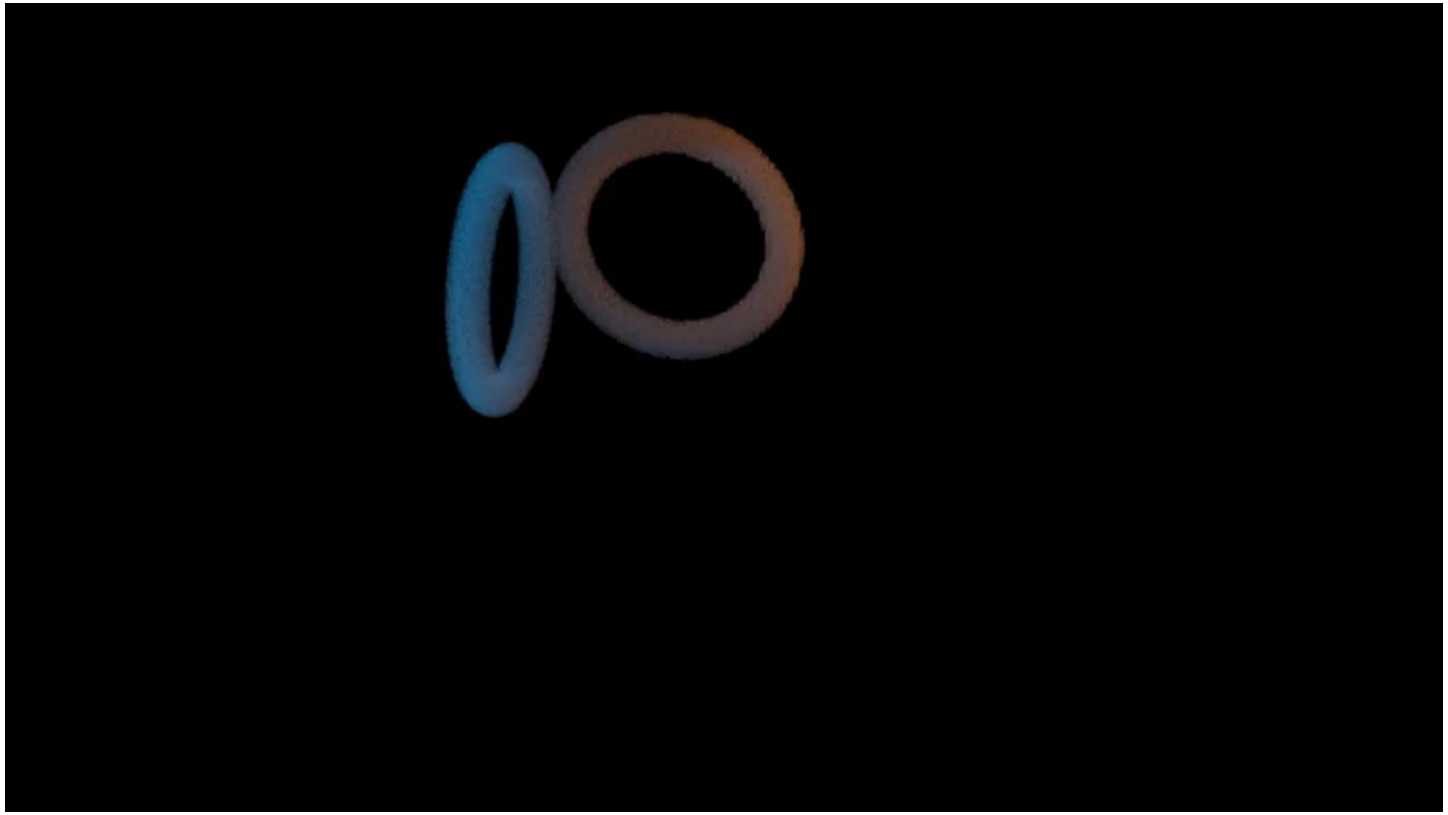
(Comparison with the standard grid-based method)

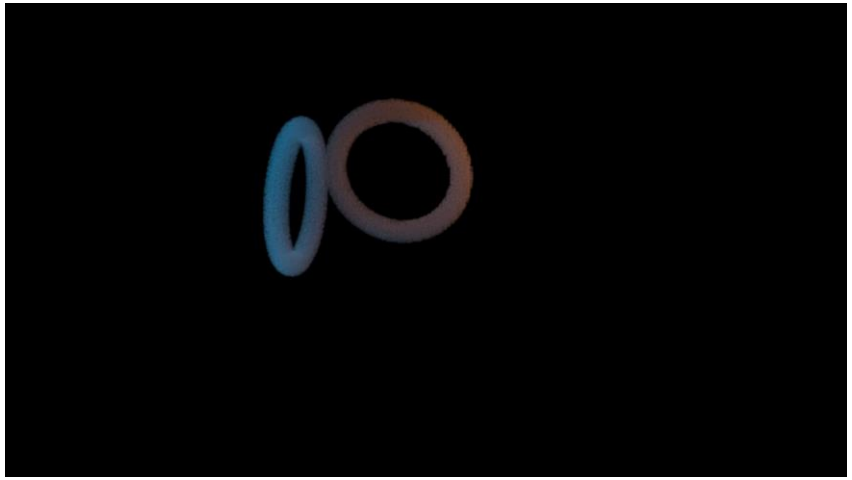
S. Yang, S. Xiong*, Y. Zhang, F. Feng, J. Liu, and B. Zhu, *SIGGRAPH*, 2021

Wave functions and VSF in Leapfrogging vortex rings

S. Yang, S. Xiong*, Y. Zhang, F. Feng, J. Liu, and B. Zhu, *SIGGRAPH*, 2021

Oblique ring collision

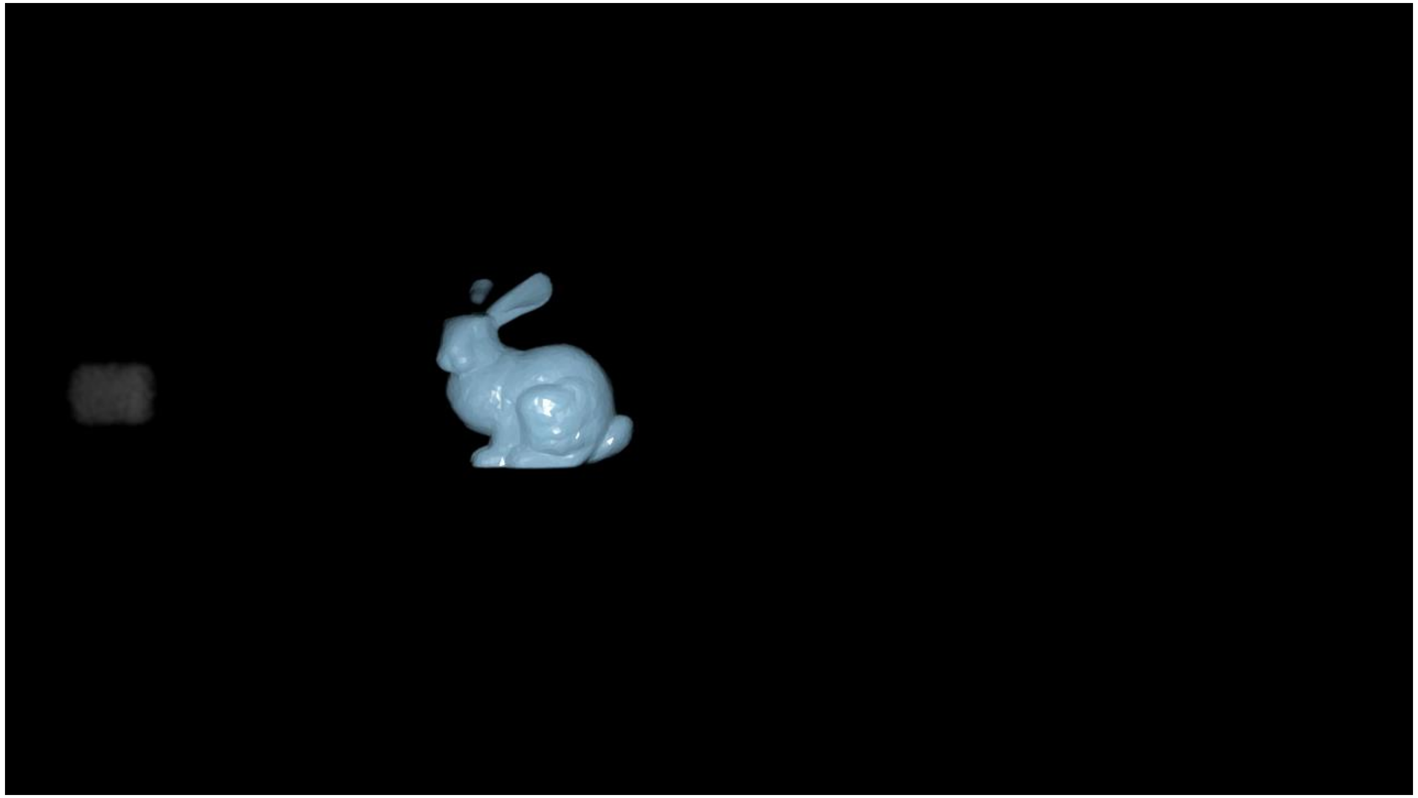




(Comparison with the standard grid-based method)

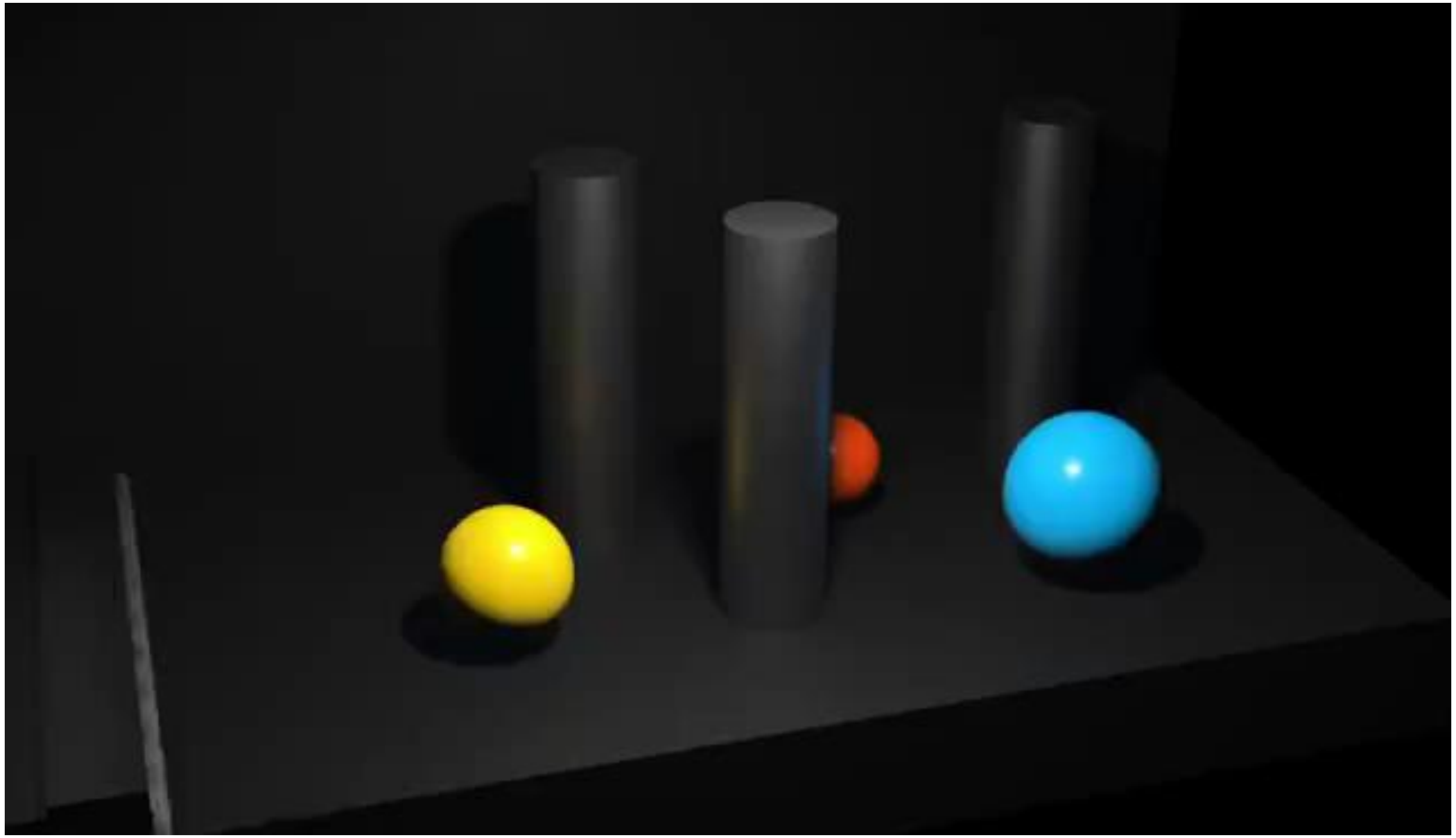
S. Yang, S. Xiong*, Y. Zhang, F. Feng, J. Liu, and B. Zhu, *SIGGRAPH*, 2021

Smoke flowing over a bunny obstacle



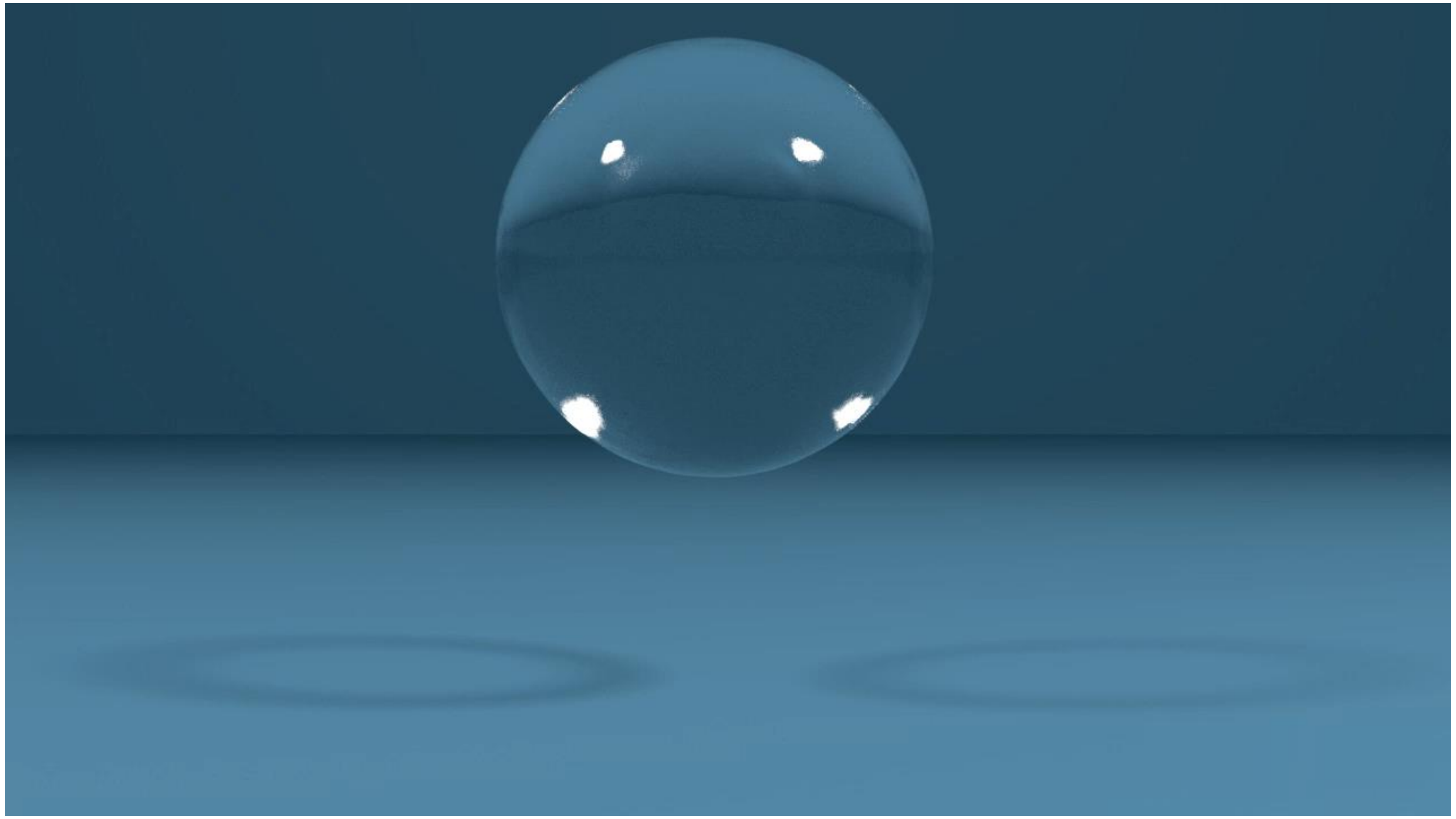
S. Yang, S. Xiong*, Y. Zhang, F. Feng, J. Liu, and B. Zhu, *SIGGRAPH*, 2021

Smoke flowing over multiple obstacles



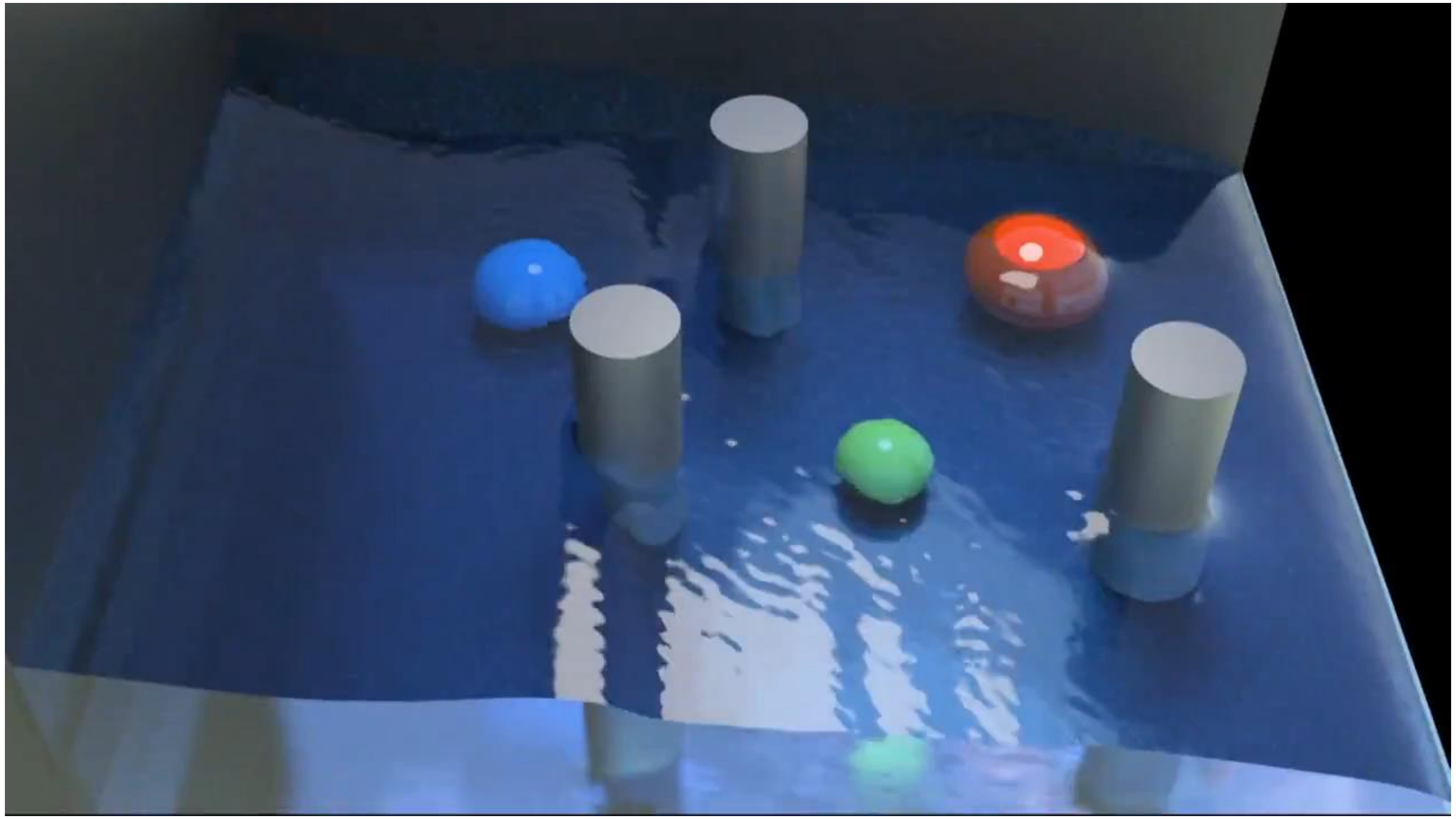
S. Yang, S. Xiong*, Y. Zhang, F. Feng, J. Liu, and B. Zhu, *SIGGRAPH*, 2021

Droplet



S. Yang, S. Xiong*, Y. Zhang, F. Feng, J. Liu, and B. Zhu, *SIGGRAPH*, 2021

Liquid flowing over multiple obstacles



S. Yang, S. Xiong*, Y. Zhang, F. Feng, J. Liu, and B. Zhu, *SIGGRAPH*, 2021

Conclusions

- We propose two feasible methods for **constructing spherical Clebsch maps** with the finite energy, arbitrary geometry, and tunable helicity

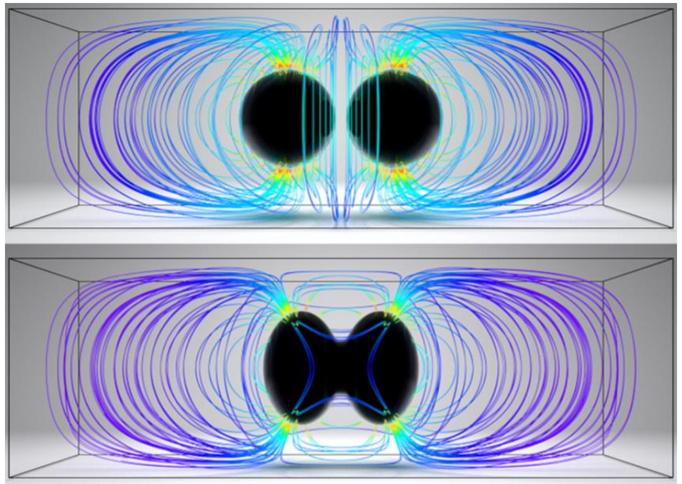
- We propose a **Clebsch gauge fluid framework** (math, algorithm, code) to solve incompressible fluid equations

- Our fluid solver can be used to solve various fluid dynamics, including complex vortex reconnection, fluids with different obstacles, and surface-tension flows

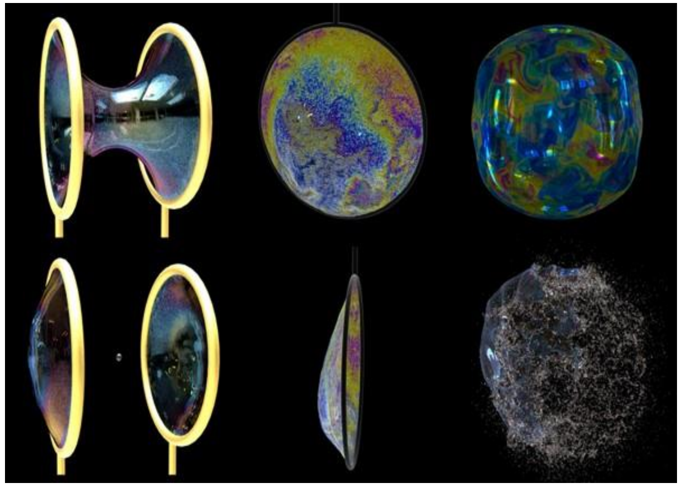
Future work

- ❑ Further study on the math foundation and numerical method for the Clebsch gauge fluid
- ❑ Extend the proposed framework to **describe different physics**, such as solid-fluid interaction, magnetohydrodynamics, and thin-film flows

Ruan *et al*., *SIGGRAPH*, 2021



Ni *et al*., *SIGGRAPH*, 2020



Wang *et al., SIGGRAPH,* 2021

Thank you!

shiying.xiong@dartmouth.edu